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# PERTURBATIVE GADGETS WITHOUT STRONG INTERACTIONS

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Perturbative gadgets are used to construct a quantum Hamiltonian whose low-energy subspace approximates a given quantum $k$-local Hamiltonian up to an absolute error $\epsilon$. Typically, gadget constructions involve terms with large interaction strengths of order $\text{poly}(\epsilon^{-1})$. Here we present a 2-body gadget construction and prove that it approximates a Hamiltonian of interaction strength $\gamma = O(1)$ up to absolute error $\epsilon \ll \gamma$ using interactions of strength $O(\epsilon)$ instead of the usual inverse polynomial in $\epsilon$. A key component in our proof is a new condition for the convergence of the perturbation series, allowing our gadget construction to be applied in parallel on multiple many-body terms.

We also discuss how to apply this gadget construction for approximating 3- and $k$-local Hamiltonians. The price we pay for using much weaker interactions is a large overhead in the number of ancillary qubits, and the number of interaction terms per particle, both of which scale as $O(\text{poly}(\epsilon^{-1}))$. Our strong-from-weak gadgets have their primary application in complexity theory (QMA hardness of restricted Hamiltonians, a generalized area law counterexample, gap amplification), but could also motivate practical implementations with several weak interactions simulating a much stronger quantum many-body interaction.

*Keywords*: local Hamiltonian, perturbation theory, quantum complexity

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## 1 Introduction

The physical properties of (quantum mechanical) spin systems can often be understood in terms of effective interactions arising from the complex interplay of microscopic interactions. Powerful methods for analyzing effective interactions have been developed, for example the renormalization group approach distills effective interactions at different length scales. Another common approach is perturbation theory – treating some interaction terms in the Hamiltonian as a perturbation to a simple original system, giving us a sense of how the fully interacting system behaves. Here, instead of trying to understand an unknown system, we ask an engineering question: how can we build a particular (many-body) effective interaction from local terms of restricted form?

The idea of *perturbative gadgets* provides a powerful answer. Initially introduced by Kempe, Kitaev and Regev [1] for showing the QMA-hardness of 2-Local Hamiltonian problem and subsequently used and developed further in numerous works [2, 3, 4, 5, 6, 7, 8], the perturbative gadgets are convenient tools by which arbitrary many-body effective interactions (which we call the *target*

*Hamiltonian*) can be obtained using a *gadget Hamiltonian* consisting of only two-body interactions. In a broader context, these gadgets have also been used to understand the computational complexity of physical systems (e.g. how hard it is to determine the ground state energy) with restricted geometry of interactions [2], locality [1, 2, 6], or interaction types [4]. Here, we choose to focus on the issue of restricted coupling strengths.

In a nutshell, perturbative gadgets allow us to map between different forms of microscopic Hamiltonians. This is an analogue of how gadgets are used in classical complexity theory, for example in reductions among NP-complete constraint satisfaction problems (e.g. 3-SAT and graph 3-coloring). In the context of combinatorial reductions in classical computation complexity theory, a *gadget* is a finite structure which maps a set of constraints from one optimization problem into a constraint of another problem. Using such gadgets, an instance of 3-SAT (an NP-complete problem) can be efficiently mapped to an instance of graph 3-coloring (also NP-complete [9]). On the other hand, more complex constructions allow us to create more frustrated instances of such problems without significant overhead, resulting in inapproximability as well as the existence of probabilistically checkable proofs [10].

For classical CSP instances, gadgets can be used to reduce the arity of clauses, to reduce the size of the alphabet, or to reduce the degree of each variable on the constraint graph. Analogously, quantum gadgets [1, 2, 5, 11] have been devised for reducing the locality of interactions (analogous to arity reduction in classical CSPs), the dimension of particles (alphabet reduction) and the degree of interaction. These reductions for quantum Hamiltonians give us tools that could help us explore the way to the quantum PCP conjecture [12]. More modestly, gadget translations between types of local Hamiltonians would have implications for the area law [13, 14, 15] and other global properties. However, generating approximate *quantum* interactions from a restricted set of terms is not straightforward.

For classical spin systems, creating effective interactions with arbitrary strength by coupling a system to several ancilla degrees of freedom is a relatively simple task. For example, we can create an effective (and twice stronger) ferromagnetic interaction between target spins $a, b$ using two ancilla spins $x, y$ and connecting them to $a, b$ as illustrated in Figure 1. The lowest energy states of this new system correspond to the lowest energy states of a system with a ferromagnetic interaction between $a$ and $b$, with doubled strength.

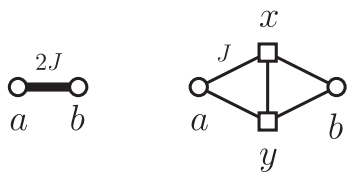


Fig. 1. A ferromagnetic interaction $E(a, b) = -2Jab$ of two classical spins $a, b \in \{-1, 1\}$ can be "built" from half-strength interactions involving two extra ancillas. The ground states of the system on the right have $a = b$, while the lowest excited states have $a \neq b$ and energy $4J$ above the ground state energy. Each edge between two classical spins $u$ and $v$ in this illustration represents a term $uv$ in the expression for energy. The ○ nodes symbolize target spins and □ nodes are ancillas.

For general quantum interactions where the target Hamiltonian consists of many-body Pauli operators, the common *perturbative gadget* introduces a strongly bound ancillary system and couples the target spins to it via weaker interactions, treating the latter as a perturbation. The target many-body Hamiltonian is then generated in some low order of perturbation theory of the combined system

of ancillary and target spins. Such gadgets first appeared in the proof of QMA-completeness of the 2-local Hamiltonian problem via a reduction from 3-local Hamiltonian [1]. There they helped build effective 3-local interactions from 2-body interactions. Perturbative gadgets can also be used for reducing a target Hamiltonian with general geometry of interactions to a planar interaction graph [2], approximating certain restricted forms of 2-body interactions using other forms of 2-body interactions [4, 16], realizing Hamiltonians exhibiting non-abelian anyonic excitations [17] and reducing $k$-local interactions to 2-local [6, 5].

For perturbation theory to apply,[a] all existing constructions of perturbative gadgets [1, 2, 6, 5, 4, 8] require interaction terms or local fields with norm much higher than the strength of the effective interaction which they generate (see Figure 2b). However, physically realizable systems often allow only limited spin-spin coupling strengths. The main result of our paper is a way around this problem.

We first build a system with a large spectral gap between the ground state and the first excited state using many relatively weak interactions: consider a collection of $n$ spins that interact with each other (i.e. $O\left(n^2\right)$ interaction terms) via $ZZ$ interactions of constant strength $J$. Then the first excited state of this $n$-spin system has energy $O(n)$ higher than the ground energy, since the ground state subspace is spanned by $\{|0\rangle^{\otimes n}, |1\rangle^{\otimes n}\}$ and flipping a spin raises the energy by $O(n)$. This way we can use weaker interactions to construct a *core* with a large spectral gap. We then use it to replace the large local field applied onto the single ancilla (Figure 2b) with weak interactions of a collection of ancillas (Figure 2c). Finally, we connect the target spins to multiple ancillas instead of just one, which allows us to use weaker $\beta$ to achieve the same effective interaction strength between the target spins (Figure 2d).

Let us review a few definitions and then state our results precisely. An $n$-qubit Hamiltonian is an $2^n \times 2^n$ Hermitian matrix; it is *$k$-local* or *$k$-body* (for a constant $k$) if it can be written as a sum of $M \leq \text{poly}(n)$ terms $H_j$, each acting non-trivially on a distinct set of at most $k$ qubits. Furthermore, we require[b] $\|H_j\| \leq \text{poly}(n)$, and that the entries of $H_j$ be specified by $\text{poly}(n)$ bits. The smallest eigenvalue of $H$ is its *ground state energy*, and we denote it $\lambda(H)$. We use $\lambda_j(H)$ to represent the $j$-th smallest eigenvalue of $H$, hence $\lambda(H) = \lambda_1(H)$. Taking a 2-local Hamiltonian acting on $n$ qubits, we can associate it with an *interaction graph* $G(V, E)$. Every vertex $v \in V$ corresponds to a qubit, and there is an edge $e \in E$ between vertices $a$ and $b$ if and only if there is a non-zero 2-local term $H_e$ on qubits $a$ and $b$ such that $H_e$ is neither 1-local nor proportional to the identity operator. More generally, we can pair a $k$-local Hamiltonian with its *interaction hypergraph* in which the $k$-local terms correspond to hyper-edges involving (at most) $k$ vertices. Note that we depict all 2-local terms on the same spins as a single edge. Next, because we can decompose any 2-local Hamiltonian term in the Pauli basis,[c] we can define a *Pauli edge* of an interaction graph $G$ as an edge between vertices $a$ and $b$ associated with an operator $\gamma_{ab}\, P_a \otimes Q_b$ where $P, Q \in \{\mathbb{I}, X, Y, Z\}$ are Pauli matrices and $\gamma_{ab}$ is a real number signifying the coupling coefficient. We refer to the maximum value of $|\gamma_{ab}|$ as the *interaction strength* of the Hamiltonian. For an interaction graph in which every edge is a *Pauli edge*, the degree of a vertex is called its *Pauli degree*. The maximum Pauli degree of a vertex in an interaction graph is the Pauli degree of the graph.

