Role of fine-grained uncertainty in determining the limit of preparation contextuality

Gautam Sharma,[1,*] Sk Sazim,[1,2,†] and Shiladitya Mal[1,‡]

[1]*Quantum Information and Computation Group, Harish-Chandra Research Institute, Homi Bhabha National Institute, Allahabad 211019 India*

[2]*RCQI, Institute of Physics, Slovak Academy of Sciences, 845 11 Bratislava, Slovakia*

(Received 20 August 2021; accepted 9 September 2021; published 28 September 2021; corrected 8 December 2021)

The optimal success probability of a communication game sets fundamental limitations on an operational theory. The quantum advantage of a parity oblivious random access code (PORAC), a communication game, over classical resources reveals the preparation contextuality of quantum theory [R. W. Spekkens *et al.*, Phys. Rev. Lett. **102**, 010401 (2009)]. The optimal quantum advantage in the N-dit PORAC game for finite dimensions is an open problem. Here, we show that the degree of uncertainty allowed in an operational theory determines the amount of preparation contextuality. We connect the upper bound of the fine-grained uncertainty relation to the success probability of a PORAC game played with the quantum resource. Subsequently, we find the optimal success probability for a 2-dit PORAC game using mutually unbiased bases for the decoding strategy. Finally, we also derive an upper bound on the quantum advantage for the N-dit PORAC game.

DOI: 10.1103/PhysRevA.104.032424

I. INTRODUCTION

Quantum physics has several fundamental no-go theorems, which reveals how radically it deviates from classical physics. Bell theorem states that quantum theory cannot be reproduced by a local realist model [1,2]. On the other hand, the Bell-Kochen-Specker theorem asserts that quantum theory is contextual [3,4]. This means that the observables cannot be assigned definite values, independent of the setting in which they are measured, i.e., the context. Later, the notion of contextuality was generalized so that it can be associated with any operational theory [5].

These no-go theorems arise out of quantum correlations[2]. In the context of spatial correlation, it is known that nonlocality of a theory is not enough that it allows signaling [6]. Quantum correlation between spacelike separated measurements is restricted by the Cirelson type bound [7]. Subsequently, it was asked whether there are some physical principles which limit the amount of nonlocality. There are approaches from information theory [8,9], communication complexity [10,11], and local quantum mechanics [12] to address this question. In Ref. [13], the authors took a very different approach, i.e., they related the limit of nonlocality to two inherent properties of any physical theory, called uncertainty [14] and steerability [15–17].

Initially, uncertainty relations were stated in terms of the product of standard deviations lower bounded by some quantity related to commutators of the observables measured [14,18]. Later, entropic uncertainty relations which are state independent were introduced [19,20]. However, entropic measures depict uncertainty in a coarse way, as they do not capture uncertainty in the realization of different outcome distributions for multiple measurements. To circumvent this, the fine-grained uncertainty relation (FUR) was introduced, which is a set of inequalities, one for each possible combination of outcomes [13]. Later this inequality was generalized for higher-dimensional systems for mutually unbiased bases (MUBs) [21].

Fundamental limiting features of a theory have often been studied through the ability of some communication games to process information [9,22]. The random access code (RAC) is a two-player communication game [23,24]; a party, say, Alice, holding a data set in the form of an n-bit string, encodes it in a state and sends it to another party, Bob, whose task is to guess any one of the bits randomly chosen from the string (see Fig. 1). The generalization of bits to higher dimensions is dits. A bit refers to a two-level system, whereas a dit is a d-level system. Therefore, instead of an n-bit string, Alice can also encode an n-dit string in a state and send it to Bob, who then tries to guess a dit from the string [25].

An interesting connection between an RAC game and contextuality was made by Spekkens *et al.* [26], invoking the parity obliviousness constraint. The constraint of parity obliviousness in an RAC game demands that encoding is such that the receiver cannot know the parity of the incoming signal x from the sender. One of the ways of defining the parity of message x is as the sum (mod d) of the bit values contained in the message. If the parity obliviousness constraint is imposed on the RAC game, which we discuss in detail, then the optimal success probability of winning with classical resources coincides with that when resources are taken from noncontextual theory. Therefore, the quantum advantage of a parity oblivious random access code (PORAC) game implies the preparation contextuality of quantum theory. It was also shown that preparation contextuality leads to nonlocality [27–30]. Following this notion connection has been made between the PORAC game and other nonlocal

[*]gautam.oct@gmail.com
[†]sk.sazimsq49@gmail.com
[‡]shiladitya@hri.res.in

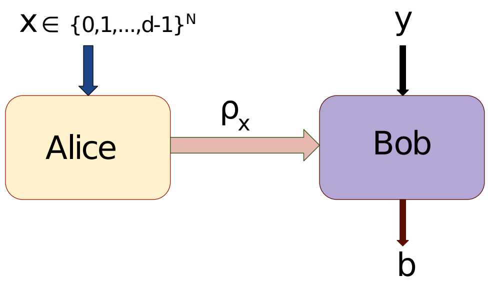

FIG. 1. In this communication game, Alice encodes the classical string $x \in \{0, 1, \dots, d-1\}^N$ in state ρ_x. Upon receiving the state ρ_x Bob performs a measurement X_i chosen uniformly from a set of N observables and tries to guess the yth dit of x using his measurement outcome b.

games [30–32] and the optimal quantum bound follows from the Cirelson-like bound associated with nonlocality [7]. To reveal preparation contextuality, the PORAC game was studied for higher-dimensional single systems and experimental realization was demonstrated as well [26,33,34]. The optimal quantum advantage of the PORAC game was derived when n-bit classical information is encoded in higher-dimensional systems [32]. Up to a few dimensions, the maximal quantum violation of preparation noncontextuality inequality was also derived numerically in [33]. In general, finding the optimal quantum bound for a high-level PORAC game or the maximal quantum violation of noncontextuality inequality is an open question.

