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Quantum addition imparts less disorder than mixing and commutes with incoherent channels

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Abstract

We prove a generalized version of a previously conjectured inequality by Zhang *et al* (Zhang *et al* 2018 *Phys. Lett.* A **382**, 1516–23) for the quantum addition operation defined by Datta *et al* (Audenaert *et al* 2016 *J. Math. Phy.* **57**, 052202) in the context of proving an entropy power inequality for qubits. We also show that the quantum addition operation commutes with an incoherent channel, which may have possible implications for resource theories of coherence in optical settings.

Shannon proposed [1] an *entropic power inequality (EPI)* for classical continuous random variables, where the addition of two random variables was taken in the sense of convolution. Shannon's conjectured entropy power inequality for continuous random variables X, Y with probability densities p_X and p_Y respectively, in terms of the differential entropy H was expressed in the following form

$$e^{H(X*Y)} \geqslant e^{H(X)} + e^{H(Y)} \tag{1}$$

where denotes the convolution operation on the probability densities p_X, p_Y. Various proofs of this inequality, beginning with the original proof by Stam [2] have come up over the years [3–5]. A quantum generalization of the above inequality was obtained for the continuous variable (CV) systems considering the beamsplitter merging operation as the analogue of convolution [6]. Datta, Ozols and Audenart [7] have recently found the EPI for finite dimensional quantum systems through the qudit addition channel [8]. A conditional EPI is also discussed in recent literature [9]. Essentially, this channel can be realized as a unitary evolution in $\mathbb{C}^d \otimes \mathbb{C}^d$ followed by tracing out the ancilla qudit. Let $\rho \in \mathcal{B}(\mathbb{C}^d)$, $\sigma \in \mathcal{B}(\mathbb{C}^d)$ be two qudit states. Then the quantum addition channel is a completely positive trace preserving (CPTP) map

$$\mathrm{tr}_2[U_\alpha(\rho \otimes \sigma)U_\alpha^\dagger] = \rho \boxplus_\alpha \sigma = \alpha\rho + (1-\alpha)\sigma - i\sqrt{\alpha(1-\alpha)}[\rho, \sigma], \tag{2}$$

where the unitary U_α is the *partial swap* channel, i.e. $U_\alpha = \sqrt{\alpha}\ \mathbb{I} + i\sqrt{1-\alpha}\ S$, where the parameter $\alpha \in [0,1]$ is the weight of the addition, and S is the two qudit swap gate $\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |j\rangle\langle i|$. The Kraus operators corresponding to this map are expressible in the form

$$K_n = \sqrt{\alpha}\mathbb{I} \otimes \langle n| + i\sqrt{1-\alpha}\,\langle n| \otimes \mathbb{I}. \tag{3}$$

It has been proven in [7] that for any concave function f which depends solely on the spectrum of a state ρ, the following relation holds true for $\alpha \in [0,1]$

$$f(\rho \boxplus_\alpha \sigma) \geqslant \alpha f(\rho) + (1-\alpha) f(\sigma). \tag{4}$$

If f is chosen as the exponential of the von Neumann entropy, we obtain the qudit analog to the classic EPI. For another suitable choice of f, the resulting inequality is a qudit analog of the entropic photon number inequality [7, 10].

1. A reverse entropy power inequality

The entropic power inequality puts a lower bound on the entropy of the output of the qudit addition channel. In this section, we attempt to find an upper bound to the entropy of the output of the qudit addition channel.

Theorem (Reverse EP equality). *If ρ and σ are two qudit states and $\boxplus_\alpha$ denotes quantum addition with weight $\alpha \in (0, 1)$, then the following equality holds*

$$S(\alpha\rho + (1-\alpha)\sigma) = S(\rho \boxplus_\alpha \sigma || \alpha\rho + (1-\alpha)\sigma) + S(\rho \boxplus_\alpha \sigma), \tag{5}$$

where $S(...)$ denotes the von Neumann entropy, and $S(...||...)$ denotes the quantum relative entropy.

