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A note on some new hook-content identities

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Abstract

Based on the work of Vershik (J Sov Math 59(5):1029–1040, 1992), we introduce two new combinatorial identities. We show how these identities can be used to prove a new hook-content identity. The main motivation for deriving this identity was a particular optimization problem in the field of quantum information processing.

Keywords Hook-content formula · Schur–Weyl duality · Symmetric group · Representation theory

1 Introduction

The representation theory of the symmetric group S_n and of the general linear group $GL(d)$ is related by the so-called Schur–Weyl duality [2]. This famous theorem proves the decomposition

$$V^{\otimes n} = \bigoplus_{\lambda \vdash n} V_\lambda \otimes S_\lambda \qquad V = \mathbb{C}^d \tag{1}$$

for the representation of $GL(d) \times S_n$, where V_λ is either zero or a polynomial irreducible representation of $GL(d)$, S_λ is an irreducible representation of S_n and λ runs over the partitions of n and is conveniently represented by Young diagram. Both the symmetric group and $GL(d)$ (and especially its compact subgroups $U(d)$ and $SU(d)$)

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are of paramount importance in theoretical physics, especially in quantum mechanics. For example, S_n is a fundamental symmetry of systems of identical particles and unitary groups represent the set of reversible (finite-dimensional) transformations. Therefore, it is not surprising that physics community keeps a steady interest in the representation theory of these groups and in the Schur–Weyl duality, from the early work of Weyl [3] up to the most recent applications in quantum computing and quantum information processing. For example, Eq. (1) denotes a subsystem decomposition (induced by the symmetry of the system–environment interaction) in which one can identify error-free subsystems [4,5], or the relevant subsystems for quantum estimation [6]. These are just a couple of examples of a much wider variety of applications (see, for example, Ref. [7] for a review). In the light of this discussion, it is clear that the dimensions of the irreducible spaces V_λ and S_λ are, more often than not, a crucial piece of information. The value of $\dim(S_\lambda)$ and $\dim(V_\lambda)$ is given by the hook length formula [8] and the hook-content formula [9], respectively. Those celebrated equations have a nice combinatorial interpretation and, since their discovery, they have been generalized (see, for example, [10–12] and references therein) and applied in different fields like algebraic geometry [13] and probability [14]. Closely related are also the Littlewood–Richardson rules[15] in the expansion $S_\lambda \otimes S_\mu = \bigoplus_\nu S_\nu^{\otimes c(\lambda,\mu,\nu)}$ and the branching rules for restricting S_λ to S_{n-1} and inducing S_λ to S_{n+1} [16].

Our work introduces a new identity, represented by Eq. (13) in Proposition 3.1, which relates the dimensions $\dim(S_\lambda)$, $\dim(V_\lambda)$, $\dim(S_{\lambda^{(j)}})$, and $\dim(V_{\lambda^{(j)}})$ for any Young diagram λ consisting of n boxes and diagrams $\lambda^{(j)}$ that can be obtained from λ by adding a single box. The proof of our result relies on a couple of combinatorial identities, Eqs. (5) and (6) in Proposition 2.2, which can be of independent interest. Our approach is modelled after the seminal work [1] of A. Vershik, which provides the essential tools used in this work.

The main motivation for the presented results originates in the problem we were solving [17] within the field of quantum information processing. While trying to derive optimal success probability for a problem with an arbitrary number of uses of a unitary transformation, n, and an arbitrary dimension of quantum systems, d, we observed several identities involving n and d, which the optimality of the solution required. By reducing the problem even further, we arrived at the necessity to prove the hook-content identity (12) that forms the core of this paper, and the needed identities for our original problem correspond to our final Proposition 3.1. After finishing the present manuscript, Sanjaye Ramgoolam and Michal Sedlák gave a representation theoretic proof [18] of the identities presented in this paper.