Here we show that the degree of uncertainty, which is a property of a theory, determines to what extent a theory would be preparation contextual. Specifically, we derive a tight FUR for any pair of measurements in any finite dimension and show that the upper bound of the FUR is closely related to the quantum advantage of the PORAC game in terms of the enhanced success probability over classical strategy. We then prove that the optimum quantum bound is reached when Alice encodes a 2-dit string in a single qudit state, which are the maximal certain states with respect to a pair of MUBs, while Bob's decoding strategy is to perform those MUBs. We also derive the quantum upper bound of the FUR for n arbitrary observables and show that the FUR upper bound also gives the upper bound of the success probability in an n-dit PORAC game, although our upper bound for an n-dit PORAC game might not be exactly reached by the quantum strategy induced by the FUR. Finally, we compare some results regarding the maximal quantum bound obtained previously with our result, for the sake of completeness.

The plan of the paper is as follows. In Sec. II, we describe preliminary ideas on the PORAC game and parity obliviousness. Then, in Sec. III, we briefly discuss fine-grained uncertainty relations and derive upper bounds of various sets of sharp measurements. In Sec. IV, we present our main result; we connect the FUR upper bound with the success probability of the PORAC game. We also compare our results with the known bounds. Finally, we conclude in Sec. V.

II. PREPARATION NONCONTEXTUALITY FROM PARITY OBLIVIOUS RANDOM ACCESS CODES

Preparation noncontextuality associated with an operational theory was first introduced in [5]. An operational theory provides the probabilities $p(k|P, M)$ of getting an outcome k given the preparation procedure P and the measurement M. Quantum theory is also an operational theory in which a preparation procedure P is represented by ρ_P and a measurement is represented by a positive operator-valued measure, $\Lambda_{M,k}$. The probability of getting an outcome k is $p(k|P, M) = \mathrm{Tr}(\rho_P \Lambda_{M,k})$.

An operational theory is said to be preparation noncontextual if two preparations yield the same measurement statistics for all possible measurements, which implies that the probability associated with two different preparations at the hidden variable level(λ) is also the same, i.e.,

$$\forall M\ \forall k; p(k|P, M) = p(k|P', M) \Rightarrow p(\lambda|P) = p(\lambda|P'), \qquad (1)$$

where λ is a hidden variable and P and P' denote two preparation procedures.

Preparation contextuality was demonstrated using the parity oblivious communication game [26,33]. In the game, Alice receives an N-dit string $x \in \{0, 1, \dots, d-1\}^N$, which she encodes in a state ρ_x and then sends to Bob, chosen uniformly. Bob's task is to guess the yth bit of string x, using his measurement outcome b obtained by a set of measurements Y, as shown in Fig. 1. There is the cryptographic constraint that Alice can encode her message under the parity obliviousness condition that no information about the parity of x can be revealed to Bob. If $s \in \mathrm{Par}$, where $\mathrm{Par} \equiv \{s|s \in \{0, 1, \dots, d-1\}^N, \zeta \leqslant d-2\}$, with ζ denoting the number of zeros appearing in a particular s, then no information about $x \cdot s = \oplus_i x_i s_i (\mathrm{mod}\, d) = l$, $\forall l \leqslant d-1$, should be revealed to Bob. We refer to this task as $N \rightarrow 1$ d-parity oblivious random access codes (d-PORACs). The parity obliviousness condition, for the set of measurements Y performed by Bob, can be cast down in the form of the equality

$$\forall s, b, l, l', y; \quad \frac{1}{p(l)} \sum_{x\cdot s=l} p(b|x, y) = \frac{1}{p(l')} \sum_{x\cdot s=l'} p(b|x, y),$$

where $p(l) = \sum_{x\cdot s=l} p(x)$. As for all l-parity strings x_l, we have d^{N-1} uniform choices, $p(l) = p(l')$. Thus, the above obliviousness condition reduces to

$$\forall s, b, l, l', y; \quad \sum_{x\cdot s=l} p(b|x, y) = \sum_{x\cdot s=l'} p(b|x, y). \qquad (2)$$

It should be noted that Eq. (1) is the most general form of parity obliviousness, whereas Eq. (2) implies obliviousness with respect to the measurements performed by Bob. For our purpose it is sufficient to consider the above condition of parity obliviousness.