Proof. Let us first analyze the difference in von Neumann entropies between the results of classical mixture and quantum addition of states ρ and σ.

$$\begin{aligned}
&S(\alpha\rho + (1-\alpha)\sigma) - S(\rho \boxplus_\alpha \sigma)\\
&= S(\rho \boxplus_\alpha \sigma + i\sqrt{\alpha(1-\alpha)}[\rho, \sigma]) - S(\rho \boxplus_\alpha \sigma)\\
&= \mathrm{tr}[(\rho \boxplus_\alpha \sigma)\log(\rho \boxplus_\alpha \sigma)] - \mathrm{tr}[(\rho \boxplus_\alpha \sigma + i\sqrt{\alpha(1-\alpha)}[\rho, \sigma])\log(\rho \boxplus_\alpha \sigma + i\sqrt{\alpha(1-\alpha)}[\rho, \sigma])]\\
&= \mathrm{tr}[(\rho \boxplus_\alpha \sigma)\log(\rho \boxplus_\alpha \sigma)] - \mathrm{tr}[(\rho \boxplus_\alpha \sigma)\log(\rho \boxplus_\alpha \sigma + i\sqrt{\alpha(1-\alpha)}[\rho, \sigma])]\\
&- \mathrm{tr}[i\sqrt{\alpha(1-\alpha)}[\rho, \sigma]\log(\alpha\rho + (1-\alpha)\sigma)]\\
&= S(\rho \boxplus_\alpha \sigma || \alpha\rho + (1-\alpha)\sigma) - \mathrm{tr}[i\sqrt{\alpha(1-\alpha)}[\rho, \sigma]\log(\alpha\rho + (1-\alpha)\sigma)]
\end{aligned} \tag{6}$$

Now, let us concentrate on the second term. Since $\alpha\rho + (1-\alpha)\sigma$ is a positive semidefinite matrix, its logarithm, say K, is Hermitian and commutes with $\alpha\rho + (1-\alpha)\sigma$. Thus,

$$[K, \rho] = -\frac{1-\alpha}{\alpha}[K, \sigma] \tag{7}$$

Armed with this result, we now simplify the second term in the following way.

$$\begin{aligned}
\mathrm{tr}[[\rho, \sigma]\log(\alpha\rho + (1-\alpha)\sigma)] &= \mathrm{tr}[\rho\sigma K] - \mathrm{tr}[\sigma\rho K]\\
&= \mathrm{tr}[\sigma K\rho] - \mathrm{tr}[\sigma\rho K]\\
&= \mathrm{tr}[\sigma[K, \rho]]\\
&= -\frac{1-\alpha}{\alpha}\mathrm{tr}[\sigma[K, \sigma]]\\
&= -\frac{1-\alpha}{\alpha}\mathrm{tr}[\sigma K\sigma - \sigma\sigma K]\\
&= 0.
\end{aligned} \tag{8}$$

Here we have used the cyclicity of trace as well as the commutation relation in (7). Thus, the second term in the remainder vanishes and the proof is complete. □

An immediate corollary to the theorem above is the following inequality for arbitrary $\alpha \in [0,1]$, which was conjectured [11] to hold for arbitrary qudit systems when $\alpha = \frac{1}{2}$.

Corollary. *If ρ and σ are two qudit states, then*

$$S(\rho \boxplus_\alpha \sigma) \leqslant S(\alpha\rho + (1-\alpha)\sigma). \tag{9}$$

This result follows from the relation (5) and the non-negativity of quantum relative entropy. It is easy to see that for $\alpha \in (0,1)$, this inequality is strict if ρ and σ do not commute. This we term as the reverse entropy power inequality. Thus quantum addition of two density operators results in a density operator whose entropy is always lower compared to the corresponding classical mixture. One may thus wonder, does this difference in entropy capture some quantumness ?

2. Quantum addition commutes with incoherent channels

Now we go on to show the second result in the paper, which proves that the quantum addition operation commutes with the free operations in the resource theory of quantum coherence [12, 13], viz. the incoherent operations (IOs).