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[a] Note that there exist special cases (e.g. Hamiltonians with all terms diagonal in the same basis) when one can analyze the Hamiltonian with non-perturbative techniques [18, 19].

[b] We use the operator norm $\|\cdot\|$, defined as $\|M\| \equiv \max_{|\psi\rangle \in \mathcal{M}} |\langle\psi|M|\psi\rangle|$ for an operator $M$ acting on a Hilbert space $\mathcal{M}$).

[c] For example, the spin chain Hamiltonian $H = \frac{1}{2}\sum_{i=1}^{n} |01-10\rangle\langle 01-10|_{i,i+1}$ has interaction edges between successive spins. Each 2-local interaction can be rewritten in the Pauli basis as $\frac{1}{4}\,(\mathbb{I}\otimes\mathbb{I} - X\otimes X - Y\otimes Y - Z\otimes Z)$. It gives us an overall energy shift (from the first term), and three Pauli edges with weight $-\frac{1}{4}$.

### *1.1 Summary*

We start in Section 2 with a theoretical framework of perturbation theory that is used throughout our discussion, and then in Section 3 we present and prove our main result – a gadget construction that simulates a target 2-local Hamiltonian using arbitrarily weak 2-local couplings and ancilla particles, summarized in the following Theorem:

**Theorem 1 (Effective 2-body interactions from weak couplings)** *Consider the Hamiltonian $H_{targ} = H_{else} + \sum_{j=1}^{M} \gamma_j A_{a_j} \otimes B_{b_j}$ on $n$ qubits, with $a_j, b_j \in [n]$ labeling the qubits that the operators $A, B$ in the $j^{th}$ term act on. $H_{targ}$ consists of*

1. *a Hamiltonian $H_{else}$ with a non-negative spectrum, obeying $\|H_{else}\| \leq poly(n)$, which corresponds to terms in the Hamiltonian that we will* not *decompose into gadgets, and*
2. *$M$ distinct 2-local interaction terms, acting on an $n$ qubit system, with an interaction graph of Pauli degree $p$, assuming $M \leq poly(n)$, and bounded interaction strength $\gamma_{max} = \max_j |\gamma_j| = O(1)$.*

*Then for any $\epsilon > 0$ and $\epsilon \ll \gamma$, there exists a Hamiltonian $\tilde{H}$ which is a sum of $H_{else}$ and a 2-local (gadgetized) Hamiltonian with interaction strengths $O(\epsilon)$, whose low-lying spectrum approximates the full spectrum of $H_{targ}$ as $|\lambda_j(\tilde{H}) - \lambda_j(H_{targ})| \leq \epsilon$ for all $j$ from 1 to $2^n$. The new Hamiltonian $\tilde{H}$ acts on $n + poly(\|H_{else}\|, \epsilon^{-1}, M)$ qubits and has an interaction graph of Pauli degree $poly(p, \|H_{else}\|, \epsilon^{-1}, M)$.*

Note that if we want to "gadgetize" the entire target Hamiltonian, $H_{\text{else}}$ is simply zero. In the remainder of the paper, for $\tilde{H}$ and $H_{\text{targ}}$ in Theorem 1, when we refer to $\tilde{H}$ approximating $H_{\text{targ}}$ *up to error* $\epsilon$, we mean the following. The low-lying eigenstates of $\tilde{H}$ are $\epsilon$-close to $|\phi_j\rangle \otimes |0 \cdots 0\rangle_{anc}$ where $|0 \cdots 0\rangle_{\text{anc}}$ is the state of the ancilla qubits of $\tilde{H}$ (the norm of the difference between the vectors is no greater than $\epsilon$), and the low-lying spectrum of $\tilde{H}$ is $\epsilon$-close to $\{\lambda_j\}$.

At first glance Theorem 1 seems naively true: we could always consider a given target term $\gamma A \otimes B$ as a sum of $m$ identical but smaller terms $\frac{\gamma}{m} A \otimes B$ and treat each small term with a separate gadget. Presumably, these gadgets are of weaker interaction strengths than a single gadget applied onto the target term directly. However, if we intend to simulate $\gamma A \otimes B$ up to error $\epsilon$, we need to simulate each of the small terms up to error $\epsilon/m$, which would translate into interaction strength in the gadget Hamiltonian scaling as $\text{poly}(\epsilon^{-1})$ regardless. Hence this idea does not improve the interaction strength asymptotically. Our contribution here is to show that we could improve the interaction strength from $\text{poly}(\epsilon^{-1})$ to $\text{poly}(\epsilon)$.

Our main Theorem 1 deals with 2-local target Hamiltonians, built from 2-local gadgets. What about gadget constructions for reducing 3-local interactions [1, 2, 6] or $k$-local interactions [5, 6] to 2-local ones? Here we generalize Theorem 1 to propose gadget constructions for 3- and $k$-body target Hamiltonians. In particular:

**Corollary 1 (3-body terms from weak 2-body interactions)** *Let us consider a Hamiltonian $H_{targ} = H_{else} + \sum_{i=1}^{M} \gamma_i A_{a_i} \otimes B_{b_i} \otimes F_{f_i}$, with Here $a_j, b_j, f_j \in [n]$ labeling the qubits that the operators $A, B, F$ in the $j^{th}$ term act on. Here $H_{targ}$ consists of*

1. *$H_{else}$ (the part we will not decompose into gadgets), a Hamiltonian with a non-negative spectrum, satisfying $\|H_{else}\| \leq poly(n)$, and*

2. *a sum of $M$ interaction terms that are 3-local, acting on an $n$ qubit system, with an interaction graph of Pauli degree $p$ and ground state energy $\lambda(H_{targ})$, assuming $M \leq poly(n)$. The interaction strength of $H_{targ}$ satisfies $\gamma_{max} = \max_j |\gamma_j| = O(1)$.*

*Then for any choice of $\epsilon > 0$, there exists a Hamiltonian $\tilde{H}$ that consists of $H_{else}$ (the part we leave intact) and a sum of $M$ terms that are 2-local, with interaction strength $O(\epsilon)$, acting on a system with $n + poly(\|H_{else}\|, \epsilon^{-1}, M)$ qubits, with an interaction graph of Pauli degree $poly(p, \|H_{else}\|, \epsilon^{-1}, M)$ and $|\lambda_j(\tilde{H}) - \lambda_j(H_{targ})| \leq \epsilon$ for all $j$.*

We outline the proof of Corollary 1 in Sec. 4. An important property of these new constructions is that they can be repeated in parallel, in essence generating arbitrary strong interactions from weak ones. Thus, we can effectively rescale interaction strengths and amplify the eigenvalue gap of a local Hamiltonian. The price we pay is the addition of many ancillas and a large increase in the number of interactions per particle.

Whereas Theorem 1 states that a 2-local target Hamiltonian can be gadgetized to a Hamiltonian with arbitrarily weak interactions, Corollary 1 states that the same could be accomplished for a 3-local target Hamiltonian. (In Section 4 we also generalize it to $k$-local Hamiltonians.)

Next, besides producing a gadget Hamiltonian with weak interactions that generates the target Hamiltonian, we could also generate the target Hamiltonian multiplied by a positive factor $\theta$. In case where $\theta > 1$, this can be viewed as a coupling strength amplification relative to the original target $k$-local Hamiltonian (see Corollary 2 below). The basic proof idea is to view the rescaled target Hamiltonian $\theta H$ (with $\theta > 1$) as a sum of $O(\theta)$ copies of itself with interaction strength $O(1)$. Using the gadget constructions from [5], we transform the $k$-local Hamiltonian $\theta H$ to a 2-local one. Finally, using our 2-body gadget construction in this work, we translate this Hamiltonian to one with only weak interactions (2-body).

