Generalized Hofmann quantum process fidelity bounds for quantum filters

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We propose and investigate bounds on the quantum process fidelity of quantum filters, i.e., probabilistic quantum operations represented by a single Kraus operator K. These bounds generalize the Hofmann bounds on the quantum process fidelity of unitary operations [H. F. Hofmann, Phys. Rev. Lett. **94**, 160504 (2005)] and are based on probing the quantum filter with pure states forming two mutually unbiased bases. Determination of these bounds therefore requires far fewer measurements than full quantum process tomography. We find that it is particularly suitable to construct one of the probe bases from the right eigenstates of K, because in this case the bounds are tight in the sense that if the actual filter coincides with the ideal one, then both the lower and the upper bounds are equal to 1. We theoretically investigate the application of these bounds to a two-qubit optical quantum filter formed by the interference of two photons on a partially polarizing beam splitter. For an experimentally convenient choice of factorized input states and measurements we study the tightness of the bounds. We show that more stringent bounds can be obtained by more sophisticated processing of the data using convex optimization and we compare our methods for different choices of the input probe states.

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I. INTRODUCTION

Characterization of quantum processes represents an indispensable tool for testing, optimization, and benchmarking of quantum information processing devices. A complete characterization of a quantum device can be provided by quantum process tomography [1,2], but this technique becomes very time-consuming with increasing complexity of the device, unless a special procedure such as compressed sensing can be applied [3,4], because the number of parameters that need to be estimated scales exponentially with the number of qubits.

Instead of full quantum tomography we may just attempt to obtain an indication of how close we are to the target operation, as quantified by the quantum process fidelity [5]. Monte Carlo sampling has been proposed for efficient estimation of the fidelity of multiqubit states and operations with resources scaling polynomially with the estimation precision [6,7]. Yet another experimentally appealing option is represented by the Hofmann bounds on the quantum process fidelity [8]. In this approach, a lower and an upper bound on fidelity with a unitary operation is determined from measurements of average output-state fidelities for input states forming two mutually unbiased bases. The latter approach is particularly efficient for characterization of few-qubit operations and it can be used, e.g., for preliminary benchmarking of a quantum device during its design and optimization before a more complete characterization is carried out at the optimal operating point. During recent years, the Hofmann bound has been successfully utilized for experimental characterization of various two-qubit and three-qubit quantum gates [9–16].

The Hofmann bound was designated to provide bounds on the quantum process fidelity with a deterministic unitary operation. Here we generalize this technique and propose and investigate Hofmann-like quantum process fidelity bounds for a special kind of probabilistic quantum operations called quantum filters. Quantum filters are completely positive trace-decreasing maps that can be represented by a single Kraus operator K. The quantum filter transforms a pure input state $|\psi\rangle$ into a pure output state $K|\psi\rangle$, and the norm of the output state $\langle\psi|K^{\dagger}K|\psi\rangle$ represents the probability of successful application of the filter. The action of a quantum filter on the input state can always be interpreted as a coherence-preserving attenuation in a specific basis, which is preceded and also followed by some unitary transformation. Mathematically, this follows from the singular value decomposition $K = UDV$, where U and V are unitary matrices, and $D \leqslant I$ is a diagonal positive semidefinite matrix. Quantum filters form an important tool in many branches of quantum information science and beyond and they find applications, e.g., in quantum state engineering, entanglement distillation [17–19], and quantum-state discrimination [20,21].

Our derivation of the generalized Hofmann bounds for quantum filters is based on operator inequalities that are at the heart of the original Hofmann bound. In contrast to unitary operations, where measurement of state fidelities for two complementary bases is sufficient, in the case of quantum filters one generally needs to perform an additional set of measurements, which essentially characterizes the performance of the filter in a basis of its eigenstates. The number of measurements can be kept the same as for unitary operations provided that one of the two input bases is formed by the right eigenstates of K. In this case it can also be proved that the resulting lower and upper bounds on the quantum process fidelity are tight in the sense that, for a perfect filter, the bounds are always equal to 1. The probabilistic and nonunitary nature of the quantum filters thus leads to a symmetry-breaking and the occurrence of a preferred set of probe input states. We explicitly consider two bases connected via Fourier transform and also two n-qubit bases connected by Hadamard transform on each qubit.

As an illustration, we theoretically investigate the characterization of a two-qubit optical quantum filter formed by interference of two photons on a partially polarizing beam

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splitter (PPBS) followed by postselection of detection of a single photon at each output port of the beam splitter. This filter is utilized in various linear optical quantum information processing devices such as linear optical quantum gates [9,15,22–24]. We consider an experimentally convenient choice of input probe states for which the required output-state fidelities can be directly determined by product single-qubit measurements. We show, that as a consequence of this basis choice, the resulting upper and lower bounds are not tight. We numerically find the ultimate upper and lower bounds for the same data using the semidefinite programming approach. In this way we illustrate that for the considered quantum filter and the available data more stringent bounds can be obtained by more sophisticated processing of the data and we compare the two methods also for a different choice of basis states.

The rest of the paper is organized as follows. In Sec. II we review the mathematical representation of quantum filters and general trace-decreasing completely positive maps, we introduce a formula for quantum filter fidelity and discuss its properties. In Sec. III we review the original Hofmann bound on the fidelity of a deterministic process with a fixed unitary process and we generalize this bound to quantum filters. In Sec. IV we study the quantum filter formed by interference of two photons on a PPBS, and Sec. V presents the conclusions. Finally, the appendices contain a proof of an alternative lower bound for n-qubit filters and some technical derivations.

II. QUANTUM FILTERS

Quantum filters $\mathcal{K}(\rho) = K\rho K^\dagger$, where $K^\dagger K \leqslant I$, represent a specific subset of general trace-decreasing completely positive maps. According to the Choi-Jamiolkowski isomorphism [25,26], any completely positive trace-nonincreasing map $\mathcal{E}$ is isomorphic to some positive-semidefinite operator χ, which can be obtained by applying the operation $\mathcal{E}$ to one part of a maximally entangled bipartite state,

$$\chi = \mathcal{I} \otimes \mathcal{E}(|\omega\rangle\langle\omega|). \tag{1}$$

Here $\mathcal{I}$ represents the identity operation,

$$|\omega\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |e_j\rangle|e_j\rangle \tag{2}$$

denotes a maximally entangled state, and $\{|e_j\rangle\}_{j=1}^d$ is an orthonormal basis of a d-dimensional Hilbert space $\mathcal{H}_d$. Using the Choi-Jamiolkowski isomorphism, the transformation $\rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}})$ of a quantum state with density matrix ρ_{in} can be expressed as

$$\rho_{\text{out}} = d\,\text{Tr}_{\text{in}}\big(\rho_{\text{in}}^T \otimes I\,\chi\big), \tag{3}$$

where T denotes transposition in the basis $|e_j\rangle$ and Tr_{in} stands for a partial trace over the input Hilbert space. For probabilistic operations, the output state is in general unnormalized, and $\text{Tr}(\rho_{\text{out}})$ represents the probability that the operation $\mathcal{E}$ is applied to the input state. Specifically, a quantum filter $\mathcal{K}$ is represented by a rank 1 operator $\chi_{\mathcal{K}} = |\omega_K\rangle\langle\omega_K|$, where $|\omega_K\rangle = I \otimes K|\omega\rangle$. Any quantum filter can be realized by using an ancillary quantum system prepared in a pure quantum state, which is projected onto a suitable pure state after it unitarily interacts with the principal system. Each admissible Kraus operator K represents some quantum filter and all Kraus operators $e^{i\phi}K$ describe the same quantum filter. This irrelevant overall phase factor $e^{i\phi}$ is the only ambiguity in the Kraus representation of $\mathcal{K}$, which disappears when dealing with $\chi_{\mathcal{K}}$.