Given the obliviousness constraint, Bob's task is to maximize the average success probability of reporting the correct output $b = x_y$. The average probability of guessing the correct bit is given by

$$p(b = x_y) = \frac{1}{d^N N} \sum_{x\in\{0,1,\dots,d-1\}^N} \sum_{y\in\{1,\dots,N\}} p(b|x, y),$$

Different operational theories provide different maximal success probabilities of the game. It was shown in [33] that for an operational theory which admits a preparation noncontextual hidden-variable model, the probability of success for an

$N \to 1$ d-PORAC is bounded by the following inequality:

$$\frac{1}{d^N N}\sum_{x\in\{0,1,\ldots,d-1\}^N}\sum_{y\in\{1,\ldots,N\}} p(b|x,y) \leqslant \frac{N+d-1}{dN}. \quad (3)$$

Any operational theory which violates this inequality is contextual.

III. FINE-GRAINED UNCERTAINTY RELATIONS

Suppose we want to measure N different observables X_i, where $i \in \{1,\ldots,N\}$, and outcomes $x_i \in \{0,\ldots,d-1\}$. One can quantify the uncertainty associated with the measurements using entropic uncertainty relations as

$$\sum_{i=1}^{N} H(X_i)_\rho \geqslant \beta,$$

where β depends on the compatibility between different observables. However, entropy is a coarse way of measuring the uncertainty and incompatibility of a set of measurements. It does not reflect the uncertainty inherent in obtaining a particular combination of outcomes x_i for different measurements X_i. To circumvent this issue, a fine-grained uncertainty relation was proposed in Ref. [13]. The uncertainty relation is a set of d^N inequalities of the form

$$P^{\text{cert}}(\rho,x) = \sum_{i=1}^{N} p(X_i)p(x_i|X_i)_\rho \leqslant C_x(\mathcal{O},\mathcal{P}), \quad (4)$$

where $C_x(\mathcal{O},\mathcal{P})$ depends on the particular combination of measurement outcomes from a set of observables $\mathcal{O}=\{X_i\}$ and chosen with distribution function $\mathcal{P}=\{p(X_i)\}$. For the set of observables $\mathcal{O}=\{X_i\}$, the state which saturates Eq. (4), is a *maximally certain state* for these observables. The quantity $C_x(\mathcal{O},\mathcal{P})$ captures the amount of uncertainty allowed in a particular physical theory. If $C_x(\mathcal{O},\mathcal{P})<1$ for any x, one cannot obtain any outcome with certainty. Later, in Ref. [21] FURs were generalized for MUBs in d-dimensional systems. For a set of N MUBs chosen with equal probability, the inequality takes the following form [21]:

$$\frac{1}{N}\sum_{i=1}^{N} p(x_i|X_i)_\rho \leqslant \frac{1}{d}\left(1+\frac{d-1}{\sqrt{N}}\right). \quad (5)$$

Now we present the FUR for a set of N arbitrary d-level observables.

Result 1. For a set of N arbitrary observables in dimension d, the FUR has the form

$$\frac{1}{N}\sum_{i=1}^{N} p(x_i|X_i) \leqslant \frac{1}{d}\left(1+\frac{(d-1)\sqrt{N+2\sum_{j>k=1}^{N}\cos(\theta_{jk})}}{N}\right), \quad (6)$$

where $\cos(\theta_{jk})$ is the angle between the Bloch vectors corresponding to eigenvectors $|x_j\rangle$ and $|x_k\rangle$.

Proof. To prove this, we need to find the state $\rho_{\max}$ which maximizes the left-hand side of Eq. (6). The eigenvectors $|x_i\rangle$ corresponding to eigenvalues x_i and the state $\rho_{\max}$ can be expressed using Bloch vector representation as [35]

$$\rho_{x_i} = \frac{1}{d}I + \vec{x_i}\cdot\vec{\Gamma} \quad \text{and} \quad \rho_{\max} = \frac{1}{d}I + \vec{b}\cdot\vec{\Gamma},$$

where $\vec{x_i}$ and $\vec{b}$ are the respective Bloch vectors and $\{\Gamma_i; i \in (0,\ldots,d-1)\}$ are the generalized Gell-Mann matrices in dimension d. The length of the Bloch vector in dimension d should be less than $\sqrt{(d-1)/2d}$, where the maximum length indicates pure states. The generalized Gell-Mann matrices are traceless, i.e., $\text{Tr}(\Gamma_i)=0$, and orthogonal, i.e., $\text{Tr}(\Gamma_i\Gamma_j)=2\delta_{ij}$ [35].

