Exploring boundaries of quantum convex structures: Special role of unitary processes

Zbigniew Puchała,[1,2] Anna Jenčová,[3] Michal Sedlák,[4,5] and Mário Ziman[5,6]

[1] *Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, ulica Bałtycka 5, 44-100 Gliwice, Poland*
[2] *Institute of Physics, Jagiellonian University, ulica Prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland*
[3] *Institute of Mathematics, Slovak Academy of Sciences, Štefánikova, 84511 Bratislava, Slovakia*
[4] *Department of Optics, Palacký University, 17 Listopadu 1192/12, 77146 Olomouc, Czech Republic*
[5] *Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 84511 Bratislava, Slovakia*
[6] *Faculty of Informatics, Masaryk University, Botanická 68a, 60200 Brno, Czech Republic*

(Received 2 April 2015; published 2 July 2015)

We address the question of finding the most unbalanced convex decompositions into boundary elements (so-called boundariness) for sets of quantum states, observables, and channels. We show that in general convex sets the boundariness essentially coincides with the question of the most distinguishable element, thus providing an operational meaning for this concept. Unexpectedly, we discovered that for any interior point of the set of channels the most unbalanced decomposition necessarily contains a unitary channel. In other words, for any given channel the most distinguishable one is some unitary channel. Further, we prove that boundariness is submultiplicative under the composition of systems and explicitly evaluate its maximal value that is attained only for the most mixed elements of the considered sets.

DOI: 10.1103/PhysRevA.92.012304 PACS number(s): 03.67.−a

I. INTRODUCTION

Convexity, rooted in the very concept of probability, is one of unavoidable mathematical features of our description of physical systems. Operationally, it originates in our ability to switch randomly between different physical devices of the same type. As a result, all elementary mathematical structures of quantum theory and most of the quantum properties can be considered convex. In particular, the sets of states, observables, and processes are all convex [1] and we will refer to them as quantum convex sets. It is of foundational interest to understand the similarities and identify the differences of their convex structures. For example, suitable convex subsets of quantum convex sets can be used to define a measure of entanglement [2] or incompatibility [3].

For any convex set, we may introduce the concept of an interior point in a natural way, as a point that can be connected to any other point by a line segment containing it in its interior [4]. We will use this concept to define mixedness and boundariness [5] as measures evaluating how much the element is not extremal or how much the element is not a boundary point, respectively. More precisely, mixedness will be determined via the highest weight occurring in decompositions into extremal points and boundariness will be determined via the highest weight occurring in decompositions into boundary points. In both cases, these numbers tell us how much randomness is needed to create the given element. In this paper we focus on quantum convex sets describing finite-dimensional quantum systems. Thus, we will work in a finite-dimensional setting. Note that similar definitions can be introduced also in infinite dimensions, although some of the facts used below are no longer true.

If the given convex set is also compact, it can be viewed as a base of a closed pointed convex cone and we may consider the corresponding base norm in the generated vector space (see, e.g., [4]). Note that the related distance between points of the base can be determined solely from the convex structure of the base (see, for instance, Ref. [6]). As it is well known for quantum states [1,7] and as has been recently proved for other quantum convex sets [8], this distance is closely related to the minimum-error discrimination problem. In particular, for quantum channels the base norm coincides with the diamond norm [9].

It was proved in Ref. [5] that for the sets of quantum states and observables, boundariness and the base norm distance are closely related. More precisely, the largest distance of a given interior point y from another point of the base is given in terms of the boundariness of y. In the present paper we show that this is true for any base of the positive cone in a finite-dimensional ordered vector space. In particular, for quantum convex sets, this property singles out a subset of extremal elements that are best distinguishable from interior points. Exploiting these results, we will point out an interesting difference between the convex sets of states and channels and also provide an unexpected operational characterization of unitary channels.

This paper is organized as follows. In Sec. II we will provide readers with basic elements of convex analysis and quantum theory relevant for the rest of the paper. The concept of boundariness will be introduced in Sec. III, where various equivalent definitions will be stated and also its operational meaning will be discussed. In Sec. IV we will investigate the boundariness for the case of quantum channels. In particular, we will prove a conjecture stated in Ref. [5]. In Sec. V we will address the question of boundariness for the composition of systems and Sec. VI is devoted to identification of elements for which boundariness achieves its maximal value. Section VII summarizes our results.

II. QUANTUM CONVEX CONES

Suppose V is a real finite-dimensional vector space and $C \subset V$ is a closed convex cone. We assume that C is pointed, i.e., $C \cap -C = \{0\}$, and generating, i.e., $V = C - C$. Then (V,C) becomes a partially ordered vector space, with C the cone of positive elements. Let V^* be the dual space with duality $\langle\cdot,\cdot\rangle$; then we may introduce a partial order in V^* as well, with the dual cone of positive functionals $C^* = \{f \in V^*, \langle f, z\rangle \geqslant 0\, \forall z \in C\}$. Note that C^* is again pointed and generating and $C^{**} = C$.

Interior points $z \in \mathrm{int}(C)$ of the cone C are characterized by the property that for each $v \in V$ there is some $t > 0$ such that $tz - v \in C$, that is, the interior points of C are precisely the order units in (V,C). Alternatively, the following lemma gives a well-known characterization of boundary points of C as elements contained in some supporting hyperplane of C (see Ref. [4], Sec. 11 for more details).

Lemma 1. An element $z \in C$ is a boundary point $z \in \partial C$ if and only if there exists a nonzero element $f \in C^*$ such that $\langle f, z\rangle = 0$. Clearly, then also $f \in \partial C^*$.

A base of C is a compact convex subset $B \subset C$ such that for every nonzero $z \in C$ there is a unique constant $t > 0$ and an element $b \in B$ such that $z = tb$. The relative interior relint(B) is defined as the interior of B with respect to the relative topology in the smallest affine subspace containing B. Note that we have $\mathrm{relint}(B) = B \cap \mathrm{int}(C)$, so the boundary points $z \in \partial B = B \setminus \mathrm{relint}(B)$ can be characterized as in the previous lemma.

There is a one-to-one correspondence between bases $B \subset C$ and order units in the dual space $e \in \mathrm{int}(C^*)$ such that $B = \{z \in C, \langle e, z\rangle = 1\}$ is a base of C if and only if e is an order unit. The order unit e determines the order unit norm in (V^*, C^*) as

$$\|f\|_e = \inf\{\lambda > 0, \lambda e \pm f \in C^*\}, \quad f \in V^*.$$

Its dual is the base norm $\|\cdot\|_B$ in (V,C). In particular, we obtain the following expression for the corresponding distance of elements of B:

$$\|x - y\|_B = 2 \sup_{g, e-g \in C^*} \langle g, x - y\rangle, \quad x, y \in B. \tag{1}$$

We will now describe the basic convex sets (see Ref. [10]) of quantum states, channels, and measurements (observables). Let us stress that each of these sets is a compact convex subset in a finite-dimensional vector space and as such forms a base of the positive cone of some partially ordered vector space, so these sets fit into the framework introduced above.