Note that two quantum filters K and aK, where a is a complex constant, produce essentially identical outputs, except for the success probability, which differs [27] by a constant factor $|a|^2$ independent on the input state. One is usually interested in the most efficient implementation of the quantum filter, i.e., in operators K whose maximum singular value is equal to 1. However, to make the experimental implementation of a given quantum filter feasible at all, one often needs to accept a reduced success probability. From this perspective, we can say that all completely positive maps of the form $q\mathcal{E}$, $0 < q \leqslant 1$, belong to the same *class* of probabilistic quantum operations, because they produce the same *normalized* output state

$$\tilde{\rho}_{\text{out}} = \frac{\rho_{\text{out}}}{\text{Tr}\rho_{\text{out}}} = \frac{d\,\text{Tr}_{\text{in}}\big(\rho_{\text{in}}^T \otimes I\,q\chi\big)}{d\,\text{Tr}\big(\rho_{\text{in}}^T \otimes I\,q\chi\big)} \tag{4}$$

for any input state ρ_{in}. On the other hand, the success probability $p_S(\mathcal{E},\rho_{\text{in}}) = \text{Tr}\rho_{\text{out}}$ will differ by a constant prefactor q again irrespectively of the input state ρ_{in}, because $p_S(q\mathcal{E},\rho_{\text{in}}) = d\,\text{Tr}(\rho_{\text{in}}^T \otimes I\,q\chi) = q p_S(\mathcal{E},\rho_{\text{in}})$. Thus, if we accept a constantly worse success probability all operations from the given class can perform the same task.

In experimental practice, we would often like to quantify the quality of the implemented quantum filter, i.e., we would like to determine the similarity between the actually implemented quantum operation $\mathcal{E}$ and the ideal quantum filter $\mathcal{K}$. While we can represent the two operations $\mathcal{E}$ and $\mathcal{K}$ by their corresponding Choi operators χ and $\chi_{\mathcal{K}}$, their comparison is somewhat complicated by the fact that the traces of these two operators can differ. To facilitate the comparison of the two quantum operations, we therefore introduce normalized Choi operators $\tilde{\chi} = \chi/\text{Tr}(\chi)$ and $\tilde{\chi}_{\mathcal{K}} = \chi_{\mathcal{K}}/\text{Tr}(\chi_{\mathcal{K}})$, which can be seen as suitable representatives of the two *classes* of operations that contain $\mathcal{E}$ and $\mathcal{K}$, respectively. After this preparatory step, we can define the fidelity between the actually implemented quantum operation and the ideal quantum filter as the fidelity between the normalized Choi operators [28–30]. Since $\tilde{\chi}_{\mathcal{K}}$ is a projector on a pure state (a rank 1 operator), the general formula for the Uhlmann-Jozsa fidelity simplifies to the overlap of the normalized Choi operators,

$$F = \frac{\langle\omega_K|\chi|\omega_K\rangle}{\text{Tr}(\chi)\langle\omega_K|\omega_K\rangle} = \frac{\text{Tr}(\chi\,\chi_{\mathcal{K}})}{\text{Tr}(\chi)\,\text{Tr}(\chi_{\mathcal{K}})}, \tag{5}$$

and it can be understood as a cosine of an angle in an operator space. It holds that $0 \leqslant F \leqslant 1$ and $F = 1$ if and only if $\chi = |a|^2|\omega_{\mathcal{K}}\rangle\langle\omega_{\mathcal{K}}|$, i.e., if the actually implemented operation is a quantum filter aK, where a is a complex constant [28]. Since $\langle\omega_K|\omega_L\rangle = \text{Tr}(K^\dagger L)/d$, a fidelity between two filters K and L can be expressed as

$$F = \frac{|\text{Tr}(K^\dagger L)|^2}{\text{Tr}(K^\dagger K)\text{Tr}(L^\dagger L)}, \tag{6}$$

which generalizes the formula for the fidelity of two unitary operations, $F = |\text{Tr}(U_1^\dagger U_2)|^2/d^2$. In particular, the fidelity between filters K and aK is equal to 1 and does not depend on a, as expected, since both these filters belong to the same class and implement essentially the same probabilistic quantum operation, only one does it more efficiently than the other.

In parallel to the fidelity, it is useful to consider also the average success probability $p_S = \int \mathrm{Tr}[\mathcal{E}(|\psi\rangle\langle\psi|)]d\psi$ of the implemented operation [27]. Since $\int |\psi\rangle\langle\psi| d\psi = I/d$ according to Schur's lemma, we obtain

$$p_S = d\,\mathrm{Tr}\left(\int |\psi\rangle\langle\psi|^T d\psi \otimes I\,\chi\right) = \mathrm{Tr}(\chi). \tag{7}$$

The average probability p_S can be, in practice, conveniently determined as an average of a discrete set of d success probabilities p_j of the operation $\mathcal{E}$ for basis states $|e_j\rangle$. In particular, we can write

$$p_S = \frac{1}{d}\sum_{j=1}^{d} p_j \tag{8}$$

and it can be shown [27] that this formula coincides with (7) and does not depend on the basis choice. For a given class of quantum operations $q\mathcal{E}$ we have $p_S = q\mathrm{Tr}(\chi)$, while the fidelity does not depend on q, $F = \mathrm{Tr}(\chi\chi_{\mathcal{K}})/[\mathrm{Tr}(\chi)\mathrm{Tr}(\chi_{\mathcal{K}})]$. The fidelity F and the success probability p_S thus represent two complementary characterizations of probabilistic quantum operations.

We emphasize that we do not claim that the fidelity provides a metric on the full space of trace-decreasing completely positive maps. Nevertheless, it does provide an upper and a lower bound on the trace distance between the normalized Choi operators,

$$2(1-F) \leqslant ||\tilde{\chi} - \tilde{\chi}_{\mathcal{K}}||_1 \leqslant 2\sqrt{1-F}, \tag{9}$$

which also upper and lower bounds the trace distance between Choi operators of the actually implemented operation $\mathcal{E}$ and some ideal filter aK. Since $p_S = \mathrm{Tr}(\chi)$ we can rewrite the inequality (9) as

$$2p_S(1-F) \leqslant ||\chi - q\chi_{\mathcal{K}}||_1 \leqslant 2p_S\sqrt{1-F}, \tag{10}$$

where $q = dp_S/\mathrm{Tr}(K^\dagger K)$, and $q\chi_{\mathcal{K}}$ is a Choi operator representing the quantum filter $\sqrt{q}K$. Let us note that the trace distance of Choi-Jamiolkowski operators is a lower bound on the diamond norm [31,32] (completely bounded norm [33]) based distance of the corresponding two completely positive maps. Thus, using the average success probability and quantum filter fidelity we can lower bound the diamond norm between the ideal quantum filter and the actually implemented quantum operation. If the goal of an experiment is to build the given quantum filter with a fixed tolerance in the diamond norm, this places a lower bound on the quantum filter fidelity with which it has to be implemented. Estimation of the quantum filter fidelity can therefore serve as a diagnostic measure [5] to guide experimentalists in choosing a suitable operating point for their experiments, where it is meaningful to perform a more detailed characterization of the implemented quantum filter, e.g., by full quantum process tomography.