**Corollary 2 (Coupling strength amplification by gadgets)** *Let $H = \sum_{j=1}^{M} H_j$ be a $k$-local Hamiltonian on $n$ qubits where $M = poly(n)$ and each $H_j$ satisfies $\|H_j\| \leq s$ for some constant $s$. Let $|\phi_j\rangle$ and $\lambda_j$ be the $j$-th eigenstate and eigenvalue of $H$. Choose a magnifying factor $\theta > 1$ and an error tolerance $\epsilon > 0$. Then there exists a 2-local Hamiltonian $\tilde{H}$ with interactions of strength $O(1)$ or weaker. The low-lying eigenstates of $\tilde{H}$ are $\epsilon$-close[d] to $|\phi_j\rangle \otimes |0 \cdots 0\rangle_{anc}$ where $|0 \cdots 0\rangle_{anc}$ is the state of the ancilla qubits of $\tilde{H}$, and the low-lying spectrum of $\tilde{H}$ is $\epsilon$-close to $\theta\{\lambda_j\}$.*

What is the efficiency of this way of amplifying the couplings? If we do it in a series of reductions from $k$ to $\lceil k/2 \rceil$ to $\lceil \lceil k/2 \rceil /2 \rceil$, etc., to 2-body interactions, the final gadget Hamiltonian will act on a system whose total number of qubits scales exponentially in $k$ (which of course is not a problem for $k = 3$).

### *1.2 Conclusions and Open Questions*

A gadget construction based on perturbation theory allows us to map between Hamiltonians of different types, with the same low-lying spectral properties. First, we replace strong interactions by *repetition* of interactions with "classical" ancillas; it works because for a low-energy state, all our extra qubits are close to the state $|0\rangle$. This is reminiscent of repetition encoding found e.g. in [20]. Second, we employ *parallelization*; it is crucial to show that the perturbation series converges even with many gadgets, relaxing the usual assumption about the norm of the perturbation.

[d] By $\epsilon$-*close* we mean the norm of the difference between the two quantities (scalar, vector or matrix operator) is $\leq \epsilon$.

This construction should find use in computer science as well as physics. First, in complexity theory, Theorem 1 together with [1] or Corollary 1 with [21] implies QMA-completeness of the 2-local Hamiltonian problem with non-repeated terms with norm at most $O(1)$ and an $O(1)$ promise gap. As a consequence, we also obtain efficient universality for quantum computation with time-independent, 2-local Hamiltonians with restricted form/strength of terms, complementing [22, 2, 23]. Second, our amplification method from Corollary 2 has been utilized in a counterexample to the generalized area law in [24]. Finally, we envision practical experimental applications of Theorem 1 – strengthening effective interactions between target (atomic) spins through many (but even for a few $R$) coupled mediator spins. In our case, these interactions need to be precisely tuned, while elsewhere we have seen disordered networks used to enhance transport between two sites in a quantum system [25].

Thinking further about interaction strengths and spectra of local Hamiltonians, we realize that Corollary 2 allows us to amplify the *eigenvalue gap* (low eigenvalue spacing) of a Hamiltonian. Does it have direct implications for hardness of Local Hamiltonian problems? When we use it on Hamiltonians appearing in QMA-complete constructions, the fractional *promise gap* (the ratio of the number of frustrated terms to the number of all terms in the Hamiltonian for a ground state of a local Hamiltonian) gets smaller. Thus, it does not directly help us move towards the quantum PCP conjecture [12]. Nevertheless, we have added another tool for mapping between Hamiltonians to our repertoire.

An important problem remains open. The price we pay for our construction is a massive blowup in the degree (the number of interactions per particle). Is there a possibility of a quantum degree-reduction gadget? One might try to use a "bad" quantum code for encoding each spin into several particles, whose encoded low-weight operators that can be implemented in *many* possible ways; this does not seem possible for both $X$ and $Z$ operators. As things stand, without a degree-reduction gadget, we do not have a way to reproduce our results in simpler geometry. It would be really interesting if one indeed could create $O(1)$-norm effective interactions from $O(1)$-terms in 3D or even 2D lattices.

We also need to think about the robustness of our results – what will change when the Hamiltonians are not exactly what we asked for? How precise do we need to be (e.g. for the 3-body to 2-body gadgets), so that the second- and first-order terms get canceled? Also, Bravyi, Terhal, DiVincenzo and Loss [6] mentioned that a $k$-body to 2-body reduction might possibly be implemented with poly$(k)$ overhead in interaction strength (instead of exponential in $k$). However, this question remains open. The exponential scaling in the overhead in [6] is due to the usual gadget constructions which require poly$\left(\epsilon^{-1}\right)$ interaction strength. We hope (but haven't proven) that with our new gadget construction, this result could be improved.

## 2 Effective interactions based on perturbation theory

The purpose of a perturbative gadget is to approximate a target $n$-qubit Hamiltonian $H_{\text{targ}}$ by a gadget Hamiltonian $\tilde{H}$ which uses a restricted form of interactions among the $n$ qubits that $H_{\text{targ}}$ acts on and poly$(n)$ additional ancilla qubits. The subspace spanned by the lowest $2^n$ eigenstates of $\tilde{H}$ should approximate the spectrum of $H_{\text{targ}}$ up to a prescribed error tolerance $\epsilon$ in the sense that the $j$-th lowest eigenvalue of $\tilde{H}$ differs from that of $H_{\text{targ}}$ by at most $\epsilon$ and the inner product between the corresponding eigenstates of $\tilde{H}$ and $H_{\text{targ}}$ (assume no degeneracy) is at least $1-\epsilon$. These error bounds can be established using perturbation theory [1, 2]. There are various versions of perturbation theory available for constructing and analyzing gadgets (for a review see [26]). For example, Jordan and Farhi [5] use Bloch's formalism, while Bravyi et al. rely on the Schrieffer-Wolff transformation [6]. For the gadgets

in Sec. 3, we use the technique from [1, 2].

Let us now review the basic ideas underlying the construction of effective Hamiltonians from gadgets. The gadget Hamiltonian $\tilde{H} = H + V$ is a sum of an unperturbed Hamiltonian $H$ and a perturbation $V$. $H$ acts only on the ancilla space, energetically penalizing certain configurations, and favoring a specific ancilla state or subspace. Second, we have a perturbation $V$ describing how the target spins interact with the ancillas.

Let us introduce the following notations: let $\lambda_j$ and $|\psi_j\rangle$ be the $j^{\text{th}}$ eigenvalue and eigenvector of $H$ and similarly define $\tilde{\lambda}_j$ and $|\tilde{\psi}_j\rangle$ for $\tilde{H}$, assuming all the eigenvalues are labeled in a weakly increasing order ($\lambda_1 \leq \lambda_2 \leq \cdots$, similarly for $\tilde{\lambda}_j$). Using a cutoff value $\lambda_*$, let us call $\mathcal{L}_- = \text{span}\{|\psi_j\rangle : \lambda_j \leq \lambda_*\}$ the *low-energy subspace* and $\mathcal{L}_+ = \text{span}\{|\psi_j\rangle : \lambda_j > \lambda_*\}$ the *high-energy subspace*. Let $\Pi_-$ and $\Pi_+$ be the orthogonal projectors onto the subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$. For an operator $O$ we define the partitioning of $O$ into these subspaces as $O_- = \Pi_- O \Pi_-$, $O_+ = \Pi_+ O \Pi_+$, $O_{-+} = \Pi_- O \Pi_+$ and $O_{+-} = \Pi_+ O \Pi_-$. We define similar notations $\tilde{\mathcal{L}}_-$ and $\tilde{\mathcal{L}}_+$ for $\tilde{H}$.

Our first goal is to understand $\tilde{H}|_{\tilde{\mathcal{L}}_-}$, the restriction of the gadget Hamiltonian to its low-energy subspace. Let us consider the operator-valued *resolvent* $\tilde{G}(z) = (z\mathbb{I} - \tilde{H})^{-1}$ where $\mathbb{I}$ is the identity operator. Similarly let us define $G(z) = (z\mathbb{I} - H)^{-1}$. Note that $\tilde{G}^{-1}(z) - G^{-1}(z) = -V$, which allows an expansion of $\tilde{G}$ in powers of $V$:

$$\tilde{G} = (G^{-1} - V)^{-1} = G(\mathbb{I} - VG)^{-1} = G + GVG + GVGVG + \cdots . \tag{1}$$

It is also standard to define the *self-energy* $\Sigma_-(z) = z\mathbb{I} - (\tilde{G}_-(z))^{-1}$. It is important because the spectrum of $\Sigma_-(z)$ gives an approximation to the spectrum of $\tilde{H}_-$, since by definition $\tilde{H}_- = z\mathbb{I} - \Pi_-(\tilde{G}^{-1}(z))\Pi_-$ while $\Sigma_-(z) = z\mathbb{I} - (\Pi_- \tilde{G}(z) \Pi_-)^{-1}$. As explained in [2], if $\Sigma_-(z)$ is roughly constant in some range of $z$ (see Theorem 2 below for details) then $\Sigma_-(z)$ is (loosely speaking) playing the role of $\tilde{H}_-$. This was formalized in Theorem 3 in [1] (and improved in Theorem A.1 in [2]). Similarly to [2], we choose to work with $H$ whose lowest eigenvalue is zero and whose spectral gap is $\Delta$. In [1], the gadget theorem (Theorem 3) is proven by establishing a sequence of Lemmas. Out of these, Lemma 5 requires the condition $\|V\| < \frac{\Delta}{2}$, with the consequence being the separation of subspaces, namely $\tilde{\mathcal{L}}_- \cap \mathcal{L}_+ = \{0\}$. Therefore, we here remove the condition $\|V\| < \frac{\Delta}{2}$ and use $\tilde{\mathcal{L}}_- \cap \mathcal{L}_+ = \{0\}$ as an alternative assumption, giving us a slightly modified Gadget approximation theorem:

**Theorem 2 (Gadget approximation theorem, modified from [1])** *Let $H$ be a Hamiltonian with a gap $\Delta$ between its ground state and first excited state. Assuming the ground state energy of $H$ is 0, let $\lambda_* = \Delta/2$. Consider a bounded norm perturbation $V$. The perturbed Hamiltonian is then $\tilde{H} = H + V$. Following the notations introduced previously, if the following holds:*

1. *$\tilde{\mathcal{L}}_- \cap \mathcal{L}_+ = \{0\}$, with $\mathcal{L}_+ = \text{span}\{|\psi_j\rangle : \lambda_j \leq \lambda_*\}$ for $|\psi_j\rangle$ eigenvectors of $H$ and $\tilde{\mathcal{L}}_- = \text{span}\{|\tilde{\psi}_j\rangle : \tilde{\lambda}_j \leq \lambda_*\}$ for $|\tilde{\psi}_j\rangle$ eigenvectors of $\tilde{H}$.*

2. *There is an effective Hamiltonian $H_{\text{eff}}$ with a spectrum contained in $[E_1, E_2]$ for some $\epsilon > 0$ and $E_1 < E_2 < \Delta/2 - \epsilon$, such that for every $z \in [E_1 - \epsilon, E_2 + \epsilon]$, the self-energy $\Sigma_-(z)$ obeys $\|\Sigma_-(z) - H_{\text{eff}}\| \leq \epsilon$.*

*then all the eigenvalues of $\tilde{H}_-$ are close to the eigenvalues of $H$, obeying*

$$|\lambda_j(H_{\text{eff}}) - \lambda_j(\tilde{H}_-)| \leq \epsilon.$$

The first, subspace condition, says that a combination of the unperturbed high-energy eigenstates of $H$ can not by themselves form a low-energy state of $\tilde{H}$. We choose to avoid the original stronger condition $\|V\| \leq \frac{\Delta}{2}$ from [1], since it imposes limitations on the *global* properties of the gadget construction, in particular the number of ancillas we use, disregarding the structure of the perturbation. One might question whether the use of perturbation theory is sensible if we assume that the perturbation $\|V\|$ is no longer necessarily small compared to the spectral gap $\Delta$ (we want to use a large number $M$ of gadgets). However, such use has been justified previously by Bravyi et al. [6] in a similar context.

To apply Theorem 2, a series expansion for the self-energy $\Sigma_-(z) = z\mathbb{I} - \tilde{G}_-^{-1}(z)$ is truncated at some low order, for which $H_{\text{eff}}$ is approximated. Using the series expansion of $\tilde{G}$ in (1), the self-energy can be expanded as (see [1] for details)

$$\Sigma_-(z) = H_- + V_- + V_{-+}G_+(z)V_{+-} + V_{-+}G_+(z)V_+G_+(z)V_{+-} + \cdots, \tag{2}$$

with $G_+(z) = \Pi_+(z\mathbb{I} - H)^{-1}\Pi_+$. The 2[nd] and higher order terms in this expansion give rise to effective many-body interactions. Introducing auxiliary spins and a suitable selection of 2-local $H$ and $V$, we can engineer $\Sigma_-(z)$ to be $\epsilon$-close to $H_{\text{eff}} = H_{\text{targ}} \otimes \Pi_-$ (here $\Pi_-$ is the projector to the ground state subspace of the ancillas) in the range of $z$ considered in Theorem 2. Therefore with $\|\Sigma_-(z) - H_{\text{eff}}\| \leq \epsilon$, condition 2 of Theorem 2 is satisfied.

In the next Section, we will look at the usual 2-body gadgets and see how the second order terms in the self-energy result in the desired effective Hamiltonian. Then we present our construction that involves more ancillas with weaker interactions, and show that the effective Hamiltonian is again what we want, and that the conditions for Theorem 2 are met.

## 3 A new gadget for 2-body interactions

We can decompose any 2-local interaction of spin-$\frac{1}{2}$ particles in the Pauli basis,[e] using terms of the form $\gamma A \otimes B$, with the operator $A$ acting on spin $a$ and $B$ acting on spin $b$, and $\gamma$ the *interaction strength*. Without loss of generality, we can also use $\pm$ Pauli matrices, and fix the coupling strengths to be positive. It will be enough to show how to replace any such "Pauli" interaction in our system by a gadget, aiming at the target interaction $H_{\text{targ}} = H_{\text{else}} + \gamma A \otimes B$, with $H_{\text{else}}$ some $O(1)$-norm, 2-local Hamiltonian. First, we briefly review the existing constructions [2, 6, 8] for generating $H_{\text{targ}}$ using a gadget Hamiltonian $\tilde{H}$. Then we present a new 2-body gadget which simulates an arbitrary $\gamma = O(1)$ strength 2-local interaction using a gadget Hamiltonian with terms of strength only $o(1)$, "building" quantum interactions from many weaker ones.

*The usual construction.* Consider a target 2-local term involving two qubits $a, b$ as depicted in Fig. 2(a). The standard construction of a gadget Hamiltonian $\tilde{H}$ that captures the 2-local target term is shown in Fig. 2(b). First, we introduce an ancilla qubit $w$ bound by a local field, with the Hamiltonian $H = -\frac{\Delta}{2}Z_w$. Alternatively, up to a spectral shift we could write $H = \Delta|1\rangle\langle 1|_w$ where $|1\rangle\langle 1|_w = \frac{1}{2}(\mathbb{I} - Z_w)$. Then we let $w$ interact with $a$ and $b$ through $\sqrt{\Delta/2}\, A \otimes \mathbb{I} \otimes X_w$ and $-\sqrt{\Delta/2}\, \mathbb{I} \otimes B \otimes X_w$, and choose $\Delta = \Theta(\epsilon^{-1})$. We can view these terms as a perturbation to $H$. The low energy effective Hamiltonian calculated from (2) is approximately $A \otimes B \otimes |0\rangle\langle 0|_w$ (up to an overall energy shift) [8]. Here "up to an error $\epsilon$" means that the $j$-th lowest eigenvalue of $\tilde{H}$ differs

[e] It is useful that the Pauli matrices $A, B \in \{\mathbb{I}, X, Y, Z\}$ square to identity, because $A^2$ and $B^2$ terms in our effective Hamiltonian will become simple overall energy shifts

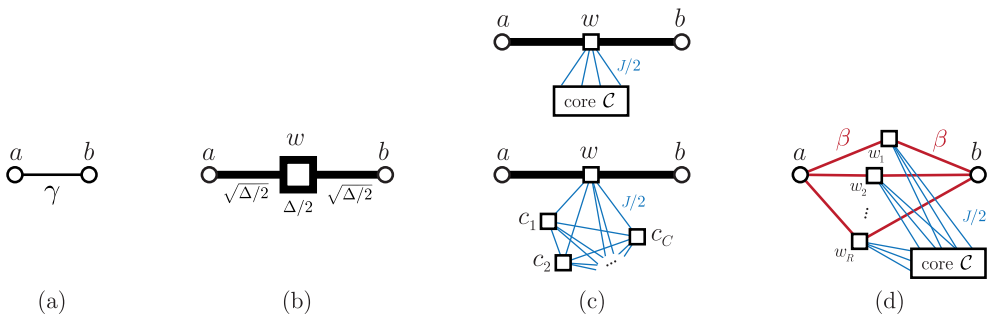


Fig. 2. Interaction graphs for effective two-body interaction mediated by ancilla qubits. Each node represents a particle. The size of the node indicates the strength of local field applied onto it. The width of each edge shows the strength of the interaction between the particles that the edge connects. (a) The desired 2-local interaction between target spins $a, b$. (b) The usual perturbative gadget uses a single ancilla $w$ in a strong local field, and large-norm interactions with the target spins. (c) We can replace the strong local field $\Delta/2$ by ferromagnetic interactions with a fixed *core* – a group of $C$ "core" ancilla qubits located in a field of strength $J/2$, interacting with each other ferromagnetically (as a complete graph), with strength $J/2$. (d) Instead of the strong interactions between target spins $a, b$ and a single ancilla $w$, we can use $R$ different "direct" ancillas (labeled as $w_1, w_2, \cdots, w_R$) and weaker interactions of strength $\beta$.

from that of $H_{\text{targ}}$ by at most $\epsilon$ and the inner product between the corresponding eigenstates of $\tilde{H}$ and $H_{\text{targ}}$ (assume no degeneracy) is at least $1 - \epsilon$.