Now, using the Bloch vector representation, we find that

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N} p(x_i|X_i) &= \frac{1}{N}\sum_{i=1}^{N}\text{Tr}[|x_i\rangle\langle x_i|\rho_{\max}] \\
&= \frac{1}{N}\sum_{i=1}^{N}\text{Tr}\left[\left(\frac{1}{d}I+\vec{x_i}.\vec{\Gamma}\right)\left(\frac{1}{d}I+\vec{b}\cdot\vec{\Gamma}\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{d}+2\vec{x_i}\cdot\vec{b}\right) \\
&= \frac{1}{d}+\frac{2}{N}\left(\sum_{i=1}^{N}\vec{x_i}\right)\cdot\vec{b}.
\end{aligned}$$

It is straightforward to see that the quantity $(\sum_{i=1}^{N}\vec{x_i})\cdot\vec{b}$ is maximum when $\vec{b}$ is collinear with $\sum_{i=1}^{N}\vec{x_i}$, i.e., $\vec{b}=\eta\sum_{i=1}^{N}\vec{x_i}$, where η is the scaling factor. For maximization, we have to find appropriate value of η so that the length of $\vec{b}$ is $\sqrt{\frac{d-1}{2d}}$ which implies that $\rho_{\max}$ has to be a pure state. On supplying this information in the equality $|\sum_{i=1}^{N}\vec{x_i}| = \sqrt{N'}\sqrt{\frac{d-1}{2d}}$, we get $\eta=\frac{1}{N'}$, where $N'=N+2\sum_{j>k=1}N\cos(\theta_{jk})$. Thus, by substituting η, we find the Bloch vector, $\vec{b}=\frac{1}{\sqrt{N'}}\sum_{i=1}^{N}\vec{x_i}$, which gives the upper bound for the considered FUR. ■

These inequalities are tight for $d=2$ but not always tight for $d\geqslant 3$. This is so because not all the points on the surface of the (n^2-1)-dimensional hypersphere correspond to a valid pure state. As a corollary of our derivation the fine-grained upper bound for MUBs can be reproduced using the following lemma.

Lemma 1. The Bloch vectors belonging to d-dimensional mutually unbiased bases are orthogonal to each other.

Proof. We note that the overlap between two mutually unbiased state vectors is

$$\frac{1}{d} = \text{Tr}\left[\left(\frac{1}{d}I+\vec{x_i}\cdot\vec{\Gamma}\right)\left(\frac{1}{d}I+\vec{x_j}\cdot\vec{\Gamma}\right)\right] = \frac{1}{d}+2\vec{x_i}\cdot\vec{x_j},$$

where we have used the tracelessness and orthogonality of the generalized Gell-Mann matrices. Therefore, we get $\vec{x_i}\cdot\vec{x_j}=0$. ■

Using Lemma 1 in Eq. (6), for any pair of mutually unbiased bases, $\cos(\theta_{jk})=0$, which gives Eq. (5). An example of the above inequality in the qubit case, for measurements σ_x

and σ_z, is given by [13]

$$\frac{1}{2}p(x_{\sigma_x}|\sigma_x)+\frac{1}{2}p(x_{\sigma_z}|\sigma_z)\leqslant\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right).$$

The above inequality is saturated for all four possible vectors $\vec{x}\in\{x_{\sigma_x},x_{\sigma_z}\}$ and the maximally certain states are given by the eigenstates of $\frac{\sigma_x\pm\sigma_z}{\sqrt{2}}$.

A. Tight fine-grained uncertainty relations for two arbitrary observables in arbitrary dimension

For two d-dimensional observables X_1 and X_2, we can prove the following fine-grained inequalities, corresponding to the combination of outcomes x_1 and x_2, respectively.

Result 2. The fine-grained inequality corresponding to obtaining outcomes x_1 and x_2 by measuring observables X_1 and X_2, respectively, in state ρ has the following form:

$$\frac{1}{2}\text{Tr}(|x_1\rangle\langle x_1|\rho)+\frac{1}{2}\text{Tr}(|x_2\rangle\langle x_2|\rho)\leqslant\frac{1+|\langle x_1|x_2\rangle|}{2}. \tag{7}$$

Proof. Again, we need to find the state ρ which maximizes the left-hand-side term in Eq. (7). For this, we use the Landau-Pollak uncertainty, which states that for two projectors $|x_1\rangle$ and $|x_2\rangle$, corresponding to outcomes x_1 and x_2, respectively, the following inequality exists:

$$\text{Arccos}\langle x_1\rangle_\rho+\text{Arccos}\langle x_2\rangle_\rho\geqslant\text{Arccos}|\langle x_1|x_2\rangle|. \tag{8}$$

Note that $\text{Tr}(|x_1\rangle\langle x_1|\rho)=\langle x_1\rangle_\rho^2$, and similarly, $\text{Tr}(|x_2\rangle\langle x_2|\rho)=\langle x_2\rangle_\rho^2$. From Eq. (8), we have $\text{Arccos}\langle x_2\rangle_\rho\leqslant\text{Arccos}|\langle x_1|x_2\rangle|-\text{Arccos}\langle x_1\rangle_\rho$. We denote $\text{Arccos}\langle x_1\rangle_\rho=\alpha$ and $\text{Arccos}|\langle x_1|x_2\rangle|=\theta$ and substitute this inequality for the left-hand-side term in Eq. (7):

$$\left(\frac{1}{2}\text{Tr}(|x_1\rangle\langle x_1|\rho)+\text{Tr}(|x_2\rangle\langle x_2|\rho)\right)$$
$$\leqslant\frac{1}{2}(\cos^2\alpha+\cos^2(\theta-\alpha)).$$

Now, finding the maximum of this expression is a simple optimization problem, which attains the maximum for $\alpha=\frac{\theta}{2}$ and gives the upper bound in Eq. (7). Thus our inequality is proved. ■

For MUBs the inequality in Eq. (7) becomes

$$\frac{1}{2}\text{Tr}(|x_1\rangle\langle x_1|\rho)+\frac{1}{2}\text{Tr}(|x_2\rangle\langle x_2|\rho)\leqslant\frac{1}{2}\left(1+\frac{1}{\sqrt{d}}\right). \tag{9}$$

In the next section, we present the states which saturate the inequalities for MUBs when we connect the FUR with a RAC game.