For deterministic unitary operations, the quantum process fidelity can be directly related to the average quantum-state fidelity [30]. However, for trace-decreasing operations there is no simple relationship between the quantum filter fidelity (5) and the average state fidelity. Moreover, even if the quantum filter fidelity F gets arbitrarily close to 1, the state fidelity can still be arbitrarily low for some particular state if the success probability of the operation is sufficiently low for that state.

III. FIDELITY BOUNDS FOR QUANTUM FILTERS

Our aim is to propose a procedure that would lower and upper bound the fidelity (5) based on measured state fidelities of output states with respect to the ideal output for several input states. To do this we first review the original Hofmann bound for deterministic operations in d dimensions [8] and then generalize it so that it will become applicable to nonunitary quantum filters.

A. Original Hofmann bound

Suppose $\{|e_j\rangle\}_{j=1}^d$ and $\{|f_k\rangle\}_{k=1}^d$ are two orthonormal bases of $\mathcal{H}_d$ that are mutually related by discrete Fourier transform, i.e.,

$$|f_k\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^{d} e^{i\frac{2\pi}{d}jk}|e_j\rangle. \tag{11}$$

Since $|\langle e_i|f_j\rangle|^2 = 1/d\ \ \forall i,j$, the two bases are mutually unbiased. We denote by $|e_j^{\mathrm{out}}\rangle \equiv U|e_j\rangle$ and $|f_k^{\mathrm{out}}\rangle \equiv U|f_k\rangle$ the ideal output states of a unitary transformation U. Density matrices ρ_j and ξ_k of output states produced by the actually implemented operation χ from the input states $|e_j\rangle$ and $|f_k\rangle$ can be expressed as

$$\begin{aligned} \rho_j &= d\,\mathrm{Tr}_{\mathrm{in}}(\chi|e_j\rangle\langle e_j|^T \otimes I), \\ \xi_k &= d\,\mathrm{Tr}_{\mathrm{in}}(\chi|f_k\rangle\langle f_k|^T \otimes I), \end{aligned} \tag{12}$$

where the transposition is taken with respect to the basis $|e_j\rangle$ used in the Choi-Jamiolkowski isomorphism. For deterministic operations $\mathrm{Tr}(\rho_j) = \mathrm{Tr}(\xi_k) = 1$ holds, while for probabilistic operations the trace of the output density matrix equals the probability of successful application of χ to a given input state, $\mathrm{Tr}(\rho_j) = p_j$. In this subsection we consider deterministic operations and review the derivation of the original Hofmann bound on the fidelity of unitary operations [8]. After these preparatory steps we then generalize this bound to probabilistic quantum filters in the next subsection.

The average output-state fidelities for the two sets of probe states are defined as follows:

$$\begin{aligned} F_1 &= \frac{1}{d}\sum_{j=1}^{d}\langle e_j^{\mathrm{out}}|\rho_j|e_j^{\mathrm{out}}\rangle, \\ F_2 &= \frac{1}{d}\sum_{k=1}^{d}\langle f_k^{\mathrm{out}}|\xi_k|f_k^{\mathrm{out}}\rangle. \end{aligned} \tag{13}$$

The state fidelities appearing in this formula can be experimentally determined by measuring the output states ρ_j (ξ_j) in a basis formed by states $|e_k^{\mathrm{out}}\rangle$ ($|f_k^{\mathrm{out}}\rangle$). The Hofmann lower bound on the quantum process fidelity [8]

$$F \geqslant F_1 + F_2 - 1 \tag{14}$$

can be proved as follows. For deterministic transformations $\mathrm{Tr}(\chi)=1$, hence the bound (14) is equivalent to

$$\mathrm{Tr}(\chi|\omega_U\rangle\langle\omega_U|) \geqslant \sum_{j=1}^{d}\mathrm{Tr}(\chi|e_j\rangle\langle e_j|^T\otimes|e_j^{\mathrm{out}}\rangle\langle e_j^{\mathrm{out}}|) + \sum_{k=1}^{d}\mathrm{Tr}(\chi|f_k\rangle\langle f_k|^T\otimes|f_k^{\mathrm{out}}\rangle\langle f_k^{\mathrm{out}}|) - \mathrm{Tr}(\chi I\otimes I). \quad (15)$$

The validity of the inequality (15) would be guaranteed by showing the positive semidefiniteness of an operator $X=I\otimes URI\otimes U^{\dagger}$, where

$$R=|\omega\rangle\langle\omega|-\sum_{j=1}^{d}|e_j\rangle\langle e_j|^T\otimes|e_j\rangle\langle e_j| - \sum_{k=1}^{d}|f_k\rangle\langle f_k|^T\otimes|f_k\rangle\langle f_k|+I\otimes I. \quad (16)$$

If $X\geqslant 0$, then $\mathrm{Tr}(\chi X)\geqslant 0$ due to $\chi\geqslant 0$, which implies inequality (15). As we show in Appendix A the operator R can be rewritten (in the term-to-term fashion) as

$$R=|\omega_{11}\rangle\langle\omega_{11}|-\sum_{j=1}^{d}|\omega_{j1}\rangle\langle\omega_{j1}|-\sum_{k=1}^{d}|\omega_{1k}\rangle\langle\omega_{1k}| + \sum_{j,k=1}^{d}|\omega_{jk}\rangle\langle\omega_{jk}|, \quad (17)$$

where $\{|\omega_{jk}\rangle\}$ is an orthonormal basis of maximally entangled states in d dimensions.

From the above equation it is clear that $R\geqslant 0$, which implies $X\geqslant 0$ and this proves the original Hofmann bound. In a similar fashion, the upper bounds on the quantum process fidelity $F\leqslant F_1$, $F\leqslant F_2$ can be derived from the following two operator inequalities

$$|\omega_{11}\rangle\langle\omega_{11}|\leqslant\sum_{j=1}^{d}|\omega_{j1}\rangle\langle\omega_{j1}|,\quad |\omega_{11}\rangle\langle\omega_{11}|\leqslant\sum_{k=1}^{d}|\omega_{1k}\rangle\langle\omega_{1k}|. \quad (18)$$

Next, we use the operator R to derive lower and upper bounds on the fidelity of quantum filters.