2 Basic identities

Let $\{a_i\}_{i=1}^{2s}$, $s \geq 1$, be elements of an arbitrary field, and let the following coefficients be well defined:

$$\begin{aligned} q^j_{m,n} &:= \prod_{i=m+1}^{j}\left(1-\frac{a_{2i-1}}{a_{2i-1}+a_{2i}+\cdots+a_{2j}}\right)\\ &\prod_{i=j+1}^{n}\left(1-\frac{a_{2i}}{a_{2j+1}+a_{2j+2}+\cdots+a_{2i}}\right)\\ &\text{for } 0\le m\le j\le n\le s. \end{aligned} \tag{2}$$

In Ref. [1], the following result is proved:

Proposition 2.1 (Vershik) *Let $q^j_{m,n}$ be defined as in Eq.* (2). *Then, for any m, n, j , $0 \le m \le j \le n \le s$ the following identities hold*

$$C_{m,n}\, q^j_{m,n} = \sum_{k=m+1}^{j} a_{2k}\, q^j_{k,n} + \sum_{l=j}^{n-1} a_{2l+1}\, q^j_{m,l} \tag{3}$$
$$C_{m,n} := a_{2m+1} + a_{2m+2} + \cdots + a_{2n},$$
$$\sum_{j=m}^{n} q^j_{m,n} = 1. \tag{4}$$

By rephrasing this result, we could say that $q^j_{m,n}$ are the solution of the recursion relation (3). Proposition 2.1 is the main tool for the proof of the following result.

Proposition 2.2 *The following identities hold:*

$$\sum_{j=m}^{n}\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right) q^j_{m,n} = 0 \tag{5}$$
$$\sum_{j=m}^{n}\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right)^2 q^j_{m,n} = \sum_{i=m+1}^{n} a_{2i} \sum_{l=m+1}^{i} a_{2l-1} \tag{6}$$

Proof of Eq. (5) The proof is by induction on $n - m$. Let us first consider the case $n - m = 1$. If we fix an arbitrary m, we have $n = m + 1$, and Eq. (5) gives:

$$\begin{aligned} -a_{2m+2}\, q^m_{m,m+1} + a_{2m+1}\, q^{m+1}_{m,m+1} &= -a_{2m+2}\frac{a_{2m+1}}{a_{2m+1}+a_{2m+2}}\\ &\quad + a_{2m+1}\frac{a_{2m+2}}{a_{2m+1}+a_{2m+2}} = 0. \end{aligned}$$

Next, we fix arbitrary m and n with $n - m > 1$ and let us suppose that the thesis holds for any m', n' such that $m' \le n'$ and $n' - m' < n - m$. By multiplying Eq. (5) by $C_{m,n}$ and by using the recursion formula (3), we obtain:

$$C_{m,n}\sum_{j=m}^{n}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)q_{m,n}^{j}=A_{m,n}+B_{m,n}$$

$$A_{m,n}:=\sum_{j=m}^{n}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)\sum_{k=m+1}^{j}a_{2k}\,q_{k,n}^{j}$$

$$B_{m,n}:=\sum_{j=m}^{n}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)\sum_{l=j}^{n-1}a_{2l+1}\,q_{m,l}^{j}$$

We start with the coefficient $A_{m,n}$. Since the term with $j=m$ is zero, we have

$$\begin{aligned}
A_{m,n}&=\sum_{j=m+1}^{n}\sum_{k=m+1}^{j}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)a_{2k}\,q_{k,n}^{j}\\
&=\sum_{k=m+1}^{n}\sum_{j=k}^{n}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)a_{2k}\,q_{k,n}^{j}\\
&=\sum_{k=m+1}^{n}\sum_{j=k}^{n}\left(\sum_{i=m+1}^{k}a_{2i-1}+\left(\sum_{i=k+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)\right)a_{2k}\,q_{k,n}^{j}\\
&=\sum_{k=m+1}^{n}a_{2k}\left\{\sum_{i=m+1}^{k}a_{2i-1}\sum_{j=k}^{n}q_{k,n}^{j}+\sum_{j=k}^{n}\left(\sum_{i=k+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)q_{k,n}^{j}\right\}\\
&=\sum_{k=m+1}^{n}a_{2k}\sum_{i=m+1}^{k}a_{2i-1}.
\end{aligned}$$

where we used Eq. (4) and the inductive hypothesis in the last equality.