*Our construction.* In the usual construction, with better precision (decreasing $\epsilon$), the spectral gap $\Delta$ (related to local field strength) and interaction strengths grow as *inverse* polynomials in $\epsilon$. We now suggest a 2-body gadget which simulates an arbitrary $O(1)$ strength target interaction using a gadget Hamiltonian of only $O(\epsilon)$ interaction strength *i.e.* without the need for large-norm terms. We build it in a sequence of steps illustrated in Fig. 2.

The first step is to reduce the large local field $\Delta$ in Fig. 2(b). Let us call the ancilla $w$ directly interacting with the target spins a *direct* ancilla. We add a *core* $\mathcal{C}$ – a set of $C$ ancilla qubits $c_1, \ldots, c_C$, with a complete graph of ferromagnetic (ZZ) interactions of strength $\frac{J}{2}$, and in a local field of strength $\frac{J}{2}$ where $J = O(\epsilon)$. We then let the direct ancilla $w$ interact (ferromagnetically) with each of the core ancillas, as in Fig. 2(c). The Hamiltonian for the direct and core ancillas then reads

$$\frac{J}{2}\sum_{c\in\mathcal{C}}(\mathbb{I} - Z_w Z_c) + \underbrace{\frac{J}{2}\sum_{c\in\mathcal{C}}(\mathbb{I} - Z_c) + \frac{J}{2}\sum_{c,c'\in\mathcal{C}}(\mathbb{I} - Z_c Z_{c'})}_{\equiv H_{\mathcal{C}}}. \tag{3}$$

$H_{\mathcal{C}}$ is the Hamiltonian describing the core $\mathcal{C}$. The ground state of this Hamiltonian is $|0\rangle_w|0\cdots 0\rangle_{\mathcal{C}}$, and the gap between its ground and first excited state $|1\rangle_w|0\cdots 0\rangle_{\mathcal{C}}$ is $\Delta = JC$. Here $C$ is the number of ancillas in the core $\mathcal{C}$.

The second step is to use $R$ direct ancillas $w_1, \ldots, w_R$ instead of just one, connecting each of them to the core ancillas as in Figure 2(d). The Hamiltonian then becomes

$$H = \frac{J}{2}\sum_{i=1}^{R}\sum_{c\in\mathcal{C}}(\mathbb{I} - Z_{w_i} Z_c) + H_{\mathcal{C}}. \tag{4}$$

Its ground state is $|0\cdots 0\rangle_w \otimes |0\cdots 0\rangle_{\mathcal{C}}$ (here we use the subscript $w$ to refer to all the direct ancillas connected to the target qubits), and the gap between the two lowest energies is still $\Delta = JC$.

We want to engineer an effective interaction $H_{\text{targ}} = \gamma A_a \otimes B_b + H_{\text{else}}$, where the first term is our desired Hamiltonian, and $H_{\text{else}}$ is a finite-norm Hamiltonian that includes all the other terms that we want to leave unchanged by this gadget. Starting with the Hamiltonian $H$ (4), we add a perturbation

$$V = H_{\text{else}} + \beta \sum_{i=1}^{R} \left( A_a \otimes X_{w_i} - B_b \otimes X_{w_i} \right), \tag{5}$$

where $\beta > 0$ is the strength of the interactions between the target spins and the direct ancillas. Showing that we can use perturbation theory to obtain the effective Hamiltonian crucially relies on Theorem 2, and we will justify that its conditions hold later. Let us now prepare the notations and tools for this. Let $\mathcal{L}_-$ be the subspace with the ancillas in the state $|0\rangle^{\otimes(R+C)}$. Denote $\mathcal{L}_+$ the subspace orthogonal to $\mathcal{L}_-$ and let $\Pi_-$ and $\Pi_+$ be the projectors onto these subspaces. We then have

$$V_- = \Pi_- V \Pi_- = H_{\text{else}} \otimes \Pi_-, \tag{6}$$

$$V_{-+} = \Pi_- V \Pi_+ = \beta(A_a - B_b) \otimes \sum_{i=1}^{R} |0\rangle\langle 1|_{w_i}, \tag{7}$$

$$V_{+-} = \Pi_+ V \Pi_- = \beta(A_a - B_b) \otimes \sum_{i=1}^{R} |1\rangle\langle 0|_{w_i}, \tag{8}$$

$$V_+ = \Pi_+ V \Pi_+ = H_{\text{else}} \otimes \Pi_+ + \beta \sum_{i=1}^{R} (A_a - B_b) \otimes \Pi_+ X_{w_i} \Pi_+. \tag{9}$$

The low-energy sector of the gadget Hamiltonian $\tilde{H} = H + V$ can be described by the self-energy expansion (2). Let us compute the terms up to the second order.

- At the 0th order, $H_- = 0$ by definition.
- At the 1st order, $V_-$ is given by (6).
- At the 2nd order, we have the term $V_{-+} G_+ V_{+-}$, where $V_{-+}$ and $V_{+-}$ can be computed from (7) and (8). We also need the operator-valued resolvent

  $$G_+(z) = \sum_{x : h(x) > 0} \frac{1}{z - h(x)\Delta} |x\rangle\langle x|.$$

  A second order transition process from the low energy subspace back to itself can only take the form $|0\rangle^{\otimes R} \to |x\rangle \to |0\rangle^{\otimes R}$, with $x$ an $R$-bit string of Hamming weight 1 (there are $R$ qubits that can be flipped there and back). Hence, the only non-trivial terms in the product $V_{-+} G_+ V_{+-}$ have the form $\beta(A_a - B_b) \cdot \frac{1}{z-\Delta} \cdot \beta(A_a - B_b)$. Altogether, we have $R$ of these terms, so the second order term becomes

  $$V_{-+} G_+ V_{+-} = \frac{1}{z - \Delta} R \beta^2 (A_a - B_b)^2.$$

We could obtain the higher order terms in a similar fashion. In the end, the self-energy expansion becomes

$$\Sigma_-(z) = \underbrace{H_{\text{else}}}_{1^{\text{st}}\text{ order}} + \underbrace{\frac{1}{z-\Delta} R\beta^2 (A_a - B_b)^2}_{2^{\text{nd}}\text{ order}} + \underbrace{\sum_{m=1}^{\infty} V_{-+} G_+ (V_+ G_+)^m V_{+-}}_{\text{error term}}. \tag{10}$$

Recall that $G(z) = (z\mathbb{I} - H)^{-1}$. The range of $z$ we consider is $|z| \leq \|H_{\text{else}}\| + |\gamma|$. We can assume $\gamma > 0$ in $H_{\text{targ}}$ (e.g. by absorbing a possible minus sign into the $A$ matrix), and choose

$$\beta = \sqrt{\frac{\gamma\Delta}{2R}} = \sqrt{\frac{\gamma JC}{2R}}. \tag{11}$$

Since $z \ll \Delta$, we can write $\frac{1}{z-\Delta} = -\frac{1}{\Delta}\left(1 - \frac{z}{\Delta}\right)^{-1} \approx -\frac{1}{\Delta} + O\left(\frac{1}{\Delta^2}\right)$. Then the 1st and 2nd order terms are approximately equal to the desired effective Hamiltonian $H_{\text{eff}} = H_{\text{targ}} \otimes \Pi_-$ up to an overall spectral shift (because $A^2 = B^2 = \mathbb{I}$). We will show later, in Claim 2, that with good choices of $R$ and $C$ we can make $\beta$ and $J$ as small as we want.

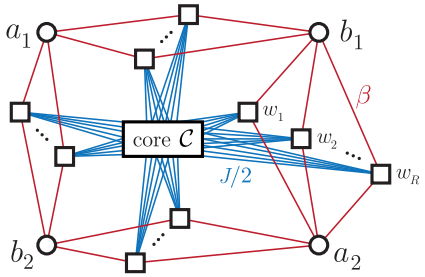


Fig. 3. Parallel composition of $M$ (here $M = 4$) two-body gadgets from Fig. 2(d), using a single common core with $C$ "core" ancillas. Each gadget has $R$ "direct" ancillas interacting with the target spins. The total number of ancillas is thus $MR + C$.