B. Generalization of the bound to quantum filters

Let us multiply operator R from Eq. (16) by $I\otimes K$ from the left and by $I\otimes K^{\dagger}$ from the right. Using Eq. (16) we obtain

$$|\omega_K\rangle\langle\omega_K|-\sum_{j=1}^{d}|e_j\rangle\langle e_j|^T\otimes K|e_j\rangle\langle e_j|K^{\dagger} - \sum_{k=1}^{d}|f_k\rangle\langle f_k|^T\otimes K|f_k\rangle\langle f_k|K^{\dagger}+I\otimes KK^{\dagger}\geqslant 0, \quad (19)$$

where the inequality follows from $R\geqslant 0$. Taking the trace with χ, Eq. (19) can be rewritten as

$$\mathrm{Tr}(\chi|\omega_K\rangle\langle\omega_K|)-\frac{1}{d}\sum_{j=1}^{d}\mathrm{Tr}(K|e_j\rangle\langle e_j|K^{\dagger}\rho_j) - \frac{1}{d}\sum_{k=1}^{d}\mathrm{Tr}(K|f_k\rangle\langle f_k|K^{\dagger}\xi_k)+\mathrm{Tr}(KK^{\dagger}\Omega)\geqslant 0, \quad (20)$$

where ρ_j and ξ_k, defined in Eq. (12), are the unnormalized output states of a probabilistic operation χ corresponding to pure input states $|e_j\rangle$ and $|f_k\rangle$, respectively, and $\Omega = d\,\mathrm{Tr}_{\mathrm{in}}(\chi\,(\frac{1}{d}I)^T\otimes I)$ is the unnormalized output state for a maximally mixed input state. To obtain a lower bound on the fidelity of a quantum filter K, we divide the inequality (20) by $\mathrm{Tr}(\chi)\,\mathrm{Tr}(\chi_K)$ and rewrite the resulting expression so that it contains normalized overlaps,

$$F\geqslant\sum_{j=1}^{d}p_j\langle\tilde{e}_j|\tilde{\rho}_j|\tilde{e}_j\rangle+\sum_{k=1}^{d}q_k\langle\tilde{f}_k|\tilde{\xi}_k|\tilde{f}_k\rangle-\Delta\,\mathrm{Tr}(KK^{\dagger}\tilde{\Omega}). \quad (21)$$

Here $\tilde{\rho}_j=\rho_j/\mathrm{Tr}(\rho_j)$, $\tilde{\xi}_k=\xi_k/\mathrm{Tr}(\xi_k)$, and $\tilde{\Omega}=\Omega/\mathrm{Tr}(\Omega)$ are normalized output states of χ, and

$$|\tilde{e}_j\rangle=\frac{K|e_j\rangle}{\sqrt{\langle e_j|K^{\dagger}K|e_j\rangle}},\quad |\tilde{f}_k\rangle=\frac{K|f_k\rangle}{\sqrt{\langle f_k|K^{\dagger}K|f_k\rangle}} \quad (22)$$

denote the normalized output states of the ideal filter K. The weight factors read

$$p_j=\Delta P_j\langle e_j|K^{\dagger}K|e_j\rangle,\quad q_k=\Delta Q_k\langle f_k|K^{\dagger}K|f_k\rangle, \quad (23)$$

where $\Delta=d/\mathrm{Tr}(K^{\dagger}K)=1/\mathrm{Tr}(\chi_K)$, and $P_j=\frac{\mathrm{Tr}(\rho_j)}{d\,\mathrm{Tr}(\chi)}$ and $Q_k=\frac{\mathrm{Tr}(\xi_k)}{d\,\mathrm{Tr}(\chi)}$ denote the relative success probabilities of operation χ for input basis states $|e_j\rangle$ and $|f_k\rangle$, respectively. These relative probabilities satisfy $\sum_j P_j=\sum_k Q_k=1$.

Formula (21) generalizes the Hofmann lower bound (14) to quantum filters and represents one of our main results. We can see that the fidelity of a quantum filter is lower bounded by an expression which contains two weighted sums of the output-state fidelities $\langle\tilde{e}_j|\tilde{\rho}_j|\tilde{e}_j\rangle$ and $\langle\tilde{f}_k|\tilde{\xi}_k|\tilde{f}_k\rangle$, which generalizes the average state fidelities F_1 and F_2 appearing in the original Hofmann bound. The last term on the right-hand side of inequality (21) provides a generalization of the factor -1 to quantum filters and depends both on the ideal filter K and on the actual operation χ through $\tilde{\Omega}$. It follows from Eq. (20) that the summation in Eq. (21) should be performed only over those terms for which the overlap $\langle e_j|K^{\dagger}K|e_j\rangle$ or $\langle f_k|K^{\dagger}K|f_k\rangle$ is nonzero. This also ensures that the normalized output states (22) are well defined.

A most straightforward way to experimentally determine the output state fidelities and relative success probabilities P_j and Q_k consists of measuring the output state ρ_j (ξ_k) in a basis including the corresponding output state $|\tilde{e}_j\rangle$ ($|\tilde{f}_k\rangle$) produced by an ideal filter. For quantum filters $\{|\tilde{e}_j\rangle\}_{j=1}^{d}$ and $\{|\tilde{f}_k\rangle\}_{k=1}^{d}$ generally do not form a basis, which means that a separate measurement basis must be set for each probe state, and these measurement bases are generally not mutually unbiased. Besides testing the unknown quantum transformation with $2d$

input states $\{|e_j\rangle\}_{j=1}^{d}$, $\{|f_k\rangle\}_{k=1}^{d}$, we also need to determine the term $\mathrm{Tr}(KK^\dagger\tilde{\Omega})$ by some measurements. To construct a suitable measurement, consider a singular value decomposition of K,

$$K = \sum_{l=1}^{d} \sqrt{\lambda_l}|v_l\rangle\langle w_l|, \tag{24}$$

where the left and right eigenvectors $|v_l\rangle$ and $|w_l\rangle$ form two orthonormal bases and the non-negative singular values were chosen in the form $\sqrt{\lambda_l}$ to simplify subsequent formulas. As a consequence, the positive-semidefinite operators $K^\dagger K$, $KK^\dagger$ have the following spectral decompositions:

$$K^\dagger K = \sum_{l=1}^{d} \lambda_l|w_l\rangle\langle w_l|, \qquad KK^\dagger = \sum_{l=1}^{d} \lambda_l|v_l\rangle\langle v_l|. \tag{25}$$

In principle, we can determine $\mathrm{Tr}(KK^\dagger\tilde{\Omega})$ from suitable measurements on any d input states forming an orthonormal basis (e.g., vectors $|u_j\rangle$). Let $\zeta_j = d\,\mathrm{Tr}_{\mathrm{in}}(\chi|u_j\rangle\langle u_j|^T \otimes I)$ and $\tilde{\zeta}_j = \zeta_j/\mathrm{Tr}(\zeta_j)$ denote the unnormalized and normalized output states corresponding to the input state $|u_j\rangle$, and similarly as before we define the relative success probability for this input state as $R_j = \frac{\mathrm{Tr}(\zeta_j)}{d\,\mathrm{Tr}(\chi)}$. The term $\mathrm{Tr}(KK^\dagger\tilde{\Omega})$ can then be expressed as

$$\mathrm{Tr}(KK^\dagger\tilde{\Omega}) = \sum_{j,l=1}^{d} \lambda_l R_j\langle v_l|\tilde{\zeta}_j|v_l\rangle. \tag{26}$$

The relative success probabilities R_j as well as the overlaps of output states $\tilde{\zeta}_j$ with $|v_l\rangle$ can be determined by measuring each output state $\tilde{\zeta}_j$ in the basis formed by the eigenstates $|v_l\rangle$.