Let us denote by $\mathcal{H}_d$ the d-dimensional Hilbert space associated with the studied physical system. Then $\mathcal{S}(\mathcal{H}_d)$ stands for the set of all density operators (positive linear operators of unit trace) representing the set of quantum states.

The statistical aspects of measurements are fully described by observables [11] that are identified with positive-operator-valued measures being determined by a collection of effects $E_1, \ldots, E_m$ $(O \leqslant E_j \leqslant I)$ normalized as $\sum_j E_j = I$. Each effect E_j defines a different measurement outcome. In particular, if the system is prepared in a state ϱ, then $p_j = \mathrm{tr}[\varrho E_j]$ is the probability of the registration of the jth outcome.

Quantum channels are modeled by completely positive trace-preserving linear maps, i.e., by transformations $\varrho \mapsto \sum_l A_l \varrho A_l^\dagger$ for any collection of operators $\{A_l\}_l$ satisfying the normalization $\sum_l A_l^\dagger A_l = I$. Define the one-dimensional projection operator $\Psi_+ = \frac{1}{d}\sum_{j,k} |jj\rangle\langle kk|$ on $\mathcal{H}_d \otimes \mathcal{H}_d$, where the vectors $|j\rangle$ form a complete orthonormal basis on $\mathcal{H}_d$. Due to the Choi-Jamiołkowski isomorphism [12,13], the set of quantum channels of a finite-dimensional quantum system is mathematically closely related to the set of density operators (states) of a composite system. In particular, a channel $\mathcal{E}$ is associated with a density operator

$$J_{\mathcal{E}} = (\mathcal{E} \otimes \mathcal{I})[\Psi_+] \in \mathcal{S}(\mathcal{H}_d \otimes \mathcal{H}_d)$$

and the normalization condition $\mathrm{tr}_1 J_{\mathcal{E}} = \frac{1}{d} I$ is the only difference between the mathematical representations of states and channels. In other words, only a special (convex) subset of density operators on $\mathcal{H}_d \otimes \mathcal{H}_d$ can be identified with quantum channels on d-dimensional quantum systems.

III. BOUNDARINESS

For any element of a compact convex subset $B \subset V$ (not necessarily a base) with boundary ∂B and a set of extremal elements ext(B) we may introduce the concepts of mixedness and boundariness evaluating the "distance" of the element from extremal and boundary points, respectively. For any convex decomposition $y = \sum_j \pi_j x_j$, where $0 \leqslant \pi_j \leqslant 1$ and $\sum_j \pi_j = 1$, we define its maximal weight $w_y(\{\pi_j, x_j\}_j) = \max_j \pi_j$. Using this quantity, we may express the mixedness of $y \in B$ as

$$m(y) = 1 - \sup_{x_j \in \mathrm{ext}(B)} w_y(\{\pi_j, x_j\}_j),$$

where the supremum is taken over all convex decompositions of y into extremal elements. In a similar way we may define the boundariness [5] of y as

$$b(y) = 1 - \sup_{x_j \in \partial B} w_y(\{\pi_j, x_j\}_j), \tag{2}$$

where the supremum is taken over all decompositions into boundary elements. By definition $m(y) \geqslant b(y)$, since the convex decompositions in (2) are less restrictive.

Let us prove that the above formula is equivalent to the original definition [5] of boundariness. We recall that for any element $y \in B$, the weight function $t_y : B \to [0,1]$ assigns for every $x \in B$ the supremum of possible weights of the point x in convex decompositions of y, i.e.,

$$t_y(x) = \sup\left\{0 \leqslant t < 1 \,\middle|\, z = \frac{y - tx}{1 - t} \in B\right\}.$$

Due to the compactness of B, the supremum is really attained and there exists some $z \in B$ such that $y = tx + (1-t)z$, where $t = t_y(x)$. Note that we must have $z \in \partial B$ and in fact for an interior point y, $t = t_y(x)$ is equivalent to $z \in \partial B$. Let us consider a convex decomposition $y = \sum_j \pi_j x_j$, $x_j \in \partial B$ and denote by k the index for which $\pi_k = \max_j \pi_j \neq 1$ [the case $\max_j \pi_j = 1$ is trivial and $b(y) = 0$ in both definitions]. If we define $\overline{x}_k = \sum_{j \neq k} \frac{\pi_j}{1 - \pi_k} x_j$ then $y = \pi_k x_k + (1 - \pi_k)\overline{x}_k$, where $\overline{x}_k \in B$. Either $\overline{x}_k \in \partial B$ and we managed to rewrite y as a two-term convex combination of elements from boundary or $\overline{x}_k \in B \setminus \partial B$, which implies $\pi_k < t_y(x_k)$ and there exists $w \in \partial B$ such that a better two-term decomposition $y = tx_k + (1-t)w$ with $t > \pi_k$ exists. This shows that the definition (2) is equivalent to

$$b(y) = 1 - \sup_{x, z \in \partial B} \{s | y = (1-s)x + sz\}$$
$$= \inf_{x \in \partial B} t_y(x).$$

Finally, we obtain the original definition [5]

$$b(y) = \inf_{x\in B} t_y(x) \tag{3}$$

because the infimum is always determined by elements $x \in \text{ext}(B)$ as we discussed in Ref. [5], Proposition 1.

Having established the cone picture of quantum convex sets (forming the bases, i.e., in what follows B is a base), it is useful to see how boundariness can be defined using this language.

Lemma 2. Let $f \in C^*$. If $\|f\|_e = 1$, then $e - f \in \partial C^*$.

Proof. Suppose $\|f\|_e = 1$; then $e - f \in C^*$. If $e - f \in \text{int}(C^*)$, then there is some $t > 0$ such that $e - f \pm tf \in C^*$. However, then $(1+t)^{-1}e - f \in C^*$, so

$$\|f\|_e \leqslant (1+t)^{-1} < 1. \qquad \blacksquare$$

We now find an equivalent expression for boundariness.

Proposition 1. Boundariness of y is equal to $b(y) = \min\{\langle f,y\rangle, f \in C^*, \|f\|_e = 1\}$.