IV. VIOLATING NONCONTEXTUALITY INEQUALITY WITH FINE-GRAINED UNCERTAINTY

In this section, we show how the FUR determines the preparation contextuality of quantum theory. As previously stated, there exist d^N such inequalities for N mutually unbiased observables, Eq. (9). If we take the average over all such inequalities for $N=2$, we obtain

$$\frac{1}{2d^2}\sum_{x\in\{0,1,\ldots,d-1\}^2}\sum_{i=1}^{2}p(x_i|X_i)_{\rho_x}\leqslant\frac{1}{2}\left(1+\frac{1}{\sqrt{d}}\right), \tag{10}$$

where $x_i\in\{0,1,\ldots,d-1\}$ are the measurement outcomes corresponding to the observable X_i. If Alice encodes the classical string x by preparing ρ_x and sends it to Bob, who measures X_i to guess the ith bit of x, then the left-hand side of the inequality, Eq. (10), becomes the success probability of a $2\rightarrow 1$ d-RAC game. Now, the right-hand side of the inequality, Eq. (10), gives the quantum upper bound for the success probability of the game. Later we also show that this encoding and decoding scheme also respects the parity obliviousness condition, with respect to Bob's choice of measurements. Now we state our result in terms of a theorem when Bob performs measurement with MUBs.

A. Preparation contextuality via a 2 → 1 *d*-PORAC game

Using a $2\rightarrow 1$ d-RAC game one can demonstrate preparation contextuality on the basis of the following theorem.

Theorem 1. In a RAC game, if Alice encodes the 2-dit classical string x in quantum states, which are maximally certain states for Bob's set of measurements (which are MUBs), then the preparation contextuality of quantum theory can be revealed. Moreover, this encoding-decoding strategy, guided by fine-grained uncertainty relations for MUBs, satisfies the parity obliviousness condition, given by Eq. (2) for a $2\rightarrow 1$ d-PORAC.

Demonstrating the preparation contextual nature with an encoding and decoding scheme requires two things: (i) The RAC game should satisfy the parity obliviousness condition and (ii) the success probability should be greater than that obtained in a noncontextual theory. First, we show that our encoding and decoding scheme respects the parity obliviousness condition, Eq. (2), given a set of measurements performed by Bob.

Proof. The $2\rightarrow 1$ d-RAC game has d sets of different parity and the number of classical messages, $x=x_0x_1$, in each set is d. We follow the encoding and decoding scheme presented in [25] for a d-level quantum random access code (QRAC) game and show that the scheme is parity oblivious and can be derived from fine-grained inequalities for MUBs. To detect the message x_0x_1 we use a mutually unbiased basis given by the computational basis $\{|p\rangle\}_p$ and Fourier basis $e_p=\frac{1}{\sqrt{d}}\sum_q^{d-1}\omega^{pq}|q\rangle$, where $\omega=\exp(\frac{2\pi i}{d})$. Alice encodes the classical signal $x_0x_1=00$ in the state $|\psi_{00}\rangle=\frac{1}{N_d}(|0\rangle+|e_0\rangle)$, where $N_d=\sqrt{2+\frac{2}{\sqrt{d}}}$. For the two projectors $|0\rangle$ and $|e_0\rangle$, this state is the maximally certain state, i.e., it saturates the fine-grained inequality in Eq. (9). Similarly, for other signals we use the encoding state $|\psi_{x_0x_1}\rangle=X^{x_0}Z^{x_1}|\psi_{00}\rangle$, where $X=\sum_{q=0}^{d-1}|q+1\rangle\langle q|$ and $Z=\sum_{q=0}^{d-1}\omega^q|q\rangle\langle q|$ are the unitary operators. To learn about the first bit x_0 Bob will do the measurement in the computational basis and to learn about x_1 he will do the measurement in the Fourier basis. Given this encoding and decoding scheme, the success probability for

Bob's determining the x_0 and x_1 bits is given by

$$P_{x_0}(p) = |\langle p|\psi_{x_0x_1}\rangle|^2 = \frac{1}{N_d^2}\left|\delta_{x_0,p} + \frac{\omega^{x_1(p-x_0)}}{\sqrt{d}}\right|^2,$$

$$P_{x_1}(p) = |\langle e_p|\psi_{x_0x_1}\rangle|^2 = \frac{1}{N_d^2}\left|\omega^{-x_0x_1}\delta_{x_1,p} + \frac{\omega^{-px_0}}{\sqrt{d}}\right|^2.$$

Bob's prediction is correct when either $p = x_0$ for P_{x_0} or $p = x_1$ for P_{x_1}. In bothcases the success probability turns out to be $\frac{1}{2}(1+\frac{1}{\sqrt{d}})$. Thus, it also saturates the fine-grained inequality for the measurements in the $\{|p\rangle\}$ basis and $\{|e_p\rangle\}$ basis.