At this point it is useful to realize that the experimental effort can be kept the same as for K being unitary at the price of a suitable choice of basis $|e_j\rangle$ and, consequently, $|f_k\rangle$. Thus, if we choose $|e_j\rangle = |w_j\rangle$ and also $|u_j\rangle = |w_j\rangle$, then

$$|\tilde{e}_j\rangle = \frac{K|w_j\rangle}{\sqrt{\langle w_j|K^\dagger K|w_j\rangle}} = |v_j\rangle \tag{27}$$

and the data for input states $|e_j\rangle$ measured after the filter in basis $|\tilde{e}_j\rangle$ can be used to determine the last term in Eq. (21), i.e., $\tilde{\zeta}_j = \tilde{\rho}_j$, $R_j = P_j\ \forall j$. After some algebra we find that the lower bound now equals

$$F \geqslant \sum_{k=1}^{d} Q_k\langle\tilde{f}_k|\tilde{\xi}_k|\tilde{f}_k\rangle - \sum_{j,l=1}^{d} \frac{\lambda_l}{\bar{\lambda}}(1-\delta_{jl})P_j\langle\tilde{e}_l|\tilde{\rho}_j|\tilde{e}_l\rangle, \tag{28}$$

where $\bar{\lambda} \equiv (\sum_{j=1}^{d}\lambda_j)/d$ and we have used the identity $\langle f_k|K^\dagger K|f_k\rangle = \bar{\lambda}$, which holds since the two bases $|e_j\rangle$ and $|f_k\rangle$ are related by quantum Fourier transform.

An important property of the lower bound is its tightness. Especially, if the implemented transformation is the desired one, then the fidelity $F = 1$ and we want our lower bound to attain the value 1 as well. If the implementation of the filter is perfect, then $\langle\tilde{f}_k|\tilde{\xi}_k|\tilde{f}_k\rangle = 1$ and $\langle\tilde{e}_l|\tilde{\rho}_j|\tilde{e}_l\rangle = \delta_{jk}$. If we insert these expressions into Eq. (28), then we get $F \geqslant 1$, which confirms that the lower bound (28) is tight for any quantum filter K. Note that this tightness is achieved due to the special choice of the probe states, where $|e_j\rangle$ coincide with the right eigenvectors of K. For other choices of the probe states the lower bound (21) is generally not tight and can be strictly lower than 1 even for a perfect filter. This should be contrasted with the original Hofmann bound (14) which always attains the value 1 if the target unitary U is implemented perfectly, irrespective of the choice of the two probe bases. The nonunitarity of the filter K thus leads to a symmetry breaking and emergence of preferred states suitable for benchmarking of the filter.

In a fashion very similar to that above, also a pair of upper bounds can be derived,

$$F \leqslant \sum_{j=1}^{d} p_j\langle\tilde{e}_j|\tilde{\rho}_j|\tilde{e}_j\rangle, \quad F \leqslant \sum_{k=1}^{d} q_k\langle\tilde{f}_k|\tilde{\xi}_k|\tilde{f}_k\rangle, \tag{29}$$

if we start from the inequalities

$$\begin{aligned} |\omega\rangle\langle\omega| &\leqslant \sum_{j=1}^{d} |e_j\rangle\langle e_j|^T \otimes |e_j\rangle\langle e_j|, \\ |\omega\rangle\langle\omega| &\leqslant \sum_{k=1}^{d} |f_k\rangle\langle f_k|^T \otimes |f_k\rangle\langle f_k|, \end{aligned} \tag{30}$$

which are equivalent to inequalities (18). For the special choice of probe states $|e_j\rangle = |w_j\rangle$ the upper bounds simplify to

$$F \leqslant \sum_{j=1}^{d} \frac{\lambda_j}{\bar{\lambda}} P_j\langle v_j|\tilde{\rho}_j|v_j\rangle, \quad F \leqslant \sum_{k=1}^{d} Q_k\langle\tilde{f}_k|\tilde{\xi}_k|\tilde{f}_k\rangle. \tag{31}$$

Similarly to the lower bound (28) also the upper bounds (31) are tight in the sense that they yield $F \leqslant 1$ if the filter is implemented perfectly.

From the application point of view, the n-qubit systems whose Hilbert space is endowed with a tensor product structure and $d = 2^n$ are particularly relevant. In this case a natural choice of $|e_j\rangle$ could be the computational basis formed by tensor products of single-qubit states $|0\rangle$ and $|1\rangle$. By its construction, the discrete quantum Fourier transform of the n-qubit computational basis states leads to product n-qubit states $|f_k\rangle$. This is good for experiments, since preparation of product states is often much simpler than preparation of entangled states. Unfortunately, for most quantum filters at least some (ideal) output quantum states for the above inputs are entangled. For that reason it might be useful to study also other pairs of input bases, which could be tested more easily. One such combination could be the computational basis and the Hadamard basis formed by tensor products of states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. As we show in Appendix B the above derived lower and upper bounds hold also for the latter setting.

IV. TWO-QUBIT QUANTUM FILTERS

The Hofmann bound (14) proved particularly useful for characterization of various linear optical quantum gates [9–13,15,16]. As a case study, we therefore investigate here a characterization of a quantum filter acting on the polarization state of two photons. The filtering is realized by interference of the photons on a PPBS followed by postselection of events when a single photon is observed at each output port of the beam splitter. In the basis of vertical and horizontal polarizations, the resulting two-qubit quantum filter reads

$$K = \begin{pmatrix} t_H^2 - r_H^2 & 0 & 0 & 0 \\ 0 & t_H t_V & -r_H r_V & 0 \\ 0 & -r_H r_V & t_H t_V & 0 \\ 0 & 0 & 0 & t_V^2 - r_V^2 \end{pmatrix}, \quad (32)$$

where t_H, t_V and r_H, r_V denote the amplitude transmittances and reflectances of PPBS for horizontal and vertical polarizations, respectively. We assume that the transmittances and reflectances are real and that the beam splitter is lossless, hence $t_j^2 + r_j^2 = 1$.

We investigate further only the case $t_H = 1$, since this element is often used in optical quantum information processing experiments and its fidelity should be assessed. In this case the filter becomes diagonal, $K = \mathrm{diag}\{1, t_V, t_V, 2t_V^2 - 1\}$. Choosing $\{|e_j\rangle\}_{j=1}^4$ as the computational basis in which the filter is diagonal would, for probe states $|f_k\rangle$, lead to measurements on output states in an entangled basis, which is problematic in many optical experimental setups. Instead, we introduce alternative probe states that require just preparations and measurements in product bases. The idea is to employ the following pair of bases:

$$\begin{aligned} |e_1\rangle &= |0\rangle|+\rangle, \quad |f_1\rangle = |+\rangle|0\rangle, \\ |e_2\rangle &= |0\rangle|-\rangle, \quad |f_2\rangle = |+\rangle|1\rangle, \\ |e_3\rangle &= |1\rangle|+\rangle, \quad |f_3\rangle = |-\rangle|0\rangle, \\ |e_4\rangle &= |1\rangle|-\rangle, \quad |f_4\rangle = |-\rangle|1\rangle. \end{aligned} \quad (33)$$