Now we evaluate the coefficient $B_{m,n}$. Since the term with $j=n$ is zero, we have

$$\begin{aligned}
B_{m,n}&:=\sum_{j=m}^{n-1}\sum_{l=j}^{n-1}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)a_{2l+1}q_{m,l}^{j}\\
&=\sum_{l=m}^{n-1}\sum_{j=m}^{l}\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{n}a_{2i}\right)a_{2l+1}\,q_{m,l}^{j}\\
&=\sum_{l=m}^{n-1}\sum_{j=m}^{l}\left(\left(\sum_{i=m+1}^{j}a_{2i-1}-\sum_{i=j+1}^{l}a_{2i}\right)-\sum_{i=l+1}^{n}a_{2i}\right)a_{2l+1}\,q_{m,l}^{j}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=m}^{n-1} a_{2l+1} \left\{ \sum_{j=m}^{l} \left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{l} a_{2i} \right) q_{m,l}^{j} - \sum_{j=m}^{l} q_{m,l}^{j} \sum_{i=l+1}^{n} a_{2i} \right\} \\
&= - \sum_{l=m}^{n-1} a_{2l+1} \sum_{i=l+1}^{n} a_{2i}
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
A_{m,n} + B_{m,n} &= \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1} a_{2k} - \sum_{l=m}^{n-1} \sum_{i=l+1}^{n} a_{2l+1} a_{2i} \\
&= \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1} a_{2k} - \sum_{l'=m+1}^{n} \sum_{i=l'}^{n} a_{2l'-1} a_{2i} \\
&= \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1} a_{2k} - \sum_{i=m+1}^{n} \sum_{l'=m+1}^{i} a_{2l'-1} a_{2i} = 0
\end{aligned}
$$

where we defined $l' = l + 1$. □

Proof of Eq. (6) The proof is again by induction and is very similar to the proof of Eq. (5). Let us first fix an arbitrary m and $n = m + 1$. Then, Eq. (6) becomes:

$$
\begin{aligned}
& a_{2m+2}^2 \, q_{m,m+1}^{m} + a_{2m+1}^2 \, q_{m,m+1}^{m+1} \\
&= a_{2m+2}^2 \frac{a_{2m+1}}{a_{2m+1} + a_{2m+2}} \\
&\quad + a_{2m+1}^2 \frac{a_{2m+2}}{a_{2m+1} + a_{2m+2}} = a_{2m+2} \, a_{2m+1}.
\end{aligned}
$$

We now fix arbitrary m and n with $n - m > 1$ and let us suppose that the thesis holds for any m', n' such that $m' \le n'$ and $n' - m' < n - m$. By multiplying the left-hand side of Eq. (6) by $C_{m,n}$ and by inserting the recursion formula (3), we obtain:

$$
\begin{aligned}
C_{m,n} \sum_{j=m}^{n} \left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 q_{m,n}^{j} &= \alpha_{m,n} + \beta_{m,n} \\
\alpha_{m,n} := \sum_{j=m}^{n} \left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 \sum_{k=m+1}^{j} a_{2k} \, q_{k,n}^{j} \\
\beta_{m,n} := \sum_{j=m}^{n} \left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 \sum_{l=j}^{n-1} a_{2l+1} \, q_{m,l}^{j}
\end{aligned}
$$

Next we rewrite coefficient $\alpha_{m,n}$. Since the term with $j=m$ is zero, we have

$$\begin{aligned}\alpha_{m,n} &= \sum_{j=m+1}^{n}\sum_{k=m+1}^{j}\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right)^2 a_{2k}\, q_{k,n}^{j} \\ &= \sum_{k=m+1}^{n}\sum_{j=k}^{n} a_{2k}\left(\sum_{i=m+1}^{k} a_{2i-1} + \left(\sum_{i=k+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right)\right)^2 q_{k,n}^{j} \\ &= \sum_{k=m+1}^{n}\sum_{j=k}^{n} a_{2k}\left(\left(\sum_{i=m+1}^{k} a_{2i-1}\right)^2 + \left(\sum_{i=k+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right)^2\right. \\ &\quad \left. + 2\left(\sum_{i=m+1}^{k} a_{2i-1}\right)\left(\sum_{i=k+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right)\right) q_{k,n}^{j} \\ &= \sum_{k=m+1}^{n} a_{2k}\left(\left(\sum_{i=m+1}^{k} a_{2i-1}\right)^2 + \sum_{i=k+1}^{n}\sum_{l=k+1}^{i} a_{2l-1}\, a_{2i}\right)\end{aligned}$$

where we used the inductive hypothesis, Eqs. (4) and (5) that was previously proved. Similarly, we rewrite the coefficient $\beta_{m,n}$. Since the term with $j=n$ is zero, we have