*Parallel 2-body gadgets.* So far, we have focused on a single 2-local term in our target Hamiltonian (see Fig. 2). Similarly to [2], we can apply our gadgets *in parallel*, which enables us to deal with a target Hamiltonian with $M$ such 2-local terms. Let us then consider a target Hamiltonian of the form

$$H_{\text{targ}} = H_{\text{else}} + \sum_{j=1}^{M} \gamma_j A_{a_j} \otimes B_{b_j} \tag{12}$$

and apply our construction to every term $\gamma_j A_{a_j} \otimes B_{b_j}$ in parallel, as in Fig. 3. Note that we save a lot of resources by using only a single core. Each target term $\gamma_j A_{a_j} \otimes B_{b_j}$ is associated with $R$ direct ancilla qubits $w_1^{(i)}, w_2^{(i)}, \cdots, w_R^{(i)}$ that are connected to target spins $a_i$ and $b_i$. All of the direct ancillas also interact with each of the $C$ core ancillas. As before, the core consists of $C$ qubits that are fully connected with ferromagnetic (ZZ) interactions of strength $\frac{J}{2}$ and also with local fields of strength $\frac{J}{2}$ on each qubit. Hence the full gadget Hamiltonian for the general 2-local target Hamiltonian in (12)

takes the form $\tilde{H} = H + V$ with

$$H = \frac{J}{2}\sum_{j=1}^{M}\sum_{i=1}^{R}\sum_{c\in\mathcal{C}}(\mathbb{I} - Z_{w_i^{(j)}}Z_c) + H_{\mathcal{C}}, \tag{13}$$

$$V = H_{\text{else}} + \sum_{j=1}^{M}\beta_j\sum_{i=1}^{R}\left(A_{a_j} - B_{b_j}\right)\otimes X_{w_i^{(j)}}.$$

where $H_{\mathcal{C}}$ is the core Hamiltonian from (4), $\beta_j = \sqrt{\frac{\gamma_j JC}{2R}}$ and the spectral gap between the ground state and the first excited state of $H$ is $\Delta = JC$. Computing the self-energy expansion as in (10) for the gadget Hamiltonian in (13) yields a contribution $-\frac{1}{z-\Delta}\sum_{j=1}^{M}\beta_j^2 R(A_{a_j} - B_{b_j})^2$ at the second order (see Claim 2 for more details). Because each term in the perturbative expansion $\Sigma_-(z)$ corresponds to a sequence of state transitions from $\mathcal{L}_-$ to $\mathcal{L}_+$ and back,[f] the second order contribution comes from those transitions where one ancilla is flipped from $|0\rangle$ to $|1\rangle$ and back to $|0\rangle$. Such transitions cannot involve more than one ancilla qubit. Hence we can regard the second order transitions involving different ancillas as occurring independently of each other (in parallel). This enables the 2-body gadgets to capture multiple 2-local target terms, and is much more effective than a "serial" approach: constructing a gadget for a single 2-body interaction, calling what we get $H_{\text{else}}$, then building another gadget for another 2-body interaction, and so on.

In order to show that the low-lying subspace of our gadget Hamiltonian $\tilde{H}$ captures the spectrum of $H_{\text{targ}}$ using Theorem 2, it is necessary to establish that $\tilde{H}$ meets both conditions of the theorem. The first condition, $\tilde{\mathcal{L}}_- \cap \mathcal{L}_+ = \{0\}$, requires the vectors the unperturbed high-energy states not to become perturbed low energy states by themselves. We will prove this as Claim 1 below. The second condition says that the self-energy expansion $\Sigma_-(z)$ can be approximated by an effective Hamiltonian when $z$ is in a certain range. We establish this as Claim 2 for $\tilde{H}$ by proving that the perturbation series converges for $\Sigma_-(z)$. Theorem 1 then follows from Theorem 2 with $\tilde{H}$ being the Hamiltonian in (13).

### *3.1 The 2-local construction satisfies the subspace condition.*

The first condition in Theorem 2 is a property of the high-energy subspace of the original Hamiltonian. We need it in order to avoid the need to bound the norm of the whole perturbation. Let us provide a high-level description of the condition and the ideas behind its proof.

Consider the gadget Hamiltonian $\tilde{H} = H + V$ defined in (13). We need to lower bound the lowest energy $E_+ = \min_\psi \langle\psi|\tilde{H}|\psi\rangle$ of a state $|\psi\rangle$ that comes from the subspace $\mathcal{L}_+$, the excited subspace of $H$, spanned by states orthogonal to the state $|0\cdots 0\rangle_w$. The terms in $H$ involve only ancilla qubits, while $V$ includes $H_{\text{else}}$, and terms that couple some computational (target) qubit $a$ and a direct ancilla $w$. These 2-local terms have form $\beta_w A_a \otimes X_w$, with interaction strengths $|\beta_w| \le \beta_{\max} = O(1)$, as in Figure 4(a). We now want to show that $E_+$ is strictly above $\lambda_* = \frac{\Delta}{2}$. To do this, we find a sequence of successively lower lower bound $E_+$ using a sequence of progressively simpler Hamiltonians, finally arriving at 1-local ones in (24), (28) and (30).

First, we will show that $E_+$ for the general Hamiltonian $\tilde{H}$ is greater or equal to the value of $E_+$ for a similar system in which all of the operators $A_a$ are the same (so that they do not compete against each other in lowering the energy) as in Figure 4(a). Second, we can only lower $E_+$ by making all

[f] Note that in fact $\mathcal{L}_- = \text{span}\{|0\cdots 0\rangle_w |0\cdots 0\rangle_{\mathcal{C}}\}$ where the subscript $w$ refers to all the ancillas $w_1^{(j)}, w_2^{(j)}, \cdots, w_R^{(j)}$, for $j = 1, 2, \cdots, M$. The transitions that contribute to the perturbative expansion $\Sigma_-(z)$ are restricted to only to the direct ancillas $|0\cdots 0\rangle_w$ since the core ancillas do not interact with the target qubits.

of the operators $A_a$ identities, and using only operators $-|\beta_w|\,X_w$ on the direct ancillas. Because the target spins are now independent from the ancillas, the contribution from $H_{\text{else}}$ is then no larger in magnitude than $\|H_{\text{else}}\|$. This is depicted in Figure 4(b).

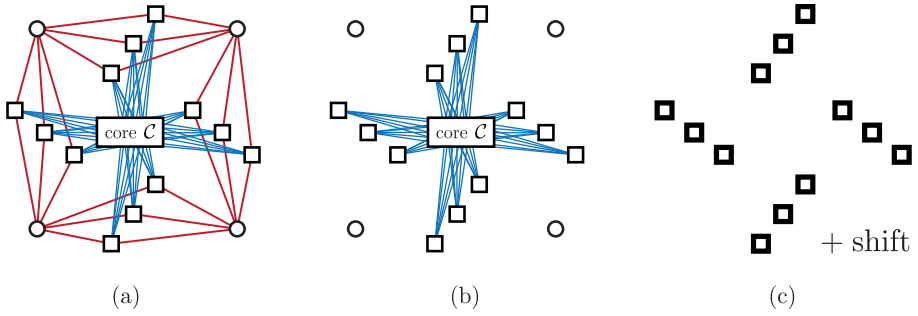


Fig. 4. A sequence of gadget Hamiltonians with progressively lower lower bounds on $E_+$. (a) Taking the terms acting on the target spins to be all the same. (b) Decoupling the target spins from the direct ancillas using $-\mathbb{I}$ operators on the target spins and (weighted) $X$-fields on the direct ancillas. (c) Replacing the interactions with core ancillas by an overall shift, and a (weighted) $Z$-field on the direct ancillas, arriving at (30).

We are now left with a Hamiltonian which is a sum of $H_{\text{else}}$, single-qubit terms on the direct ancilla qubits, and their interactions[g] with the core ancillas. The Hilbert space divides into a direct sum of invariant subspaces labeled by the state of the core ancillas. These subspaces are decoupled (the original Hamiltonian $H$ and the perturbation $V$ do not flip the core ancillas), so we can analyze them one by one. We do so for the subspaces with $a \geq 1$ core ancillas flipped to $|1\rangle$, and then finally for the subspace with all core ancillas equal to $|0\rangle$. It turns out that in each such subspace we can map the terms $Z_w Z_c$, $Z_c Z_{c'}$, $Z_c$ and $H_{\text{else}}$ of the Hamiltonian[h] to one that is simply an overall shift, and a $-\frac{\Delta_a}{2} Z_w$ term on each of the direct ancillas, with $\Delta_a$ a function of how many ancillas were flipped. The resulting 1-local Hamiltonian illustrated in Figure 4(c) can be analyzed, and yields the desired lower bound on $E_+$. Let us then state and prove our first Claim.