The choice of these probe states is motivated by previous experiments, where bounds on the fidelity of a quantum controlled-NOT gate and controlled-Z gate were determined [9–12,16]. The two bases (33) are related via a Hadamard transform on each qubit, $|f_j\rangle = H \otimes H|e_j\rangle$, and this relation together with the factorized form of the basis states $|e_j\rangle$ and $|f_k\rangle$ ensures that the lower and upper fidelity bounds, (21) and (29), are applicable (cf. also Appendix B). The practical advantage of the probe states (33) is that the filter K maps them on product states,

$$\begin{aligned} |\tilde{e}_1\rangle &= |0\rangle|a_+\rangle, \quad |\tilde{f}_1\rangle = |a_+\rangle|0\rangle, \\ |\tilde{e}_2\rangle &= |0\rangle|a_-\rangle, \quad |\tilde{f}_2\rangle = |b_+\rangle|1\rangle, \\ |\tilde{e}_3\rangle &= |1\rangle|b_+\rangle, \quad |\tilde{f}_3\rangle = |a_-\rangle|0\rangle, \\ |\tilde{e}_4\rangle &= |1\rangle|b_-\rangle, \quad |\tilde{f}_4\rangle = |b_-\rangle|1\rangle, \end{aligned}$$

where

$$|a_\pm\rangle = \frac{1}{\sqrt{1+t_V^2}}|0\rangle \pm \frac{t_V}{\sqrt{1+t_V^2}}|1\rangle,$$

$$|b_\pm\rangle = \frac{t_V}{\sqrt{t_V^2 + (2t_V^2-1)^2}}|0\rangle \pm \frac{2t_V^2-1}{\sqrt{t_V^2 + (2t_V^2-1)^2}}|1\rangle.$$

Since the filter K is not diagonal in either of the two probe bases (33), additional measurements are required to estimate the term $\mathrm{Tr}(KK^\dagger\tilde{\Omega})$. A natural choice is to employ the computational basis states as additional probes $|u_j\rangle$ and measure the resulting output states in the computational basis. Mathematically, we have $|u_j\rangle = |w_j\rangle = |v_j\rangle$ and

$$|v_1\rangle = |00\rangle, \quad |v_2\rangle = |01\rangle, \quad |v_3\rangle = |10\rangle, \quad |v_4\rangle = |11\rangle. \quad (34)$$

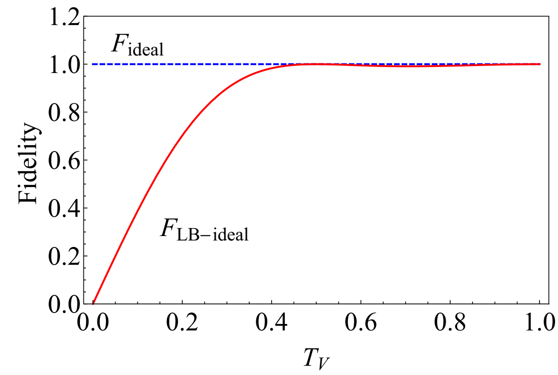

FIG. 1. A lower bound (21) on the fidelity F of a two-qubit quantum filter K is plotted dependent on the intensity transmittance $T_V = t_V^2$ of the partially polarizing beam splitter. Ideal implementation of the filter is assumed, hence the true fidelity $F = 1$ and it is depicted by the dashed blue line. The gap between the two lines illustrates the tightness of the bound.

The above outlined procedure to test the unknown quantum filter requires $3d = 12$ product input states and each output state needs to be measured in a single product basis. For comparison, full quantum process tomography of a two-qubit quantum filter [34] would typically involve about 144 different combinations of input states and output measurements. Thus, characterization of the filter via the generalized Hofmann bounds requires far fewer resources.

Let us now discuss the tightness of the lower bound (21) for our choice of the filter and the probe states. A detailed analysis reveals that in the present case this bound is tight for the ideal quantum filter, i.e., equal to 1, only if $T_V = \frac{1}{2}$ or $T_V = 1$, where $T_V = t_V^2$. Otherwise there is a gap, which is smaller than 1% if the desired PPBS has $T_V > \frac{1}{2}$ (see Fig. 1). We also considered the situation where the actual value of the transmittance T_V differs from the desired one, which we denote $\tilde{T}_V$. The dependence of the lower bound on the actual value of T_V is plotted in Fig. 2 for four desired values of $\tilde{T}_V$. The results indicate that the proposed lower bound works well for $\tilde{T}_V > \frac{1}{2}$. However, for $\tilde{T}_V < \frac{1}{2}$ the bound quickly becomes too loose to be useful.

A. Semidefinite programming approach

The above considerations open the question what are the best upper and lower bounds on fidelity given certain data from experiments [35]. Finding the best bounds requires the solution of a convex optimization problem, namely, a so-called semidefinite program [36]. Since the fidelity and the constraints given by the measured data are linear in $A = \frac{\chi}{\mathrm{Tr}(\chi)}$,

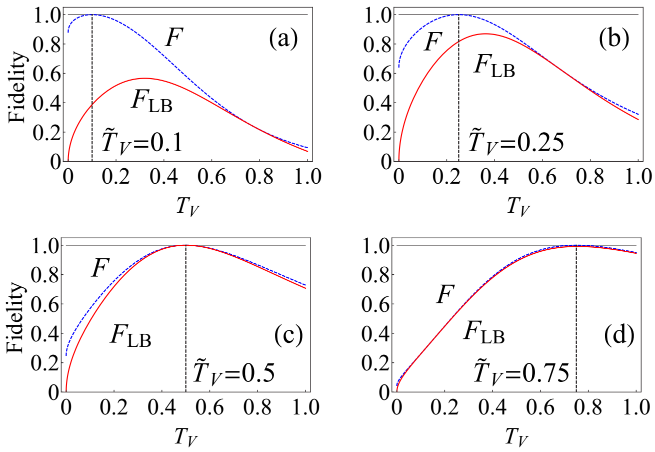

FIG. 2. The solid red lines show the dependence of the lower bound of the fidelity of a quantum filter K on the actual intensity transmittance $T_V = t_V^2$ of the PPBS for four target transmittances: (a) $\tilde{T}_V = 0.1$, (b), $\tilde{T}_V = 0.25$, (c) $\tilde{T}_V = 0.5$, and (d) $\tilde{T}_V = 0.75$. The true fidelity of the filter is depicted by the dashed blue lines.

we might write

$$F_{\rm UB} = \max_{\substack{A \geqslant 0,\ {\rm Tr}(A)=1 \\ \forall k\, {\rm Tr}(AM_k)=r_k}} \Delta\, {\rm Tr}(A|\omega_K\rangle\langle\omega_K|),$$
$$F_{\rm LB} = \min_{\substack{A \geqslant 0,\ {\rm Tr}(A)=1 \\ \forall k\, {\rm Tr}(AM_k)=r_k}} \Delta\, {\rm Tr}(A|\omega_K\rangle\langle\omega_K|), \quad (35)$$

where the matrices M_k and parameters r_k capture the linear constraints provided by the data, and the optimization is carried over all semidefinite operators A that represent equivalence classes of all trace-decreasing completely positive maps χ (consistent with the data) with respect to the fidelity (5). To make our formulation of these constraints sufficiently general, consider a set of input probe states $\{|m_j\rangle\}_{j=1}^d$ forming a basis, where each output state is measured in a generally different basis $\{|n_{jk}\rangle\}_{k=1}^d$. Let f_{jk} denote the measured frequencies which sample the theoretical probabilities

$$p_{jk} = d\, {\rm Tr}(\chi\, |m_j\rangle\langle m_j|^T \otimes |n_{jk}\rangle\langle n_{jk}|). \quad (36)$$