We note that since the success probability of P_{x_0} is independent of the dit at position x_1, and similarly the success probability P_{x_1} is independent of dit x_0, our encoding and decoding scheme is parity oblivious in this scenario. ■

Next, we show that the success probability of our encoding and decoding scheme exceeds the noncontextual bound of a PORAC.

Proof. The maximum success probability of the $2 \to 1$ d-level RAC game in quantum theory is exactly the right-hand side of Eq. (10). Comparing the upper bound of the $N \to 1$ d-PORAC game with that of the FUR, we find that $\frac{1}{d}(1+\frac{d-1}{\sqrt{2}}) \geqslant \frac{2+d-1}{2d} = \frac{1}{d}(1+\frac{d-1}{2})$. Therefore, we have obtained a violation of the preparation noncontextuality inequality. ■

It should be noted that a full set of MUBs for an arbitrary dimension d is not known. But in the $2 \to 1$ d-PORAC game, we need only two such observables of dimension d for our scheme to work.

Example 1. First, we present the simplest example of a $2 \to 1$ 2-PORAC. This has been presented earlier [26], so we only highlight how the fine-grained uncertainty relation enters the picture. The classical signals {00,01,10,11} are encoded in the states with Bloch vectors $(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$, because for σ_x and σ_y these states saturate the fine-grained uncertainty relation. To decode the signal Bob uses σ_x to measure the first bit and σ_y to measure the second bit. Using this method he detects the correct signal with probability $\frac{1}{2}(1+\frac{1}{\sqrt{2}}) = 0.853\,555\,3 \geqslant \frac{n+1}{2n} = \frac{3}{4}$ and, thus, violates the inequality in Eq. (3). The parity obliviousness condition is also respected, since the parity 0 and 1 states are represented by the same density matrix operator, i.e., $\frac{1}{2}\rho_{00} + \frac{1}{2}\rho_{11} = \frac{1}{2}\rho_{10} + \frac{1}{2}\rho_{01} = \frac{I}{2}$. Thus, by using the fine-grained uncertainty relation we obtain a violation of preparation noncontextuality.

Example 2. In [25], the authors found a violation of the $2 \to 1$ 3-PORAC game analytically as well as numerically. Although they did not call it a parity oblivious game, the analytical value of the success probability is $\frac{1}{2}(1+\frac{1}{\sqrt{3}}) = 0.788\,675$. As we have shown, their encoding and decoding scheme is parity oblivious also.

B. Parity obliviousness in a 3 → 1 2-PORAC game

In this section, we use the fine-grained inequality in Eq. (5) for demonstrating preparation contextuality. Since this inequality is tight only for $d = 2$ we limit ourselves to that. In the $3 \to 1$ 2-PORAC game, Alice encodes the classical signal {000,001,010,011,100,101,110,111} in qubit quantum states and sends them to Bob. Following the fine-grained inequality in Eq. (6), if Alice encodes these states with Bloch vectors $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$, which saturate the fine-grained uncertainty for three observables, σ_x, σ_y, and σ_z, with mutually unbiased bases. Bob employs the σ_x, σ_y, and σ_z operators to detect the first, second, and third bits, respectively, and obtains the correct signal with probability $\frac{1}{2}(1+\frac{1}{\sqrt{3}}) = 0.788\,675 \geqslant \frac{n+1}{2n} = \frac{2}{3}$. It has been shown that this encoding scheme is also parity oblivious [26]. Moreover, this is the optimal success probability for this game [30,32].

Now, we find the upper bound of the quantum violation of the preparation noncontextuality inequality.

Theorem 2. The encoding-decoding strategy based on the FUR for MUBs gives the upper bound of the quantum violation of the preparation noncontextuality inequality; specifically, it gives the upper bound of the success probability of quantum theory in the $N \to 1$ d-PORAC game, in which decoding is done using rank 1 projective measurements.

We prove Theorem 2 in Appendix B.

V. CONCLUSION

The optimal success probabilities of certain communication games reveal the fundamental limitations of different operational theories. The quantum advantage of a random access code game with the additional constraint of parity obliviousness asserts that quantum theory is preparation contextual. Here, we show that the success probability of a parity oblivious RAC game is determined by the amount of fine-grained uncertainty for Bob's choice of rank 1 projective measurements. To show this, we have derived an upper bound for fine-grained uncertainty relations of N arbitrary observables of dimension d. In addition, we have also found the tight fine-grained inequalities for two observables, which provide the optimal encoding and decoding strategy for a $2 \to 1$ d-PORAC. Subsequently, we find analytically the quantum violation of the preparation contextuality inequality for the $2 \to 1$ d-PORAC game. Some partial results of optimal violations were known up to a few dimensions with the help of numerical methods, i.e., semidefinite programming [31]. Our results are derived under the condition that the dimension of the resource states corresponding to the d-PORAC game is also d in classical or quantum theory. In future, one can try to find the violation of preparation noncontextuality for $N \to 1$ d-PORAC games for $N > 2$ also.

ACKNOWLEDGMENTS

G.S. would like to acknowledge the Department of Atomic Energy, Government of India, for providing research fellowships. The research of G.S. was also supported in part by the INFOSYS scholarship for senior students.