Proof. Let us denote the minimum on the right-hand side by $\tilde{b}(y)$. Let $x \in B$ and $y = tx + (1-t)z$, with $t = t_y(x)$. Then $z \in \partial B$, so there is some nonzero $f \in C^*$ such that $\langle f,z\rangle = 0$. If we set $\tilde{f} = \|f\|_e^{-1} f$, then $\tilde{f} \in C^*$ and $\|\tilde{f}\|_e = 1$ and we have

$$\tilde{b}(y) \leqslant \langle \tilde{f},y\rangle = t_y(x)\langle \tilde{f},x\rangle \leqslant t_y(x).$$

Since this holds for all $x \in B$, we obtain $\tilde{b}(y) \leqslant b(y)$.

For the converse, let $f \in C^*$ and $\|f\|_e = 1$; then $e - f \in \partial C^*$. Hence there is some element $x \in B$ such that $\langle e - f,x\rangle = 0$. If we set $s = t_y(x)$, then $y = sx + (1-s)z$ for some $z \in \partial B$. We have

$$\begin{aligned}\langle f,y\rangle &= 1 - \langle e-f,y\rangle = 1 - (1-s)\langle e-f,z\rangle \geqslant s\\ &= t_y(x) \geqslant b(y),\end{aligned}$$

hence $\tilde{b}(y) \geqslant b(y)$. $\blacksquare$

Let $x,y \in B$ and take $z \in \partial B$ such that $y = sx + (1-s)z$, where $s = t_y(x)$. Then

$$\begin{aligned}\|x-y\|_B &= \|x - sx - (1-s)z\|_B\\ &= (1-s)\|x-z\|_B \leqslant 2[1-b(y)]\end{aligned} \tag{4}$$

constitutes the upper bound derived in [5].

Proposition 2. Let $y \in \text{relint}(B)$ and let $x \in B$. The following are equivalent.

(i) $\|y-x\|_B = 2[1-b(y)]$.

(ii) $t_y(x) = b(y)$.

(iii) There is some $f \in C^*$, with $\|f\|_e = 1$ and $\langle f,y\rangle = b(y)$, such that $\langle f,x\rangle = 1$.

Proof. Suppose (i) and let $y = sx + (1-s)z$ with $s = t_y(x)$. Then

$$2[1-b(y)] = \|x-y\|_B = (1-s)\|x-z\|_B.$$

Since both $(1-s) \leqslant 1 - b(y)$ and $\|x-z\|_B \leqslant 2$, the equality implies that $t_y(x) = s = b(y)$.

If we suppose (ii), then $y = b(y)x + [1-b(y)]z$ for some $z \in \partial B$. There is some nonzero $f \in C^*$ such that $\langle f,z\rangle = 0$ and we may clearly suppose that $\|f\|_e = 1$. By Proposition 1, $b(y) \leqslant \langle f,y\rangle = b(y)\langle f,x\rangle \leqslant b(y)$. Since y is an interior point, $b(y) > 0$, so we must have $\langle f,y\rangle = b(y)$ and $\langle f,x\rangle = 1$.

Finally, if we suppose (iii), then using inequalities (1) and (4),

$$\begin{aligned}2[1-b(y)] &\geqslant \|x-y\|_B \geqslant 2\langle e-f,y-x\rangle = 2\langle e-f,y\rangle\\ &= 2[1-b(y)].\end{aligned} \qquad \blacksquare$$

We now resolve the conjecture of the tightness of the upper bound (4) by showing that it can be always saturated.

Theorem 1. For any $y \in B$, there exists some $x_0 \in \text{ext}(B)$ such that

$$\|y - x_0\|_B = \sup_{x\in B}\|y-x\|_B = 2[1-b(y)].$$

Proof. Note first that since $x \mapsto \|y-x\|_B$ is a convex function, the supremum over B is attained at some $x_0 \in \text{ext}(B)$. It is therefore enough to prove that the equality in (4) holds for some $x \in B$. If y is an interior point, then by Proposition 2, the equality is attained for any x such that $t_y(x) = b(y)$ and we know from the results in [5] that this is achieved in B. If $y \in \partial B$, then there exists some $f \in C^*$ and $\|f\|_e = 1$ such that $\langle f,y\rangle = 0$ and since $e - f \in \partial C^*$, there is some $x \in B$ such that $\langle e-f,x\rangle = 0$. Then

$$2 \geqslant \|y-x\|_B \geqslant 2\langle e-f,y-x\rangle = 2 = 2[1-b(y)]. \qquad \blacksquare$$

IV. BOUNDARINESS FOR QUANTUM CHANNELS

In Ref. [5] it was shown that the inequality (4) is saturated for states and observables, however, the case of channels remained open. Theorem 1 shows that this saturation holds also in this remaining case. In particular, for any interior point $Y \in \mathcal{Q}$, where $\mathcal{Q}$ is either the set of quantum states, channels, or observables, the identity

$$||X - Y||_B = 2[1 - b(Y)]$$

holds for a suitable $X \in \text{ext}(\mathcal{Q})$. In what follows we will make a bit stronger and surprising observation that in the case of channels, X needs to be a unitary channel. We will prove a theorem indicating that unitary channels are somehow special from the perspective of boundariness and minimum-error discrimination. Moreover, the value of boundariness, in this case, is a function of maximally entangled numerical radius [14] of the inverse of the corresponding Choi-Jamiołkowski state.

Lemma 3. Let D be a positive operator on $\mathcal{H}_d \otimes \mathcal{H}_d$ and define

$$\mathcal{R} = \left\{|y\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d : \text{tr}_1|y\rangle\langle y| \leqslant \frac{1}{d}I\right\}. \tag{5}$$

Denote by $|y_D\rangle \in \mathcal{R}$ a vector that maximizes the overlap with D, i.e., $\langle y_D|D|y_D\rangle = \max_{|y\rangle\in\mathcal{R}}\langle y|D|y\rangle$. Then $|y_D\rangle$ is a unit vector, hence it is maximally entangled.