Using the identity $\sum_{j,k=1}^d p_{jk} = d\,{\rm Tr}(\chi)$ we can express the set of constraints on A imposed by the data f_{jk} as follows:

$${\rm Tr}(A|m_j\rangle\langle m_j|^T \otimes |n_{jk}\rangle\langle n_{jk}|) = \frac{f_{jk}}{\sum_{l,m=1}^d f_{lm}}. \quad (37)$$

In our present case, each probe basis $|e_j\rangle$, $|f_k\rangle$, and $|v_l\rangle$ provides a set of 15 linearly independent constraints.

To test the performance of this approach, we used it to find the best upper and lower bounds from the data that we generated for 1000 randomly chosen quantum operations χ that were constructed as a random mixture of an ideal quantum filter K and some other randomly chosen filter K'. The optimization, (35), was performed numerically with the use of CVX, a package for specifying and solving convex programs [37,38]. For a given random quantum filter χ we also compute the analytical lower and upper fidelity bounds according to our procedure and the true fidelity. In Fig. 3 we plot the obtained results as a function of the true fidelity, hence the graph of the true fidelity forms a straight line with a unit slope. For comparison, we consider probing with two sets of input states. The first set is specified by Eq. (33) and requires only product measurements on output states, but the analytical bounds are not tight. The second set of probe states is constructed such that it leads to tight analytical bounds (28) and (31) at the price of the requirement of measurements in an entangled basis for some of the output states. Interestingly, the approach based on semidefinite programming provides tight

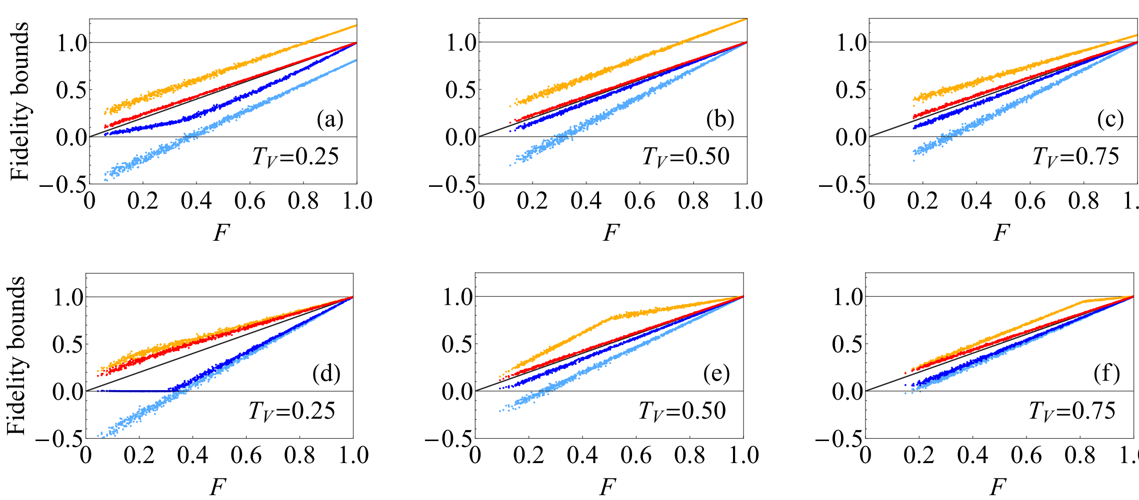

FIG. 3. Graphs of upper and lower bounds on the fidelity of two-qubit quantum filters. Upper and lowermost dots in each graph correspond to analytical bounds for randomly generated quantum operations constructed as a mixture of a fixed ideal filter with transmittance T_V specified in each panel and a randomly chosen filter. The dots just above and below the solid line correspond to the bounds obtained from the same data, but using a semidefinite programming approach. Results in the upper row correspond to probing with states (33) for which measurements in the product basis are sufficient. For comparison, the three graphs in the lower row were obtained for a different set of probe states with $|e_j\rangle$ chosen to be the computational basis, where the ideal filter K is diagonal. The latter choice of probe states requires that some of the output states are measured in an entangled basis.

bounds in both cases; i.e., for ideal implementation of the filter the upper and lower bounds coincide and $F_{\rm LB} = F_{\rm UB} = 1$.

V. CONCLUSIONS

In summary, we have designed and analyzed bounds on the quantum process fidelity F of a specific type of nondeterministic operations called quantum filters. These operations are mathematically characterized as completely positive maps, which can be expressed by a single Kraus operator K. Operationally they correspond to operations which succeed only with a limited probability that depends on the input state and they map any pure state again to a pure state. The proposed bounds represent a generalization of the original Hofmann bounds on the fidelity of unitary transformations [8]. For quantum filters, the average state fidelities are replaced with specific weighted averages, and the lower bound contains also an additional term that depends both on the desired and on the actual operation. As a consequence, in addition to the determination of relative success rates and output-state fidelities for two sets of input basis states, further measurements are generally needed. Nevertheless, we show that the number of input states and measurements can be kept the same as for unitary operations if one of the two input bases is formed by the right eigenstates of K. An important property of any bound is its tightness. In particular, for quantum process fidelity bounds we would like to have both the upper and the lower bounds equal to 1 if the actual and the desired quantum transformations coincide, because in that case the fidelity is $F = 1$. We demonstrated that our bound is tight if one set of probe states is formed by right eigenstates of K and the other by their quantum Fourier transform or by their Hadamard transform. The proposed bounds extend the toolbox of efficient methods of characterization of quantum operations [3,4,6–8,39–41] and provide a method for quick checking of the quality of quantum filters before their more comprehensive characterization, e.g., by quantum process tomography.

As an illustration, we have theoretically investigated the application of the proposed fidelity bounds to characterization of a specific two-qubit linear-optical quantum filter. This filter is implemented by two-photon interference on a partially polarizing beam splitter followed by conditioning on the emergence of a single photon at each output port of the beam splitter, and it is often utilized in optical quantum information processing with polarization-encoded qubits. We consider experimentally convenient choices of product input probe states for which the required output-state fidelities can be directly determined by product single-qubit measurements. It turns out that the price to pay for this experimental convenience is that the resulting bounds are generally not tight. We compare our analytical bounds with ultimate lower and upper bounds that can be obtained from given experimental data with the help of convex optimization. The experimental data represent a set of linear constraints and we numerically solve a so-called semidefinite program that, among all quantum operations satisfying given linear constraints, finds an operation with minimum and maximum overlap with the target quantum filter K. We observe that the ultimate lower and upper fidelity bounds obtained in this way are tight, i.e., equal to 1 for perfect filters, even if the analytical bounds are not. In this way we illustrate that for the considered quantum filter and the available data more stringent bounds can be obtained by more sophisticated data processing.