$$\begin{aligned}\beta_{m,n} &= \sum_{j=m}^{n-1}\sum_{l=j}^{n-1}\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i}\right)^2 a_{2l+1}\, q_{m,l}^{j} \\ &= \sum_{l=m}^{n-1}\sum_{j=m}^{l}\left(\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{l} a_{2i}\right) - \sum_{i=l+1}^{n} a_{2i}\right)^2 a_{2l+1}\, q_{m,l}^{j} \\ &= \sum_{l=m}^{n-1} a_{2l+1}\sum_{j=m}^{l}\left(\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{l} a_{2i}\right)^2 - 2\left(\sum_{i=l+1}^{n} a_{2i}\right)\left(\sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{l} a_{2i}\right)\right. \\ &\quad \left. + \left(\sum_{i=l+1}^{n} a_{2i}\right)^2\right) q_{m,l}^{j} = \sum_{l=m}^{n-1} a_{2l+1}\left(\sum_{i=m+1}^{l}\sum_{k=m+1}^{i} a_{2k-1}\, a_{2i} + \left(\sum_{i=l+1}^{n} a_{2i}\right)^2\right),\end{aligned}$$

where we have used the inductive hypothesis, Eqs. (4) and (5). Let us inspect the product of the right-hand side of Eq. (6) and $C_{m,n}$. We obtain

$$C_{m,n}\left(\sum_{i=m+1}^{n} a_{2i}\sum_{l=m+1}^{i} a_{2l-1}\right) = \gamma_{m,n}^{(1)} + \gamma_{m,n}^{(2)}, \tag{7}$$

$$\gamma_{m,n}^{(1)} := \sum_{i=m+1}^{n}\sum_{l=m+1}^{i}\sum_{k=m+1}^{n} a_{2i}a_{2l-1}a_{2k-1},\ \gamma_{m,n}^{(2)} := \sum_{i=m+1}^{n}\sum_{l=m+1}^{i}\sum_{k=m+1}^{n} a_{2i}a_{2l-1}a_{2k}. \tag{8}$$

Let us define

$$\alpha^{(1)}_{m,n} := \sum_{k=m+1}^{n}\sum_{i=m+1}^{k}\sum_{j=m+1}^{k} a_{2i-1}\,a_{2j-1}\,a_{2k}, \quad \beta^{(1)}_{m,n} := \sum_{l=m}^{n-1}\sum_{i=m+1}^{l}\sum_{k=m+1}^{i} a_{2k-1}a_{2i}a_{2l+1},$$

$$\alpha^{(2)}_{m,n} := \sum_{k=m+1}^{n}\sum_{i=k+1}^{n}\sum_{l=k+1}^{i} a_{2l-1}\,a_{2i}\,a_{2k}, \quad \beta^{(2)}_{m,n} := \sum_{l=m}^{n-1}\sum_{i=l+1}^{n}\sum_{j=l+1}^{n} a_{2i}a_{2j}a_{2l+1}$$

$$\alpha_{m,n} = \alpha^{(1)}_{m,n} + \alpha^{(2)}_{m,n} \qquad \beta_{m,n} = \beta^{(1)}_{m,n} + \beta^{(2)}_{m,n}.$$

We then have

$$\begin{aligned}\gamma^{(1)}_{m,n} - \beta^{(1)}_{m,n} &= \sum_{l=m}^{n-1}\sum_{i=m+1}^{n}\sum_{k=m+1}^{i} a_{2l+1}a_{2i}a_{2k-1} - \sum_{l=m}^{n-1}\sum_{i=m+1}^{l}\sum_{k=m+1}^{i} a_{2k-1}a_{2i}a_{2l+1}\\ &= \sum_{l=m}^{n-1}\sum_{i=l+1}^{n}\sum_{k=m+1}^{i} a_{2k-1}a_{2i}a_{2l+1} = \sum_{l=m+1}^{n}\sum_{i=l}^{n}\sum_{k=m+1}^{i} a_{2k-1}a_{2i}a_{2l-1}\\ &= \sum_{i=m+1}^{n}\sum_{l=m+1}^{i}\sum_{k=m+1}^{i} a_{2k-1}a_{2i}a_{2l-1} = \alpha^{(1)}_{m,n}\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\alpha^{(2)}_{m,n} + \beta^{(2)}_{m,n} &= \sum_{j=m+1}^{n}\sum_{i=j+1}^{n}\sum_{l=j+1}^{i} a_{2j}a_{2i}a_{2l-1} + \sum_{l=m+1}^{n}\sum_{i=l}^{n}\sum_{j=l}^{n} a_{2l-1}a_{2i}a_{2j}\\ &= \sum_{j=m+1}^{n}\sum_{l=j+1}^{n}\sum_{i=l}^{n} a_{2j}a_{2l-1}a_{2i} + \sum_{i=m+1}^{n}\sum_{l=m+1}^{i}\sum_{j=l}^{n} a_{2i}a_{2l-1}a_{2j}\\ &= \sum_{i=m+1}^{n}\sum_{l=m+1}^{n}\sum_{j=l}^{n} a_{2i}a_{2l-1}a_{2j} = \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}\sum_{l=m+1}^{j} a_{2i}a_{2l-1}a_{2j} = \gamma^{(2)}_{m,n}\ .\end{aligned}$$