APPENDIX A

Lemma 2. The sum of the Bloch vectors corresponding to eigenvectors of an observable is 0.

Proof. The eigenvectors $|v_i\rangle$ of an observable $\mathcal{O}$ satisfy $\sum_i |v_i\rangle\langle v_i| = I$. In terms of Bloch vectors $\vec{b}$, one can write

$$|v_i\rangle\langle v_i| = \frac{1}{d}I + \vec{b}_i \cdot \vec{\Gamma}.$$

By taking the sum over all the eigenvectors, we get $\sum_i |v_i\rangle\langle v_i| = I + (\sum_i \vec{b}_i)\cdot\vec{\Gamma} = I$, which gives $\sum_i \vec{b}_i = 0$. ■

APPENDIX B

Here, we prove Theorem 2 and find the maximal success probability of an $N \to 1$ d-PORAC game, over all possible measurement settings. The success probability of the $N \to 1$ d-PORAC is given by

$$p_{\text{succ}} = \frac{1}{d^N N} \sum_{x\in\{0,1,\ldots,d-1\}^N} \sum_{i=1}^{N} p(x_i|X_i)_\rho,$$

where we have not specified the measurement setting chosen by Bob. For any arbitrary measurement performed by Bob, $p(x_i|X_i)_\rho = \frac{1}{d} + 2\vec{x}_i\cdot\vec{b}$, where $\vec{x}_i$ is the Bloch vector corresponding to the outcome x_i, and $\vec{b}$ is the Bloch vector of the encoding state. For optimal encoding it will depend on the $\vec{x}_i$'s, which we prove now. Substituting this probability into the above equation we get

$$\begin{aligned} p_{\text{succ}} &= \frac{1}{d^N N} \sum_{x\in\{0,1,\ldots,d-1\}^N} \sum_{i=1}^{N} \left(\frac{1}{d} + 2\vec{x}_i\cdot\vec{b}\right) \\ &= \frac{1}{d} + \frac{2}{Nd^N} \sum_{x\in\{0,1,\ldots,d-1\}^N} \sum_{i=1}^{N} \vec{x}_i\cdot\vec{b} \\ &= \frac{1}{d} + \frac{2\,\Phi(\vec{X},\vec{b})}{Nd^N}, \end{aligned} \tag{B1}$$

where $\Phi(\vec{X},\vec{b}) = \sum_{x\in\{0,1,\ldots,d-1\}^N} \sum_{i=1}^{N} \vec{x}_i\cdot\vec{b}$. To get the optimal success probability, we need to maximize $\Phi(\vec{X},\vec{b})$ over all possible measurements X_i and encodings $\vec{b}$. We denote the maximum value $\Phi(N)$:

$$\Phi(N) = \max_{\vec{X},\vec{b}} \Phi(\vec{X},\vec{b}) = \max_{\vec{X}} \sum_{x\in\{0,1,\ldots,d-1\}^N} \max_{\vec{b}} \vec{b}\cdot\sum_i \vec{x}_i.$$

The second maximization can be easily done by choosing $\vec{b}$ in the direction of $\sum_i \vec{x}_i$, so that $\vec{b}\cdot\sum_i\vec{x}_i = \sqrt{\frac{d-1}{2d}}||\sum_i \vec{x}_i||$. Then

$$\Phi(N) = \max_{\vec{X},\vec{b}} \Phi(\vec{X},\vec{b}) = \sqrt{\frac{d-1}{2d}} \max_{\vec{X}} \sum_{x\in\{0,1,\ldots,d-1\}^N} \left|\left|\sum_i \vec{x}_i\right|\right|.$$

To find the value of $\Phi(N)$, we use the following lemma.

Lemma 3. For vectors $\vec{x}_i$, we have $\sum_{x\in\{0,1,\ldots,d-1\}^N} ||\sum_{i=1}^N \vec{x}_i||^2 = \frac{(d-1)}{2d} N d^N$.

Proof. We prove Lemma 3 by induction. For $N = 1$, we have

$$\sum_{x\in\{0,1,\ldots,d-1\}} ||\vec{x}_1||^2 = \frac{d(d-1)}{2d}.$$

Assuming that our lemma holds for $N = m$, then for $N = m+1$ we have

$$\begin{aligned} &\sum_{x\in\{0,1,\ldots,d-1\}^{m+1}} \left|\left|\sum_{i=1}^{m+1} \vec{x}_i\right|\right|^2 \\ &= \sum_{x\in\{0,1,\ldots,d-1\}^{m+1}} ||\vec{x}_1 + \vec{x}_2 + \ldots + \vec{x}_{m+1}||^2. \end{aligned}$$

By summing over the $m+1$ index, we get

$$\begin{aligned} \sum_{x\in\{0,1,\ldots,d-1\}^m} &\big(||\vec{x}_1 + \ldots + \vec{x}_m||^2 + ||\vec{x}_{m+1}||^2 \\ &+ 2\vec{x}_{m+1}(\vec{x}_1 + \ldots + \vec{x}_m)\big). \end{aligned}$$