Proof. Let us note that $|y\rangle \in \mathcal{R}$ is normalized to one if and only if $|y\rangle$ is maximally entangled, i.e., $\text{tr}_1|y\rangle\langle y| = \frac{1}{d}I$. Suppose $|y_D\rangle$ has the Schmidt decomposition $|y_D\rangle = \sum_j \sqrt{\mu_j}|e_j\rangle|f_j\rangle$ and assume that for some k we have $\mu_k < 1/d$, thus it is not normalized. Then

$$\begin{aligned}\langle y_D|D|y_D\rangle = {}& \mu_k\langle e_k f_k|D|e_k f_k\rangle + \sum_{j,l\neq k}\sqrt{\mu_j\mu_l}\langle e_j f_j|D|e_l f_l\rangle\\ &+ 2\sqrt{\mu_k}\sum_{j\neq k}\sqrt{\mu_j}\,\text{Re}\langle e_k f_k|D|e_j f_j\rangle.\end{aligned}$$

In what follows we will construct a vector from $\mathcal{R}$ that has a greater overlap with D. First, we introduce a vector $|\tilde{e}_k\rangle$ that differs from $|e_k\rangle$ only by a sign

$$|\tilde{e}_k\rangle = \text{sgn}_+\left(\sum_{j\neq k}\sqrt{\mu_j}\,\text{Re}\langle e_k f_k|D|e_j f_j\rangle\right)|e_k\rangle, \qquad (6)$$

where $\text{sgn}_+(x)$ equals 1 for non-negative x and -1 for negative x. Using this vector we write

$$\begin{aligned}
&\mu_k\langle e_k f_k|D|e_k f_k\rangle + 2\sqrt{\mu_k}\sum_{j=1,j\neq k}^{d}\sqrt{\mu_j}\,\text{Re}\langle e_k f_k|D|e_j f_j\rangle\\
&\leqslant \mu_k\langle e_k f_k|D|e_k f_k\rangle\\
&\quad + 2\sqrt{\mu_k}\left|\sum_{j=1,j\neq k}^{d}\sqrt{\mu_j}\,\text{Re}\langle e_k f_k|D|e_j f_j\rangle\right|\\
&= \mu_k\langle \tilde{e}_k f_k|D|\tilde{e}_k f_k\rangle\\
&\quad + 2\sqrt{\mu_k}\sum_{j=1,j\neq k}^{d}\sqrt{\mu_j}\,\text{Re}\langle \tilde{e}_k f_k|D|e_j f_j\rangle. \qquad (7)
\end{aligned}$$

In the last line above, μ_k is multiplied by strictly positive factor (D is a positive matrix) and $\sqrt{\mu_k}$ is multiplied by a non-negative factor, so we will (strictly) increase the value of the products if we replace μ_k with $\frac{1}{d}$. Finally we obtain

$$\langle y|D|y\rangle < \langle\tilde{y}|D|\tilde{y}\rangle \qquad (8)$$

for $|\tilde{y}\rangle = \sum_{i=1,i\neq k}^{d}\sqrt{\mu_i}|e_i f_i\rangle + \sqrt{\frac{1}{d}}|\tilde{e}_k f_k\rangle$. Since $|\tilde{y}\rangle \in \mathcal{R}$, we obtained a contradiction. ■

Theorem 2. Suppose $\mathcal{F}$ is an interior element of the set of channels $\mathcal{Q}$. Then

$$b(\mathcal{F}) = \left[\max_{\mathcal{U}}\lambda_1\left(J_{\mathcal{F}}^{-1}J_{\mathcal{U}}\right)\right]^{-1} = \frac{d}{\max_U\langle\langle U|J_{\mathcal{F}}^{-1}|U\rangle\rangle}, \qquad (9)$$

where the optimization runs over all unitary channels $\mathcal{U}:\rho\mapsto U\rho U^\dagger$ and $|U\rangle\rangle = (U\otimes I)\sum_j|jj\rangle$. Moreover, if $\mathcal{F} = b(\mathcal{F})\mathcal{E} + [1-b(\mathcal{F})]\mathcal{G}$ for some $\mathcal{E}\in\mathcal{Q}$ and $\mathcal{G}\in\partial\mathcal{Q}$, then $\mathcal{E}$ must be a unitary channel.

Proof. Let us denote by $J_{\mathcal{E}}$ and $J_{\mathcal{F}}$ Choi-Jamiołkowski operators for channels $\mathcal{E}$ and $\mathcal{F}$, respectively. We assume that $\mathcal{F}$ is an interior element, thus $J_{\mathcal{F}}$ is invertible. Then $t_{\mathcal{F}}(\mathcal{E}) = \sup\{0\leqslant t<1, J_{\mathcal{F}} - tJ_{\mathcal{E}}\geqslant 0\}$. It follows that for all $|x\rangle$, $\langle x|J_{\mathcal{F}}|x\rangle \geqslant t\langle x|J_{\mathcal{E}}|x\rangle$. Setting $|y\rangle = \sqrt{J_{\mathcal{F}}}|x\rangle$ we obtain

$$\frac{1}{t}\geqslant\frac{\langle y|\sqrt{J_{\mathcal{F}}}^{-1}J_{\mathcal{E}}\sqrt{J_{\mathcal{F}}}^{-1}|y\rangle}{\langle y|y\rangle}. \qquad (10)$$

The maximum value on the right-hand side equals $\lambda_1(\sqrt{J_{\mathcal{F}}}^{-1}J_{\mathcal{E}}\sqrt{J_{\mathcal{F}}}^{-1}) = \lambda_1(J_{\mathcal{F}}^{-1}J_{\mathcal{E}}) = \lambda_1(\sqrt{J_{\mathcal{E}}}J_{\mathcal{F}}^{-1}\sqrt{J_{\mathcal{E}}})$, where $\lambda_1(X)$ denotes the maximal eigenvalue of X. In conclusion, $t_{\mathcal{F}}(\mathcal{E}) = 1/\lambda_1(J_{\mathcal{F}}^{-1}J_{\mathcal{E}})$ and

$$b(\mathcal{F}) = \inf_{\mathcal{E}} t_{\mathcal{F}}(\mathcal{E}) = \left[\max_{\mathcal{E}}\lambda_1\left(J_{\mathcal{F}}^{-1}J_{\mathcal{E}}\right)\right]^{-1}, \qquad (11)$$

where the optimization runs over all channels.