Therefore, $\alpha^{(1)}_{m,n} + \alpha^{(2)}_{m,n} + \beta^{(1)}_{m,n} + \beta^{(2)}_{m,n} = \gamma^{(1)}_{m,n} + \gamma^{(2)}_{m,n}$ which finally proves the thesis. □

3 Hook-content identities and *GL*(*n*)

For a natural number n, we denote with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ $\lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, $\sum_{i=1}^{k} \lambda_i = n$ a *partition* of n and we write $\lambda \vdash n$. Any partition corresponds to a *Young diagram*, which is an array of boxes, in the plane, left-justified, with λ_i cells in the i-th row from the top (English convention). A Greek letter λ denotes both the partition and the corresponding Young diagram. A box $b \in \lambda$ of a Young diagram can be labelled by a pair of integer numbers $b = (i, j)$ where i denotes the row and j denotes the column. We denote with $h(b)$ the *hook length* of the box $b = (i, j)$, i.e.

the number of boxes $b' = (i', j')$ such that $i = i'$ and $j' \geq j$ or $j = j'$ and $i' \geq i$. For example, if $\lambda = (4, 3, 1)$ and $b = (1, 2)$, we have $h(b) = 4$. The *content* of the box $b = (i, j)$ is defined as $c(b) := j - i$. A *Young tableau* of shape λ is a Young diagram λ in which each box is filled with an integer number. A *semistandard* Young tableau of parameters (d, λ) is a Young tableau of shape λ such that the entries are positive integers no greater that d and they weakly increase along rows and strictly increase along columns. For example, the following tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array},$$

is a semistandard Young tableau of parameters $(4, (4, 2, 1))$. The Stanley's *hook-content formula* [9] gives the number of semistandard Young tableaux of parameters (λ, d) (denoted with $SSYT(\lambda, d)$), namely

$$SSYT(\lambda, d) = \prod_{b \in \lambda} \frac{d + c(b)}{h(b)}. \tag{9}$$

The number $SSYT(\lambda, d)$ is the dimension $\dim(V_\lambda)$ of the vector space V_λ, which is the irreducible polynomial representation of $GL(d)$ labelled by the partition λ.

A *standard* Young tableau of shape $\lambda \vdash n$ is semistandard Young tableau of parameters (n, λ) such that the filling is a bijective assignment of $1, 2..., n$. For example, the following tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array},$$

is a standard Young tableau of shape $(4, 2, 1)$. The number of standard Young tableaux of shape $\lambda \vdash n$ (denoted with $SYT(\lambda)$) is given by the *hook length formula* [8]

$$SYT(\lambda) = \prod_{b \in \lambda} \frac{n!}{h(b)}. \tag{10}$$

The number $SYT(\lambda)$ is the dimension $\dim(S_\lambda)$ of the Specht module S_λ, i.e. the irreducible representation of the symmetric group S_n labelled by the partition λ.

We now introduce the *step coordinates* for a Young diagram λ. First, let us define the following notation for a partition $\lambda = ((\lambda'_1, k'_1), (\lambda'_2, k'_2), \ldots, (\lambda'_s, k'_s))$, where k'_i denotes the multiplicity of the number λ'_i and $\lambda'_1 > \lambda'_2 > \cdots > \lambda'_s$. For example, we have $(4, 4, 4, 3, 3, 1, 1, 1,) = ((4, 3), (3, 2), (1, 3))$. Then, we define $p_1 = \lambda'_s$, $p_i := \lambda'_{s-i+1} - \lambda'_{s-i+2}$ for $i = 2, \ldots s$ and $k_i := k'_{s-i+1}$ for $i = 1, \ldots, s$. The numbers $(p_1, k_1, p_2, k_2, \ldots, p_s, k_s)$ are the step coordinates of the Young diagram λ; the reason for this notation is clear by looking at the example in Fig. 1.