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APPENDIX A: POSITIVITY OF OPERATOR *R*

Let us define an orthonormal basis $\{|\omega_{jk}\rangle\}_{j,k=1}^{d}$ of maximally entangled states via the action of a pair of operators

$$Z = \sum_j |e_{j\oplus 1}\rangle\langle e_j|, \quad W = \sum_j e^{i\frac{2\pi j}{d}} |e_j\rangle\langle e_j| \tag{A1}$$

on the state $|\omega\rangle$ as

$$|\omega_{jk}\rangle = Z^{j-1}W^{k-1} \otimes I\ |\omega\rangle. \tag{A2}$$

By definition $|\omega_{11}\rangle = |\omega\rangle$ and

$$\sum_{j=1}^{d} |\omega_{1j}\rangle\langle\omega_{1j}| = \frac{1}{d}\sum_{j,k,l=1}^{d} e^{i\frac{2\pi j(k-l)}{d}} |e_k\rangle|e_k\rangle\langle e_l|\langle e_l|$$
$$= \sum_{j=1}^{d} |e_j\rangle\langle e_j|^T \otimes |e_j\rangle\langle e_j|, \tag{A3}$$

because here the transposition is taken with respect to the basis $|e_j\rangle$. Similarly,

$$\sum_{j=1}^{d} |\omega_{j1}\rangle\langle\omega_{j1}| = \frac{1}{d}\sum_{j,k,l=1}^{d} |e_{k\oplus j}\rangle|e_k\rangle\langle e_{l\oplus j}|\langle e_l|. \tag{A4}$$

On the other hand,

$$\sum_{k=1}^{d} |f_k\rangle\langle f_k|^T \otimes |f_k\rangle\langle f_k|$$
$$= \frac{1}{d}\sum_{j,k,l,m,n=1}^{d} e^{i\frac{2\pi k(l-j+m-n)}{d}} |e_j\rangle|e_l\rangle\langle e_m|\langle e_n|$$
$$= \sum_{\substack{j,l,m,n \\ j-l=m-n}} |e_j\rangle|e_l\rangle\langle e_m|\langle e_n|, \tag{A5}$$

which is clearly equivalent to (A4). Thus the operator R defined by Eq. (16) can be expressed as

$$R = |\omega_{11}\rangle\langle\omega_{11}| - \sum_{j=1}^{d} |\omega_{j1}\rangle\langle\omega_{j1}| - \sum_{k=1}^{d} |\omega_{1k}\rangle\langle\omega_{1k}|$$
$$+ \sum_{j,k=1}^{d} |\omega_{jk}\rangle\langle\omega_{jk}|, \tag{A6}$$

where we have used the identity $I \otimes I = \sum_{j,k=1}^{d} |\omega_{jk}\rangle\langle\omega_{jk}|$. Since we managed to rewrite R as a sum of projectors, $R = \sum_{j,k=2}^{d} |\omega_{jk}\rangle\langle\omega_{jk}|$, this proves that $R \geqslant 0$.

APPENDIX B: ALTERNATIVE LOWER BOUND FOR *n*-QUBIT FILTERS

Our goal is to prove the positivity of operator R defined in Eq. (16) for a different pair of orthonormal bases $\{|e_j\rangle\}_{j=1}^d$, $\{|f_k\rangle\}_{k=1}^d$. This would allow us to exactly repeat the same steps as in the text and thus we could use all the derived lower and upper bounds, but for a different choice of $|e_j\rangle$, $|f_k\rangle$. Specifically, we consider systems of n qubits, hence $d=2^n$. Let $\{|e_j\rangle\}_{j=1}^d$ be the computational basis, where $|e_j\rangle = |j_1\rangle|j_2\rangle\dots|j_n\rangle$ and j_m is the mth digit in the binary representation of number $j-1$. Similarly, let $\{|f_k\rangle\}_{k=1}^d$ be the computational basis transformed by the Hadamard transform [acting as $H|j_m\rangle = (|0\rangle + (-1)^{j_m}|1\rangle)/\sqrt{2}$] on every qubit, $|f_k\rangle = H|k_1\rangle \otimes H|k_2\rangle \dots \otimes H|k_n\rangle$. The operator R is acting on $2n$ qubits, which are ordered as n qubits related to the input state tensored with another n qubits related to the output state of the quantum filter. It is useful to divide the $2n$-qubit Hilbert space on which the operator R acts into two-qubit subsystems formed by the mth qubit of the input and the mth qubit of the output. We introduce a unitary operator W which groups together the mth input and output qubits [15],

$$W|j_1\dots,j_n\rangle|k_1,\dots,k_n\rangle = |j_1,k_1\rangle\dots|j_n,k_n\rangle. \tag{B1}$$

In this way the maximally entangled state can be written as $W|\omega\rangle = |\Phi^+\rangle_1\cdots|\Phi^+\rangle_n$, where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ are the Bell states and the subscripts indicate the two-qubit subsystems. It is not difficult to show that the summations over all projectors $|e_j\rangle\langle e_j|^T \otimes |e_j\rangle\langle e_j|$ and $|f_k\rangle\langle f_k|^T \otimes |f_k\rangle\langle f_k|$ in Eq. (16) factorize into products of n summations over two-qubit subsystems consisting of a single input and output qubit. The summations over the two-qubit subspaces can be performed with the use of the identities

$$\begin{aligned} |00\rangle\langle 00| + |11\rangle\langle 11| &= \Phi^+ + \Phi^-, \\ |++\rangle\langle ++| + |--\rangle\langle --| &= \Phi^+ + \Psi^+, \end{aligned} \tag{B2}$$

where $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are the other two Bell states and we used the notation $\Phi^+ \equiv |\Phi^+\rangle\langle\Phi^+|$. Identities (B2) allow us to rewrite the operator R as

$$\begin{aligned} W R W^\dagger = (\Phi^+)^{\otimes n} - (\Phi^+ + \Phi^-)^{\otimes n} - (\Phi^+ + \Psi^+)^{\otimes n} \\ +(\Phi^+ + \Phi^- + \Psi^+ + \Psi^-)^{\otimes n}, \end{aligned} \tag{B3}$$

where we have used the four Bell states to express the identity in the two-qubit Hilbert space as $\mathbb{I} = \Phi^+ + \Phi^- + \Psi^+ + \Psi^-$. Since W is unitary, the operator $WRW^\dagger$ has the same eigenvalues as R. Moreover, $WRW^\dagger$ is diagonal in the basis formed by tensor products of Bell states, hence the eigenvalues can be directly determined from expression (B3). All the projectors contained in the first three terms on the right-hand side of (B3) determine the $(2^{n+1}-1)$–dimensional zero-eigenvalue subspace and it is easy to see that in the remaining subspace the eigenvalue is 1. Thus, all the eigenvalues of $WRW^\dagger$ are nonnegative, which proves that R is a positive semidefinite operator. This proves that the fidelity bound (21) holds also for the case, when the bases $\{|e_j\rangle\}_{j=1}^d$ and $\{|f_k\rangle\}_{k=1}^d$ are the computational basis and its Hadamard transform on every qubit.

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