For any Choi-Jamiołkowski state $J_{\mathcal{E}}$ and an arbitrary unit vector $|x\rangle\in\mathcal{H}_d\otimes\mathcal{H}_d$ we have $\sqrt{J_{\mathcal{E}}}|x\rangle\langle x|\sqrt{J_{\mathcal{E}}}\leqslant J_{\mathcal{E}}$. The complete positivity of the partial trace implies that $\text{tr}_1(J_{\mathcal{E}} - \sqrt{J_{\mathcal{E}}}|x\rangle\langle x|\sqrt{J_{\mathcal{E}}})\geqslant 0$ and since $\text{tr}_1 J_{\mathcal{E}} = \frac{1}{d}I$ it follows

$$\text{tr}_1\sqrt{J_{\mathcal{E}}}|x\rangle\langle x|\sqrt{J_{\mathcal{E}}}\leqslant\frac{1}{d}I.$$

In other words, $\sqrt{J_{\mathcal{E}}}|x\rangle\in\mathcal{R}$, defined in Lemma 3. Consequently, $\lambda_1(J_{\mathcal{F}}^{-1}J_{\mathcal{E}}) = \max_{|x\rangle}\langle x|\sqrt{J_{\mathcal{E}}}J_{\mathcal{F}}^{-1}\sqrt{J_{\mathcal{E}}}|x\rangle\leqslant\max_{|y\rangle\in\mathcal{R}}\langle y|J_{\mathcal{F}}^{-1}|y\rangle$ for every channel $\mathcal{E}$ and using Eq. (11) we obtain

$$b(\mathcal{F}) = \left[\max_{\mathcal{E},|x\rangle}\langle x|\sqrt{J_{\mathcal{E}}}J_{\mathcal{F}}^{-1}\sqrt{J_{\mathcal{E}}}|x\rangle\right]^{-1}\geqslant\left[\max_{|y\rangle\in\mathcal{R}}\langle y|J_{\mathcal{F}}^{-1}|y\rangle\right]^{-1}. \qquad (12)$$

Since $J_{\mathcal{F}}^{-1}$ is a positive operator Lemma 3 implies that the maximum over $|y\rangle$ is achieved only by unit (hence maximally entangled) vectors. For every such vector $|y_{\mathcal{F}}\rangle$ there exists a unitary matrix U such that $|y_{\mathcal{F}}\rangle = \frac{1}{\sqrt{d}}\sum_j U|j\rangle\otimes|j\rangle$. Moreover, the choice of $|x\rangle = |y_{\mathcal{F}}\rangle$ and $\mathcal{E} = \mathcal{U}$, where $J_{\mathcal{U}} = |y_{\mathcal{F}}\rangle\langle y_{\mathcal{F}}|$, proves that the lower bound (12) is tight. Finally, the achievability of the maximum on the right-hand side of Eq. (12) requires by Lemma 3 that the norm of $\sqrt{J_{\mathcal{E}}}|x\rangle$ is one, which in turn implies that $\mathcal{E}$ is a unitary channel. Otherwise $t_{\mathcal{F}}(\mathcal{E}) > b(\mathcal{F})$ [see Eq. (11)] and decompositions of the form $\mathcal{F} = b(\mathcal{F})\mathcal{E} + [1-b(\mathcal{F})]\mathcal{G}$ ($\mathcal{G}\in\partial\mathcal{Q}$) cannot exist. ■

Corollary 1. Suppose $\mathcal{F}$ is an interior element of the set of channels. Then there exists a unitary channel $\mathcal{U}$ such that $||\mathcal{F}-\mathcal{U}||_B = 2[1-b(\mathcal{F})]$. Moreover, if $\mathcal{E}\in\mathcal{Q}$ is not a unitary channel, then $\|\mathcal{F}-\mathcal{E}\|_B < 2[1-(b(\mathcal{F})]$.

Proof. Combining Proposition 2 and Theorem 2, we conclude that the equality $||\mathcal{F}-\mathcal{U}||_B = 2[1-b(\mathcal{F})]$ holds precisely for unitary channels $\mathcal{U}$ such that $\frac{b(\mathcal{F})}{d} = \langle\langle U|J_{\mathcal{F}}^{-1}|U\rangle\rangle^{-1}$. ■

In what follows we will explicitly evaluate the boundariness formula determined in Eq. (9) for the families of qubit and erasure channels (on an arbitrary-dimensional system).

A. Qubit channels

Theorem 3. Suppose $\mathcal{F}$ is an interior element of the set of qubit channels. Then

$$b(\mathcal{F}) = \frac{2}{\lambda_1\left(W^\dagger J_{\mathcal{F}}^{-1}W + (W^\dagger J_{\mathcal{F}}^{-1}W)^T\right)}, \qquad (13)$$

where W is a unitary matrix (called sometimes a magic basis) [15]

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1 & i\\ -1 & i & 0 & 0\\ 1 & i & 0 & 0\\ 0 & 0 & 1 & -i\end{pmatrix}. \qquad (14)$$

Proof. For any quantum channel $\mathcal{F}$ with the Choi-Jamiołkowski state $J_{\mathcal{F}}$, boundariness $b(\mathcal{F})$ is given by [see Eq. (9)]

$$b(\mathcal{F}) = \frac{1}{\max_{\psi\in\mathcal{S}_{\text{ME}}}\langle\psi|J_{\mathcal{F}}^{-1}|\psi\rangle}\equiv\frac{1}{r^{\text{ent}}\left(J_{\mathcal{F}}^{-1}\right)}, \qquad (15)$$

where $\mathcal{S}_{\text{ME}} = \{|\psi\rangle\in\mathcal{H}_d\otimes\mathcal{H}_d, \text{tr}_1|\psi\rangle\langle\psi| = \frac{1}{d}I\}$ and $r^{\text{ent}}(A)$ is a maximally entangled numerical radius for the matrix A. We know from the literature [14] that the maximally entangled numerical range for the 4×4 matrix A is equal to the real numerical range of the matrix $W^\dagger A W$. From the above we note that

$$r^{\text{ent}}\left(J_{\mathcal{F}}^{-1}\right) = \lambda_1\left(\frac{W^\dagger J_{\mathcal{F}}^{-1} W + \left(W^\dagger J_{\mathcal{F}}^{-1} W\right)^T}{2}\right), \tag{16}$$

which together with Eq. (15) finishes the proof. ■

In the case of the qubit channel $\mathcal{F}$ we can specify the unitary channel $\mathcal{U}$, for which $||\mathcal{F} - \mathcal{U}||_B = 2[1 - b(\mathcal{F})]$. It follows from the reasoning above that the unitary matrix U, which defines the channel, can be written as

$$|U\rangle\rangle = \sqrt{2} W |v\rangle. \tag{17}$$

The vector $|v\rangle$ above is the leading eigenvector of the real symmetric matrix $W^\dagger J_{\mathcal{F}}^{-1} W + (W^\dagger J_{\mathcal{F}}^{-1} W)^T$.