**Claim 1** *Consider the 2-body gadget Hamiltonian $\tilde{H} = H + V$ from* (13)*, corresponding to a target Hamiltonian $H_{targ} = H_{else} + \sum_{i=1}^{M} \gamma_i A_i \otimes B_i$ with $\gamma_j \leq O(1)$ and $H_{else}$ positive semi-definite. Let $\Delta$ be the spectral gap between the ground and the first excited subspace of $H$, and define a cutoff $\lambda_* = \Delta/2$. Following Section 2, we define $\mathcal{L}_+ = span\{|\psi_j\rangle : \lambda_j > \lambda_*\}$ for $|\psi_j\rangle$ eigenvectors of $H$, and $\tilde{\mathcal{L}}_- = span\{|\tilde{\psi}_j\rangle : \tilde{\lambda}_j < \lambda_*\}$, for $|\tilde{\psi}_j\rangle$ eigenvectors of $\tilde{H}$. Then if $\Delta \geq 160 M \gamma_{max}$, with $\gamma_{max} = \max_{j=1,\cdots,M} |\gamma_j|$, we have*

$$\tilde{\mathcal{L}}_- \cap \mathcal{L}_+ = \{0\}.$$

We start the proof by exhibiting a sequence of Hamiltonians with progressively lower $E_+$, and then showing Claim 1 for the last of them.

Let $|\psi\rangle \in \mathcal{L}_+$ be the state with minimum energy for the perturbed Hamiltonian $\tilde{H}$, and let us label this minimum energy $E_+ = \langle\psi|\tilde{H}|\psi\rangle$. The Hamiltonian $\tilde{H}$ connects target spins to direct ancillas via terms of the type $A_a \otimes X_j$. We now argue that $E_+$ can be only lowered if we decouple the target spins from the direct ancillas, and simply use $-\mathbb{I} \otimes X_j$ instead of $A_a \otimes X_j$.

[g] If the values of $\beta$ are different for different terms, we still use a single core with a fixed $J$, fixed $C$, fixed $\Delta = JC$, and adjust each $\beta_w$ for each target interaction individually so that the resulting effective interaction strength $\beta_w^2 R/\Delta = O(1)$ is what we desire.

[h] There is no $Z_w$ term on the direct ancillas, so that a single direct ancilla flip increases the energy by $\Delta = JC$.

The expectation value $\langle\psi|\tilde{H}|\psi\rangle = E_H + E_V$ comes from the expectation value of $H$ which is diagonal in the computational basis (the $Z$ and $ZZ$ terms involving the ancillas) and the expectation of $V$, which includes the interactions with target spins as well as the term $H_{\text{else}}$. Let us rewrite the state $|\psi\rangle$ as

$$|\psi\rangle = \sum_w c_w |w\rangle \otimes |\phi_w\rangle, \tag{14}$$

where $w$ is a binary string labeling computational basis state of all the ancillas. The expectation value of the term $H$ depends only on the magnitudes of the $c_w$'s. The contribution from $H_{\text{else}}$ is

$$\sum_w |c_w|^2 \langle\phi_w|H_{\text{else}}|\phi_w\rangle. \tag{15}$$

Finally, each term in $V$ of the form $A_a \otimes X_j$ contributes

$$c_v^* c_{v'} \langle v|X_j|v'\rangle\langle\phi_v|A_a|\phi_{v'}\rangle \tag{16}$$

for every pair $v, v'$ that differ only at bit $j$. This expression can be positive or negative, depending on $c_v$ and $c_{v'}$. More crucially, its magnitude will depend on $\langle\phi_v|A_a|\phi_{v'}\rangle$. Because $A_a$ is a Pauli operator, this magnitude can never exceed 1. Let us now consider a state

$$|\psi'\rangle = \left(\sum_w |c_w|\,|w\rangle\right) \otimes |\phi\rangle, \tag{17}$$

with positive coefficients $|c_w|$, and a particular state $|\phi\rangle$ chosen to minimize $\langle\phi|H_{\text{else}}|\phi\rangle$. The expectation value of $H$ does not change, while the contribution from $H_{\text{else}}$ can only decrease, because we have chosen $|\phi\rangle$ to minimize it. In other words, $\langle\psi'|H|\psi'\rangle \le \langle\psi|H|\psi\rangle$ and $\langle\psi'|H_{\text{else}}|\psi'\rangle \le \langle\psi|H_{\text{else}}|\psi\rangle$. Finally, the expectation value of the interaction terms in $V'$ (when we set $A_a = -\mathbb{I}$) like (16) now become

$$-|c_v| \cdot |c_{v'}| \langle v|X_u|v'\rangle \le -c_v^* c_{v'} \langle v|X_u|v'\rangle\langle\phi_v|A_a|\phi_{v'}\rangle. \tag{18}$$

Thus, $\langle\psi'|V'|\psi'\rangle \le \langle\psi|V|\psi\rangle$ and we conclude that the new minimum energy of $\tilde{H}'$ restricted to $\mathcal{L}_+$ is $E'_+ \le \langle\psi'|H|\psi'\rangle + \langle\psi'|V'|\psi'\rangle \le E_+$. It means that when we replace the Hamiltonian $V$ with one that has no interactions between the direct ancillas and the target spins, and uses operators $-X_w$ on the direct ancillas, $E_+$ decreases (or remains what it was).

Therefore, we can assume without loss of generality that the Hamiltonian $\tilde{H}$ has this special form. We will continue the proof by showing that if $|\psi\rangle \in \mathcal{L}_+$ then $\langle\psi|\tilde{H}|\psi\rangle > \lambda_*$ for any normalized state $|\psi\rangle$.

The subspace $\mathcal{L}_+$ is spanned by (direct + core) ancilla qubit states with at least one $|1\rangle$. Let $\mathcal{K}_- = |0\cdots0\rangle_w$ be the all-zero state the direct ancillas, and let $\mathcal{S}_a = \text{span}\{|x\rangle_{\mathcal{C}} : h(x) = a\}$ be the subspace of the core ancillas with exactly $a$ qubits[i] in the state $|1\rangle$. Thus, the subspace $\mathcal{L}_+$ splits into two parts as $\mathcal{L}_+ = \mathcal{L}_1 \oplus \mathcal{L}_2$, where[j]

$$\mathcal{L}_1 = \mathcal{H}_w \otimes \left(\bigoplus_{a=1}^{C} \mathcal{S}_a\right), \qquad \mathcal{L}_2 = \mathcal{K}_-^{\perp} \otimes \mathcal{S}_0. \tag{19}$$

[i] Here $h(x)$ is the Hamming weight of the binary string $x$.
[j] Here $\mathcal{H}_w$ is the Hilbert space of the direct ancillas.

The first part $\mathcal{L}_1$ spanned by all the states where the core has at least one qubit $|1\rangle$, while the second part $\mathcal{L}_2$ is spanned by all the states with the core ancillas all $|0\rangle$, and at least one direct ancillas being $|1\rangle$. We now first show that $\forall|\psi\rangle \in \mathcal{L}_1$, $\langle\psi|\tilde{H}|\psi\rangle > \lambda_*$, and then similarly for $\mathcal{L}_2$.

**(1)** *If* $|\psi\rangle \in \mathcal{L}_1$*, then* $\langle\psi|\tilde{H}|\psi\rangle > \lambda_*$.

Let us first consider $|\psi_a\rangle \in \mathcal{H}_w \otimes \mathcal{S}_a$ for some fixed $a \in \{1, 2, \cdots, C\}$. Then $|\psi_a\rangle$ is a (linear combination of) state(s) where $a$ ancillas in the core are $|1\rangle$ and the other $C-a$ core ancillas are $|0\rangle$. We will find a lower bound for $\tilde{E}_{+,a} = \langle\psi_a|\tilde{H}|\psi_a\rangle$ by considering each component of $\tilde{H}$. Recall from (3) the definition of the core Hamiltonian

$$H_{\mathcal{C}} = \frac{J}{2}\sum_{c\in\mathcal{C}}(\mathbb{I} - Z_c) + \frac{J}{2}\sum_{c,c'\in\mathcal{C}}(\mathbb{I} - Z_c Z_{c'}). \tag{20}$$

Then the energy of $|\psi_a\rangle$ with respect to the core Hamiltonian is $E_{\mathcal{C},a} = \langle\psi_a|H_{\mathcal{C}}|\psi_a\rangle = Ja(C-a+1) \geq JC$. Let

$$H_w = \frac{J}{2}\sum_{j=1}^{M}\sum_{i=1}^{R}\sum_{c\in\mathcal{C}}(\mathbb{I} - Z_{w_i^{(j)}}Z_c) = \frac{J}{2}\sum_{j=1}^{M}\sum_{i=1}^{R}\left(C\mathbb{I} - Z_{w_i^{(j)}}\sum_{c\in\mathcal{C}}Z_c\right) \tag{21}$$

be the interaction Hamiltonian between the direct ancillas and the core ancillas. Recall from (13) that $H = H_w + H_{\mathcal{C}}$. The second equality in (21) indicates that $H_w$ consists of a sum of terms of the form $C\mathbb{I} - Z_{w_i^{(j)}}\sum_{c\in\mathcal{C}} Z_c$. Let us focus on such a term for a particular direct ancilla $w$. Consider the states $|0\rangle_w \otimes |a\rangle_{\mathcal{C}}$ and $|1\rangle_w \otimes |a\rangle_{\mathcal{C}}$ with $|a\rangle_{\mathcal{C}} \in \mathcal{S}_a$ and look at the term $Z_w\sum_{c\in\mathcal{C}} Z_c$. Its expectation value in these states is $C-2a$ and $2a-C$, regardless of the state of the core ancillas. Thus, we get an effective Hamiltonian

$$h'_w = C\mathbb{I} - (C-2a)Z_w \tag{22}$$

for each direct ancilla $w$. Collecting these effective Hamiltonians for each direct ancilla, we get

$$H'_w = \frac{J}{2}\sum_{j=1}^{M}\sum_{i=1}^{R} h'_{w_i^{(j)}} = \frac{J}{2}\sum_{k=1}^{N} h'_k, \tag{23}$$

whose lowest energy in the subspace $\mathcal{H}_w \otimes \mathcal{S}_a$ is equal to that of $H_w$. For convenience, we relabel the direct ancillas by $k = 1, \ldots N$ with $N = MR$ (we are simulating $M$ two-body interactions using $R$ direct ancillas per interaction), and replace the sum over $i$ and $j$ with a single index summation over $k$.