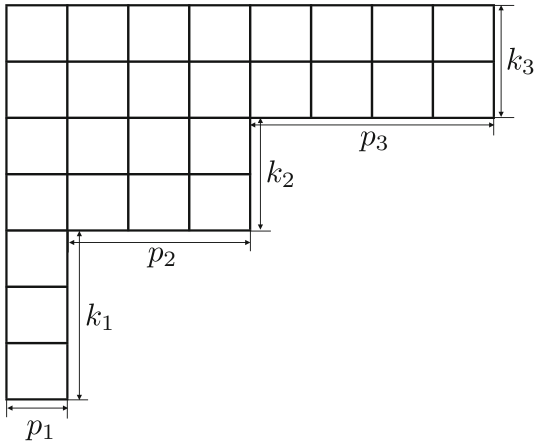
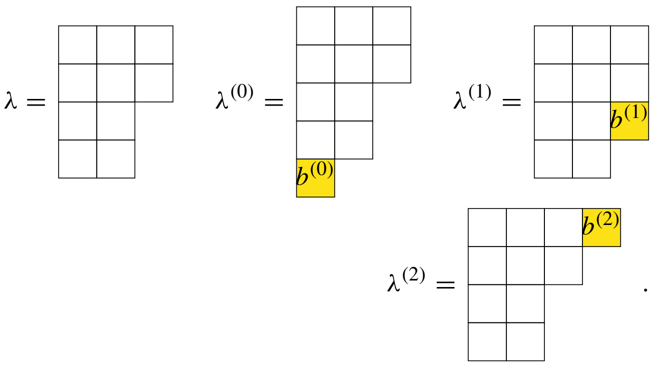


Fig. 1 The Young diagram $\lambda = (8, 8, 4, 4, 1, 1, 1)$ has step coordinates $(1, 3, 3, 2, 4, 2)$

For a given Young diagram λ, we denote with $\lambda^{(j)}$ the Young diagram that can be obtained from λ by the addition of the box $b^{(j)}$, for example, if $\lambda = (3, 3, 2, 2)$ we have



This is clearly equivalent to say that, for a Young diagram $\lambda \vdash n$, $\lambda^{(j)}$ is a partition of $n+1$ such that the Young diagram of λ fits inside that of $\lambda^{(j)}$, and we write $\lambda^{(j)} \leftarrow \lambda$.

The connection between combinatorial identities studied in Section 2 and the representation theory is provided by the following observation. Let us now consider an arbitrary Young diagram λ with step coordinates $(p_1, k_1, p_2, k_2, \ldots, p_s, k_s)$, and let us apply Eq. (2) to the sequence $a_{2i-1} := p_i, a_{2i} := k_i \; i = 1, \ldots, s$. Then, using the definition of hook length, we have[1]

$$q^j_{0,s} = \frac{\prod_{b\in\lambda} h(b)}{\prod_{b\in\lambda^{(j)}} h(b)} \qquad \text{for } j = 0, \ldots, s, \tag{11}$$

[1] This observation appears in the work of Vershik [1] (see Eq. (17) therein), with different notations for the step coordinates.

for any Young diagram $\lambda \vdash n$, and $\lambda^{(j)} \vdash n+1$, $\lambda^{(j)} \leftarrow \lambda$. Applying this observation to combinatorial identities (5) and (6) and realizing that

$$\sum_{i=1}^{j} a_{2i-1} - \sum_{i=j+1}^{s} a_{2i} = \sum_{i=1}^{j} p_i - \sum_{i=j+1}^{s} k_i = c(b^{(j)})$$
$$\sum_{i=1}^{s} a_{2i} \sum_{l=1}^{i} a_{2l-1} = \sum_{i=1}^{s} k_i \sum_{l=1}^{i} p_l = n \quad ,$$

we obtain:

$$F_k := \sum_{\lambda^{(j)} \leftarrow \lambda} (c(b^{(j)}))^k \frac{\prod_{b\in\lambda} h(b)}{\prod_{b\in\lambda^{(j)}} h(b)}, \qquad F_0 = 1,\ F_1 = 0,\ F_2 = n, \tag{12}$$

where the identity $F_0 = 1$ is Vershik's original result (4). As a consequence of Eq. (12), we also have:

Proposition 3.1 *For any Young diagram $\lambda \vdash n$, and $\lambda^{(j)} \vdash n+1$, $\lambda^{(j)} \leftarrow \lambda$, we have*

$$\sum_{\lambda^{(j)} \leftarrow \lambda} \left(\frac{SSYT(\lambda^{(j)}, d)}{SSYT(\lambda, d)} \right)^2 \frac{SYT(\lambda)}{SYT(\lambda^{(j)})} = \frac{n+d^2}{n+1} \tag{13}$$

Proof By expanding Eq. (13), we obtain

$$(n+1) \sum_{\lambda^{(j)} \leftarrow \lambda} \left(\frac{SSYT(\lambda^{(j)}, d)}{(SSYT(\lambda, d)} \right)^2 \frac{SYT(\lambda)}{SYT(\lambda^{(j)})} = \sum_{\lambda^{(j)} \leftarrow \lambda} (d + c(b^{(j)}))^2 \frac{\prod_{b\in\lambda} h(b)}{\prod_{b\in\lambda^{(j)}} h(b)}$$
$$= \sum_{\lambda^{(j)} \leftarrow \lambda} \left(d^2 + 2dc(b^{(j)}) + (c(b^{(j)}))^2 \right) \frac{\prod_{b\in\lambda} h(b)}{\prod_{b\in\lambda^{(j)}} h(b)} = d^2 + n \ ,$$

where the last equality follows from Eqs. (4), (11), and (12). □

We conclude this section by noticing that the identity $F_1 = 0$ of Eq. (12) can be alternatively proved by combining the hook-content formula (9) and easy consideration of representation theory. Indeed, let us consider the trivial identity

$$d \dim(V_\lambda) = \sum_{\lambda^{(j)} \leftarrow \lambda} \dim(V_{\lambda^{(j)}}) \tag{14}$$

where V_λ is an irreducible polynomial representation of $GL(d)$, V is the fundamental (or defining) representation of $GL(d)$, and $V_{\lambda^{(j)}}$ are the irreducible inequivalent polynomial representations in the decomposition $V_\lambda \otimes V = \sum_{\lambda^{(j)} \leftarrow \lambda} V_{\lambda^{(j)}}$. From the hook-content formula, Eq. (14) becomes

$$d = \sum_{\lambda^{(j)} \leftarrow \lambda} \frac{\dim(V_{\lambda^{(j)}})}{\dim(V_\lambda)} = \textstyle\sum_{\lambda^{(j)} \leftarrow \lambda} (d + c(b^{(j)})) \frac{\prod_{b\in\lambda} h(b)}{\prod_{b\in\lambda^{(j)}} h(b)} \implies \textstyle\sum_{\lambda^{(j)} \leftarrow \lambda} c(b^{(j)}) \frac{\prod_{b\in\lambda} h(b)}{\prod_{b\in\lambda^{(j)}} h(b)} = 0$$

where we used Eqs. (11) and (4) in the final step. Alternatively, a combinatorial proof of the same identity follows by applying the Robinson–Schensted–Knuth correspondence.

4 Conclusion

In this paper, we proved some new combinatorial identities, Eqs. (5) and (6), which can be proved following the techniques of Ref.[1]. These identities lead to a couple of hook-content identities (Eq. (12)). The $F_1 = 0$ identity can be proved with arguments from representation theory, and our approach provides an alternative proof. On the other hand, the $F_2 = n$ identity and Eq. (13) are new. As possible generalization of this result, one could consider the quantities F_n for n greater than 2. A preliminary analysis suggests that F_n could be a homogeneous polynomial of degree n for any n. We leave this conjecture as an open problem.

The representation theory of the symmetric group and of the general linear group plays a significant role in many areas of quantum information, as discussed, for example, in Ref. [19]. In particular, Eq. (13) is directly linked to the optimal solution of the perfect probabilistic storage and retrieval of an unknown unitary transformation [17], which was the problem that led us to prove the presented hook-content identities.

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