Lemma. *If σ is an incoherent state and Λ denotes an incoherent channel, then the following equality holds*

$$\Lambda(\rho) \boxplus_\alpha \Lambda(\sigma) = \Lambda(\rho \boxplus_\alpha \sigma) \tag{10}$$

Proof. The Kraus operators $\{K_i\}$ corresponding to the IO Λ are known to be expressible in the form $K_i = \sum_k d_{ik}|\pi_k\rangle\langle k|$, where π is a one-to-one mapping [14]. LHS of the equality to be proved reads as

$$\Lambda(\rho) \boxplus_\alpha \Lambda(\sigma) = \alpha\Lambda(\rho) + (1-\alpha)\Lambda(\sigma) - i\sqrt{\alpha(1-\alpha)}[\Lambda(\rho), \Lambda(\sigma)] \tag{11}$$

. Now, let us focus on the quantity $\Lambda(\rho)\,\Lambda(\sigma)$, which equals

$$\begin{aligned}
\Lambda(\rho)\Lambda(\sigma) &= \sum_{ij} K_i\rho K_i^\dagger K_j\sigma K_j^\dagger \\
&= \sum_{ij}\sum_{mnt} K_i\rho d_{im}^*|m\rangle\langle\pi_m|d_{jn}|\pi_n\rangle\langle n|\sigma d_{jt}^*|t\rangle\langle\pi_t| \\
&= \sum_{ij}\sum_{mnt} K_i\rho d_{im}^* d_{jn} d_{jt}^*|m\rangle\langle\pi_m|\pi_n\rangle\langle n|\sigma|t\rangle\langle\pi_t| \\
&= \sum_{ij}\sum_{mt} K_i\rho d_{im}^* d_{jm} d_{jt}^*|m\rangle\langle m|\sigma|t\rangle\langle\pi_t| \\
&= \sum_{ij}\sum_{mt} K_i\rho\sigma d_{im}^* d_{jm} d_{jt}^*|m\rangle\langle m|t\rangle\langle\pi_t| \because \sigma \text{ is incoherent} \\
&= \sum_{ij}\sum_{m} K_i\rho\sigma d_{im}^* d_{jm} d_{jm}^*|m\rangle\langle\pi_m| \\
&= \sum_{i}\sum_{m} K_i\rho\sigma d_{im}^*\left(\sum_j d_{jm} d_{jm}^*\right)|m\rangle\langle\pi_m| \\
&= \sum_i\left(K_i\rho\sigma\sum_m d_{im}^*|m\rangle\langle\pi_m|\right) \\
&= \sum_i K_i\rho\sigma K_i^\dagger \\
&= \Lambda(\rho\sigma)
\end{aligned} \tag{12}$$

Similarly one can prove that $\Lambda(\sigma\,\rho) = \Lambda(\sigma)\,\Lambda(\rho)$. The equality to be proved now follows in a straightforward way. □

This result above shows that the qudit addition a.k.a. partial swap channel commutes with an IO. For CV settings, the beamsplitter operates as a similar channel to the qudit addition channel described here. Thus, it may be interesting to check whether the free operations in resource theory of coherence for CV systems do also commute with the beamsplitter channel, when one of the inputs in the beamsplitter happen to be free in that formulation of the resource theory of coherence, e.g, a quantum optical coherent state or a thermal state.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary information files).

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References

[1] Shannon C E 2001 *SIGMOBILE Mob. Comput. Commun. Rev.* **5** 3–55
[2] Stam A 1959 *Inf. Control* **2** 101–12
[3] Blachman N 2006 *IEEE Trans. Inf. Theor.* **11** 267–71
[4] Rioul O 2007 arXiv:cs/0701050
[5] Madiman M, Melbourne J and Xu P (arXiv:1604.04225)
[6] Konig R and Smith G 2014 *IEEE Trans. Inf. Theory* **60** 1536–48
[7] Audenaert K, Datta N and Ozols M 2016 *J. Math. Phy.* **57** 052202
[8] De Palma G, Mari A, Lloyd S and Giovannetti V 2015 *Phys. Rev.* A **91** 032320
[9] Jeong K, Lee S and Jeong H 2018 *J. Phys. A: Math. Theor.* **51** 145303
[10] Guha S, Erkmen B I and Shapiro J H 2008 The entropy photon-number inequality and its consequences *2008 Information Theory and Applications Workshop* pp. 128–30
[11] Zhang L, Wang J and Chen Z 2018 *Phys. Lett.* A **382** 1516–23
[12] Baumgratz T, Cramer M and Plenio M B 2014 *Phys. Rev. Lett.* **113** 140401
[13] Streltsov A, Adesso G and Plenio M B 2017 *Rev. Mod. Phys.* **89** 041003
[14] Winter A and Yang D 2016 *Phys. Rev. Lett.* **116** 120404