By using Lemma 2, we note that

$$\sum_{x\in\{0,1,\ldots,d-1\}^N} 2\vec{x}_{m+1}(\vec{x}_1 + \vec{x}_2 + \ldots + \vec{x}_m) = 0.$$

Since we have assumed that the lemma holds for $N = m$, the above expression simplifies to $d(m\cdot d^m + d^m)\frac{(d-1)}{2d} = \frac{(d-1)}{2d}(m+1)d^{m+1}$. ■

Now $\Phi(N)$ can be seen as an inner product between $\sum_{x\in\{0,1,\ldots,d-1\}^N}\sum_{i=1}^N \vec{x}_i$ and the vector $(1,1,\ldots,1) \in \mathcal{R}^{d^N}$, hence we can apply the Cauchy-Schwarz inequality to get the upper bound on $\Phi(N)$, so that

$$\Phi(N) \leqslant \sqrt{\frac{(d-1)}{2d}}\sqrt{d^N}\sqrt{\frac{(d-1)}{2d}Nd^N} = \frac{\sqrt{N}(d-1)}{2d}d^N.$$

By substituting $\Phi(N)$ in Eq. (B1), we get the maximum success probability as

$$p_{\text{succ}} = \frac{1}{d}\left(1 + \frac{d-1}{\sqrt{N}}\right).$$

Note. We are finding the upper bound of an $N \to 1$ d-RAC game, based on projective measurements. Therefore, to find the quantum upper bound we consider FUR inequalities involving projective measurements. The maximum success probability for a $2 \to 1$ RAC game is obtained by encoding the signal in a two-dimensional quantum state. Therefore, for an $N \to 1$ d-level RAC game also, we have restricted the dimension of the encoding state (classical/quantum) to be equal to dimension d.

[1] J. S. Bell, Phys. Phys. Fiz. **1**, 195 (1964).
[2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[3] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[4] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[5] R. W. Spekkens, Phys. Rev. A **71**, 052108 (2005).
[6] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).
[7] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).
[8] G. Ver Steeg and S. Wehner, Quantum Info. Comput. **9**, 801 (2009).
[9] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, Nature (London) **461**, 1101 (2009).
[10] G. Brassard, H. Buhrmann, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, Phys. Rev. Lett. **96**, 250401 (2006).
[11] W. van Dam, Nat. Comput. **12**(1), 9 (2013); http://arxiv.org/abs/quant-ph/0501159.

[12] H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner, Phys. Rev. Lett. **104**, 140401 (2010).
[13] J. Oppenheim and S. Wehner, Science **330**, 1072 (2010).
[14] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[15] M. D. Reid, Phys. Rev. A **40**, 913 (1989).
[16] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[17] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Rev. Mod. Phys. **92**, 015001 (2020).
[18] E. H. Kennard, Z. Phys. **44**, 326 (1927).
[19] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988).
[20] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Rev. Mod. Phys. **89**, 015002 (2017).
[21] A. E. Rastegin, Quant. Info. Proc. **14**, 783 (2015).
[22] A. Grudka, M. Horodecki, R. Horodecki, and A. Wójcik, Phys. Rev. A **92**, 052312 (2015).
[23] A. Ambainis, A. Nayak, A. Ta-Shma, and U. Vazirani, in *Proceedings of the Thirty-first Annual ACM Symposium on Theory of Computing*, *STOC '99* (Association for Computing Machinery, New York, 1999), pp. 376–383.
[24] A. Ambainis, D. Leung, L. Mancinska, and M. Ozols, arXiv preprint: arXiv:0810.2937 (2008).
[25] A. Tavakoli, A. Hameedi, B. Marques, and M. Bourennane, Phys. Rev. Lett. **114**, 170502 (2015)
[26] R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner, and G. J. Pryde, Phys. Rev. Lett. **102**, 010401 (2009).
[27] M. S. Leifer and O. J. E. Maroney, Phys. Rev. Lett. **110**, 120401 (2013).
[28] M. Banik, S. S. Bhattacharya, S. K. Choudhary, A. Mukherjee, and A. Roy, Found. Phys. **44**, 1230 (2014).
[29] A. Tavakoli and R. Uola, Phys. Rev. Research **2**, 013011 (2020).
[30] S. Ghorai and A. K. Pan, Phys. Rev. A **98**, 032110 (2018).
[31] A. Ambainis, M. Banik, A. Chaturvedi, D. Kravchenko, and A. Rai, Quant. Info. Proc. **18**, 111 (2019).
[32] A. Chailloux, I. Kerenidis, S. Kundu, and J. Sikora, New J. Phys. **18**, 045003 (2016).
[33] A. Hameedi, A. Tavakoli, B. Marques, and M. Bourennane, Phys. Rev. Lett. **119**, 220402 (2017).
[34] H. Anwer, N. Wilson, R. Silva, S. Muhammad and A. Tavakoli, arXiv:1904.09766 (2019).
[35] R. A. Bertlmann and P. Krammer, J. Phys. A: Math. Theor. **41**, 235303 (2008).

Correction: The affiliation associated with all authors was presented incorrectly and has been fixed. A second affiliation for the second author has been inserted, necessitating a change in the setup of the affiliations.