B. Erasure channels

Erasure channels transform any input state ρ onto a fixed output state $\mathcal{F}_\sigma(\rho) = \sigma$. For such a channel $\mathcal{F}_\sigma$ the Choi-Jamiołkowski state reads

$$J_{\mathcal{F}_\sigma} = \frac{1}{d}\sigma \otimes I. \tag{18}$$

If the state σ belongs to the boundary of the set of states $\mathcal{S}(\mathcal{H}_d)$ then also the channel $\mathcal{F}_\sigma$ belongs to the boundary of the set of channels and $b(\mathcal{F}_\sigma) = 0$. This holds because the Choi-Jamiołkowski state of $\mathcal{F}_\sigma$ has zero in its spectrum and boundary elements of compact convex sets have zero boundariness [see Appendix C, Eq. (C3) in Ref. [5]]. For the remaining case we provide the following proposition.

Proposition 3. The boundariness of the erasure channel $\mathcal{F}_\sigma$, which maps everything to a fixed interior point σ in the set of states $\mathcal{S}(\mathcal{H}_d)$, is given by

$$b(\mathcal{F}_\sigma) = \frac{1}{\text{tr}[\sigma^{-1}]}. \tag{19}$$

Proof. Since σ is an interior element of the set of states, $J_{\mathcal{F}_\sigma}^{-1} = d\,\sigma^{-1} \otimes I$ is well defined. Using Theorem 2 we obtain

$$b(\mathcal{F}_\sigma) = \frac{1}{\max_U \sum_{j,k} \langle jj|(U^\dagger \sigma^{-1} U) \otimes I|kk\rangle} = \frac{1}{\text{tr}[\sigma^{-1}]},$$

where we used $UU^\dagger = I$ and the cyclic invariance of the trace. ■

Let us note that in the special case of a qubit erasure channel $\mathcal{F}_\sigma$ with $\sigma = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ we find $b(\mathcal{F}_\sigma) = p(1-p)$ in accordance with the results of [5].

V. BOUNDARINESS UNDER COMPOSITION

The composition of quantum systems is intimately related to the tensor product. The so-called factorized elements (being tensor products of elements associated with individual subsystems) have a clear physical interpretation of the independent preparation of states of each subsystem, uncorrelated single-partite measurements, and uncorrelated channels acting on each subsystem independently. Mathematically, they form a special nonconvex subset of all elements included in the larger convex set of bipartite states, observables, and channels. Therefore, there is no *a priori* reason to expect any deeper relation between the bipartite boundariness of these factorized elements and individual values of boundariness for subsystems. Knowing that the boundariness is related to the minimum-error discrimination we may ask what the best discriminable elements from the factorized elements are; investigation of the behavior of the boundariness under the tensor product will help us answer this question.

Suppose $\mathcal{E}$ and $\mathcal{F}$ are channels on systems described in Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_d$, respectively. Denote by $b(\mathcal{E})$ and $b(\mathcal{F})$ the values of their boundariness. We address the question of the relation between the boundariness of channel composition $b(\mathcal{E} \otimes \mathcal{F})$ and the boundariness for individual channels.

Proposition 4. For channels the boundariness is submultiplicative, i.e., $b(\mathcal{E} \otimes \mathcal{F}) \leqslant b(\mathcal{E}) b(\mathcal{F})$.

Proof. Let us consider some decomposition of channels $\mathcal{E}$ and $\mathcal{F}$ into boundary elements with the weight equal to their boundariness:

$$J_{\mathcal{E}} = b(\mathcal{E}) J_{\mathcal{E}+} + [1 - b(\mathcal{E})] J_{\mathcal{E}-},$$
$$J_{\mathcal{F}} = b(\mathcal{F}) J_{\mathcal{F}+} + [1 - b(\mathcal{F})] J_{\mathcal{F}-}.$$

This allows us to write

$$J_{\mathcal{E}} \otimes J_{\mathcal{F}} = b(\mathcal{E})\, b(\mathcal{F})\, J_{\mathcal{E}+} \otimes J_{\mathcal{F}+} + [1 - b(\mathcal{E})\, b(\mathcal{F})\,]\, J_{\mathcal{T}}, \tag{20}$$

where

$$\begin{aligned} J_{\mathcal{T}} = {} & [1 - b(\mathcal{E}) b(\mathcal{F})]^{-1} \{ b(\mathcal{E})[1 - b(\mathcal{F})] J_{\mathcal{E}+} \otimes J_{\mathcal{F}-} \\ & + [1 - b(\mathcal{E})] b(\mathcal{F}) J_{\mathcal{E}-} \otimes J_{\mathcal{F}+} \\ & + [1 - b(\mathcal{E})][1 - b(\mathcal{F})] J_{\mathcal{E}-} \otimes J_{\mathcal{F}-} \} \end{aligned} \tag{21}$$

is a Choi-Jamiołkowski state of a channel. Let us recall that a channel is on the boundary of the set of channels if and only if its Choi-Jamiołkowski state has a nonempty kernel (see, e.g., [5]). It is easy to realize that if $\mathcal{E}_+$ and $\mathcal{F}_+$ are boundary elements of the respective sets of channels, $\mathcal{E}_+ \otimes \mathcal{F}_+$ lies on the boundary as well. Similarly, taking vectors $|\varphi\rangle$ and $|\psi\rangle$ from the kernel of $J_{\mathcal{E}\_}$ and $J_{\mathcal{F}\_}$, respectively, we can immediately see that $|\varphi\rangle \otimes |\psi\rangle$ belongs to the kernel of $J_{\mathcal{T}}$. This shows that Eq. (20) provides a valid convex decomposition of a channel $\mathcal{E} \otimes \mathcal{F}$ into two boundary elements and we conclude that $t_{\mathcal{E}\otimes\mathcal{F}}(\mathcal{E}_+ \otimes \mathcal{F}_+) = b(\mathcal{E}) b(\mathcal{F})$. Due to definition of boundariness from Eq. (3) we obtain the upper bound from the proposition. ■

Proposition 5. For states and observables the boundariness is multiplicative, i.e., $b(x \otimes y) = b(x) b(y)$, where x, y stands for any pair of states or observables.

Proof. The equality in Proposition 5 is fulfilled because for states and observables the boundariness is given by the smallest eigenvalue and eigenvalues of the tensor products are products of the eigenvalues. ■

We have numerical evidence suggesting that the equality holds also in the case of channels, but we have no proof of such a conjecture. Using Eq. (9), this is equivalent to the equality of $\max_\xi \langle \xi | J_{\mathcal{E}}^{-1} \otimes J_{\mathcal{F}}^{-1} | \xi \rangle$ and $\max_\chi \langle \chi | J_{\mathcal{E}}^{-1} | \chi \rangle \max_\omega \langle \omega | J_{\mathcal{F}}^{-1} | \omega \rangle$, where ξ, χ, and ω are maximally entangled states on the corresponding systems.

Below we prove this equality for the case of qubit channels when one of the channels is the "maximally mixed" channel $\mathcal{F}$, hence, for this pair of channels, the boundariness is multiplicative.

Proposition 6. Let $\mathcal{E}$ be an arbitrary qubit channel and let $\mathcal{F}$ be the erasure channel mapping any input to $\frac{1}{d}I$. Then $b(\mathcal{E}\otimes\mathcal{F}) = b(\mathcal{E})b(\mathcal{F})$.

Proof. By proposition 4, $b(\mathcal{E}\otimes\mathcal{F}) \leqslant b(\mathcal{E})b(\mathcal{F})$, so we have to show the opposite inequality. Let $\mathcal{E}:\mathcal{B}(\mathcal{H}_A)\to\mathcal{B}(\mathcal{H}_B)$ and $\mathcal{F}:\mathcal{B}(\mathcal{H}_{A'})\to\mathcal{B}(\mathcal{H}_{B'})$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ denote copies of $\mathcal{H}_2$, and $\mathcal{H}_{A'}$ and $\mathcal{H}_{B'}$ denote copies of $\mathcal{H}_d$. Since $J_{\mathcal{F}}^{-1} = d^2 I_{B'A'}$, then by Theorem 2 we want to prove the inequality

$$\max_{V\in\mathcal{U}(\mathcal{H}_{BB'})}\langle\langle V|J_{\mathcal{E}}^{-1}\otimes I_{B'A'}|V\rangle\rangle \leqslant d \max_{U\in\mathcal{U}(\mathcal{H})}\langle\langle U|J_{\mathcal{E}}^{-1}|U\rangle\rangle.$$

For $V\in\mathcal{U}(\mathcal{H}_{BB'})$, let $X_V = \mathrm{tr}_{B'A'}|V\rangle\rangle\langle\langle V|$. Then X_V is a positive operator on $\mathcal{H}_{BA}$ and we have

$$\mathrm{tr}_B X_V = \mathrm{tr}_{A'}\mathrm{tr}_{BB'}|V\rangle\rangle\langle\langle V| = dI_A.$$

Similarly, $\mathrm{tr}_A X_V = dI_B$. It follows that $\frac{1}{2d}X_V$ is the Choi-Jamiołkowski matrix of a unital qubit channel. As it is well known, any such channel is a random unitary channel, so there are some unitaries $U_i\in\mathcal{U}(\mathcal{H}_2)$ and probabilities p_i such that $X_V = d\sum_i p_i|U_i\rangle\rangle\langle\langle U_i|$. It follows that

$$\langle\langle V|J_{\mathcal{E}}^{-1}\otimes I_{B'A'}|V\rangle\rangle = \mathrm{tr}\big[J_{\mathcal{E}}^{-1}X_V\big] \leqslant d\max_{U\in\mathcal{U}(\mathcal{H})}\langle\langle U|J_{\mathcal{E}}^{-1}|U\rangle\rangle.$$

■

VI. MAXIMAL VALUE OF BOUNDARINESS

By definition, boundariness takes values between zero and $\frac{1}{2}$, but all values in this interval are not necessarily attainable. A simple example is the triangle [see Fig. 1(a) and the Appendix], for which $\frac{1}{3}$ is the maximal value. In this section we will investigate what is the highest achievable value of boundariness for particular quantum convex sets. In fact, we will see that the elements maximizing the boundariness are unique and coincide with the so-called maximally mixed elements playing the role of white noise.

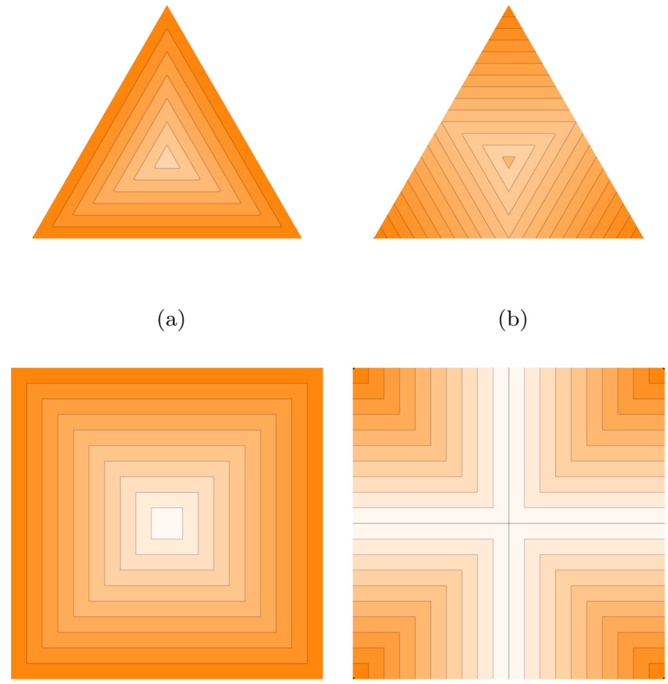
As for the other questions addressed in this paper, it is straightforward to evaluate the maximal value for states and measurements, but the case of channels is more involved.

Proposition 7. The maximal value of boundariness for quantum convex sets is given as follows: For states, $b^s_{\max} = 1/d$ is achieved for a completely mixed state $\varrho = \frac{1}{d}I$; for observables, $b^o_{\max} = 1/n$ is achieved for an n-outcome (uniformly) trivial observable $\{E_j = \frac{1}{n}I\}_{j=1}^n$; and for channels, $b^c_{\max} = 1/d^2$ is achieved for a completely depolarizing channel mapping all states into a completely mixed state $\frac{1}{d}I$.

Proof. For states and measurements [5] the highest boundariness means the highest value of the lowest eigenvalue, which leads to a maximally mixed state $\rho = \frac{1}{d}I$ and a (uniform) trivial observable $\{E_i = \frac{1}{N}I\}_{i=1}^N$, respectively. The case of channels is more subtle. From the formula (9) giving the boundariness of a channel it is clear that we search for a channel $\mathcal{F}$ such that $\max_U\langle\langle U|J_{\mathcal{F}}^{-1}|U\rangle\rangle$ is minimized. We construct a simple lower bound using an orthonormal basis $\{|v_i\rangle\}_{i=1}^{d^2}$ of maximally entangled states

$$\mathrm{tr}\big[J_{\mathcal{F}}^{-1}\big] = \sum_{i=1}^{d^2}\langle v_i|J_{\mathcal{F}}^{-1}|v_i\rangle \leqslant d\max_U\langle\langle U|J_{\mathcal{F}}^{-1}|U\rangle\rangle. \quad (22)$$

Such a basis $\{|v_{pq}\rangle = (Z^pW^q\otimes I)\frac{1}{\sqrt{d}}\sum_j|jj\rangle\}$ can be constructed using shift and multiply unitary operators $Z=\sum_j|j\oplus 1\rangle\langle j|$ and $W=\sum_j\omega^j|j\rangle\langle j|$, where $\omega = e^{2\pi i/d}$. On the other hand, from spectral decomposition $J_{\mathcal{F}} = \sum_i\lambda_i|a_i\rangle\langle a_i|$, where $\sum_i\lambda_i = 1$, we have $\mathrm{tr}[J_{\mathcal{F}}^{-1}] = \sum_i\frac{1}{\lambda_i}\geqslant d^4$. Combining this with Eq. (22), we get $d^3\leqslant \max_U\langle\langle U|J_{\mathcal{F}}^{-1}|U\rangle\rangle$. Inserting this into Eq. (9), we finally obtain $b(\mathcal{F})\leqslant\frac{1}{d^2}$. It is easy to see that the inequalities can be made tight only by a single channel, which maps everything to a complete mixture. ■



(a) (b)

(c) (d)

FIG. 1. (Color online) Illustration of (a) and (c) boundariness and (b) and (d) mixedness for simple convex sets. In general, mixedness and boundariness differ; exceptions are sets like an ellipse or sphere.

VII. SUMMARY

This paper completes and extends previous work [5] in which the concept of boundariness was introduced. We proved that for compact convex sets evaluation of boundariness of y coincides with the question of the best distinguishable element from y, i.e.,

$$2[1-b(y)] = \max_x ||x-y||,$$

where $||\cdot||$ denotes the so-called base norm (being the trace norm for states and the completely bounded norm, also known as the diamond norm, for channels and observables). This identity was formulated in Ref. [5] as an open conjecture for the case of quantum channels and is confirmed by our results presented in this paper. In fact, we have discovered that the optimum is attained only for unitary channels. This surprising result provides quite unexpected operational characterization of unitary channels and exhibits their specific role among boundary elements and in minimum-error discrimination questions. The unique role of unitary channels is noticeable also in the explicit formula that we derived for the evaluation of boundariness of channels. We note that these results hold also for channels, for which their inputs are less dimensional than their outputs. It can be checked that in such a case the role of unitary channels is played by isometries, which is in accordance with the intuition. However, if the dimension of the input is larger than the dimension of the output system, the generalization is not that straightforward and is left for future research.

Further, we investigated how the boundariness behaves under the tensor product. We have shown that boundariness is a multiplicative quantity for states and observables, however, for channels we proved only the submultiplicativity

$$b(\mathcal{E} \otimes \mathcal{F}) \leqslant b(\mathcal{E})b(\mathcal{F}).$$

However, our numerical analysis suggests that the boundariness is multiplicative also for the case of channels.

Exploiting the relation between the boundariness and the discrimination, the multiplicativity implies that the most distinguishable element from $x \otimes y$ is still a factorized element $x_0 \otimes y_0$, where x_0 and y_0 stand for the most distinguishable elements from x and y, respectively. For channels this would mean that factorized unitaries are the most distant ones for all factorized channels. However, whether this is the case is left open.

In the remaining part of the paper we evaluated explicitly the maximal value of boundariness. We found that this maximum is achieved intuitively for the maximally mixed elements, i.e., for a completely mixed state, uniformly trivial observables, and channel contracting state space to the completely mixed state. In particular, for d-dimensional quantum systems we found for states $b^s_{\max} = 1/d$, for observables $b^o_{\max} = 1/n$, which is independent of the dimension (only the number of outcomes n matters), and for channels $b^c_{\max} = 1/d^2$. Let us stress that these numbers also determine the optimal values of error probability for related discrimination problems.

ACKNOWLEDGMENTS

We thank Errka Haapasalo for discussions and the workshop ceqip.eu for motivating this work. This work was supported by Project No. VEGA 2/0125/13 (QUICOST). Z.P. acknowledges support from the Polish National Science Centre through Grant No. DEC-2011/03/D/ST6/00413. A.J. acknowledges support from Research and Development Support Agency under Contract No. APVV-0178-11 and Project No. VEGA 2/0059/12. M.S. acknowledges support from the Operational Program Education for Competitiveness, European Social Fund (Project No. CZ.1.07/2.3.00/30.0004) of the Ministry of Education, Youth and Sports of the Czech Republic. M.Z. acknowledges support from GAČR Project No. P202/12/1142 and COST Action No. MP1006.

APPENDIX: BOUNDARINESS OF A SQUARE AND A TRIANGLE

If we consider a centroid of a square, then any line going through this point gives us a two-term decomposition of the central point into two boundary points with both weights being $\frac{1}{2}$. Thus, the most unbalanced decomposition into two boundary points has the smaller weight equal to $\frac{1}{2}$, which is the highest achievable value of boundariness by definition.

In contrast, for the centroid of a triangle the most unbalanced two-term decompositions into boundary points are formed by lines passing through the edges of the triangle. However, the lower weight is in this case only $\frac{1}{3}$, because the ratio between the distance from the centroid to the side and to the edge is $\kappa = \frac{1}{2}$ and this can be converted to a mixing coefficient by $t = \frac{\kappa}{\kappa+1} = \frac{1}{3}$.

[1] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[2] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).
[3] E. Haapasalo, J. Phys. A: Math. Theor. **48**, 255303 (2015).
[4] R. T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, 1970).
[5] E. Haapasalo, M. Sedlák, and M. Ziman, Phys. Rev. A **89**, 062303 (2014).
[6] D. Reeb, M. J. Kastroyano, and M. M. Wolf, J. Math. Phys. **52**, 082201 (2011).
[7] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
[8] A. Jenčová, J. Math. Phys. **55**, 022201 (2014).
[9] A. Kitaev, Russ. Math. Surv. **52**, 1191 (1997).
[10] T. Heinosaari and M. Ziman, *The Language of Quantum Theory* (Cambridge University Press, Cambridge, 2013).
[11] P. Busch, M. Grabowski, and P. J. Lahti, *Operational Quantum Physics*, Lecture Notes in Physics Monographs (Springer, Berlin, 1995).
[12] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).
[13] M.-D. Choi, Linear Algebra Appl. **10**, 285 (1975).
[14] C. F. Dunkl, P. Gawron, Ł. Pawela, Z. Puchała, and K. Życzkowski, Linear Algebra Appl. **479**, 12 (2015).
[15] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).