Let us now add the perturbation $V$ (13). For each direct ancilla $k$ there is a term in $V$ of the form $v_k = \beta_k(A\otimes X_k - B\otimes X_k) = \beta_k O_{AB}\otimes X_k$, and we have shown by a sequence of reductions that the lowest energy of $v_k$ in $\mathcal{H}_k \otimes \mathcal{S}_a$ is lower bounded by that of $v'_k = 2\beta_k X_k$. Thus, when we label $V' = \sum_{k=1}^{N} v'_k = \sum_{k=1}^{N} 2\beta_k X_k$, we get a 1-local Hamiltonian

$$\begin{aligned}\tilde{H}' &= E_{\mathcal{C},a}\mathbb{I} + H'_w + V' = E_{\mathcal{C},a}\mathbb{I} + \sum_{k=1}^{N}\left(2\beta_k X_k + \frac{JC}{2}\mathbb{I} - \frac{J(C-2a)}{2}Z_k\right)\\ &= \left(Ja(C-a+1) + \frac{JCN}{2}\right)\mathbb{I} + \sum_{k=1}^{N}\sqrt{4\beta_k^2 + \frac{J^2}{4}(C-2a)^2}P_k\end{aligned} \tag{24}$$

acting only on the direct ancillas, which gives us a lower bound on $E_+$, i.e. for any $|\psi_a\rangle \in \mathcal{H}_w \otimes \mathcal{S}_a$,

$$\min_{\mathcal{H}_w \otimes \mathcal{S}_a} \langle \psi_a | \tilde{H} | \psi_a \rangle \geq \min_{\mathcal{H}_w \otimes \mathcal{S}_a} \langle \psi_a | \tilde{H}' | \psi_a \rangle, \tag{25}$$

with $P_k$ a single qubit operator of the form $\hat{p} \cdot \vec{\tilde{\sigma}}$, with $\vec{\tilde{\sigma}} = \{X, Y, Z\}$ and unit vector $\hat{p}$. Note that the lower bound (25) does not include $H_{\text{else}}$ in $\tilde{H}'$; because $H_{\text{else}} \geq 0$, we are only lowering the right side by omitting it.

Note that the above argument can be generalized to $\mathcal{L}_1 = \mathcal{H}_w \otimes (\oplus_{a=1}^{C} \mathcal{S}_a)$. For a general $|\psi\rangle \in \mathcal{L}_1$, $|\psi\rangle$ must take the form

$$|\psi\rangle = |\phi\rangle_w \otimes \sum_{a=1}^{C} \eta_a |a\rangle_{\mathcal{C}}, \qquad \text{where} \qquad |a\rangle = \sum_{h(x)=a} c_{a,x} |x\rangle, \quad x \in \{0,1\}^C \tag{26}$$

for some sets of complex coefficients $\{\eta_a\}$ and $\{c_{a,x}\}$ that are both normalized. Then $\langle \psi | H_{\mathcal{C}} | \psi \rangle = \sum_{a=1}^{C} |\eta_a|^2 Ja(C-a+1)$. Let $\mathcal{A}$ be the set of $a$ for which $\eta_a \neq 0$. Let $a_{\max}$ be the value of $a$ in $\mathcal{A}$ that maximizes $(C-2a)^2$. Define

$$\begin{aligned} |\psi'\rangle &= |\phi\rangle_w \otimes |a_{\max}\rangle_{\mathcal{C}}, \\ h'_{w,a_{\max}} &= C\mathbb{I} - (C - 2a_{\max}) Z_w, \\ H'_{w,a_{\max}} &= \frac{J}{2} \sum_{k=1}^{N} h'_k. \end{aligned} \tag{27}$$

Then $\langle \psi | H_w | \psi \rangle \geq \langle \psi_a | H'_{w,a_{\max}} | \psi_a \rangle$ for any $|\phi\rangle_w \in \mathcal{H}_w$. Since the generalization from $\mathcal{H}_w \otimes \mathcal{S}_a$ to $\mathcal{H}_w \otimes (\oplus_{a=1}^{C} \mathcal{S}_a)$ does not concern the direct ancillas, we can use the same argument as before to construct a 1-local Hamiltonian

$$\tilde{H}'_{a_{\max}} = \sum_{a=1}^{C} |\eta_a|^2 \left( Ja(C-a+1) + \frac{JCN}{2} \right) \mathbb{I} + \sum_{k=1}^{N} \sqrt{4\beta_k^2 + \frac{J^2}{4}(C - 2a_{\max})^2} P_{k,a_{\max}} \tag{28}$$

such that for any $|\psi_a\rangle \in \mathcal{L}_1$, there always exists a value $a_{\max}$ such that $\min_{|\psi\rangle \in \mathcal{L}_1} \langle \psi | \tilde{H} | \psi \rangle \geq \min_{|\psi\rangle \in \mathcal{L}_1} \langle \psi | \tilde{H}'_{a_{\max}} | \psi \rangle$.

Let us now find a lower bound on $\langle \psi | \tilde{H}'_{a_{\max}} | \psi \rangle$. Note that $Ja(C-a+1) \geq JC$ for any $a = 1, 2, \cdots, C$. Let $\beta_{\max} = \max_{k=1,2,\cdots,N} |\beta_k|$. Noting that $P_{k,a_{\max}}$ in (28) is a unit-norm operator, for any $|\psi\rangle \in \mathcal{L}_1$ we get

$$\begin{aligned} \langle \psi | \tilde{H}'_{a_{\max}} | \psi \rangle &\geq \left( JC + \frac{JCN}{2} \right) - N \sqrt{4\beta_{\max}^2 + \frac{J^2}{4}(C - 2a_{\max})^2} \\ &= JC + \frac{JCN}{2} - \frac{JCN}{2} \sqrt{1 + \frac{16\beta_{\max}^2}{J^2 C^2}} \\ &\geq JC - \frac{JCN}{2} \cdot \frac{16\beta_{\max}^2}{2J^2C^2} = JC - \frac{4N\beta_{\max}^2}{JC} \\ &= \Delta - \frac{4MR\beta_{\max}^2}{\Delta} \geq \frac{79\Delta}{80} > \frac{\Delta}{2} = \lambda_*, \end{aligned} \tag{29}$$

where we have used $2R\beta_{\max}^2/\Delta = \gamma_{\max}$ from (11) and asked for $\Delta \geq 160 M \gamma_{\max}$ in the last line. Here $\gamma_{\max} = \max_{j=1,\cdots,M} |\gamma_j|$ where $\gamma_j$ are coefficients in the target Hamiltonian. Putting (29) into (25),

we get $E_+ > \frac{\Delta}{2} = \lambda_*$. We have thus shown the desired lower bound on $E_+$ in the subspace $\mathcal{L}_1$. Let us now deal with the other part, $\mathcal{L}_2$.

**(2)** *If* $|\psi\rangle \in \mathcal{L}_2$*, then* $\langle\psi|\tilde{H}|\psi\rangle > \lambda_*$.

Any state in the subspace $\mathcal{L}_2 = \mathcal{K}_-^{\perp} \otimes \mathcal{S}_0$ has the core ancillas in the state $|0\cdots 0\rangle_{\mathcal{C}}$, hence $\langle\psi|H_{\mathcal{C}}|\psi\rangle = 0$. To find a lower bound for the energy of $H_w$ in this subspace, we use the construction $H'_w$ in (23) with $a = 0$. For the energy of $V$ we use the same simplifying argument and obtain (again) a 1-local Hamiltonian acting only on the $N = MR$ direct ancillas (cf. Equation 24)

$$\tilde{H}'_0 = \sum_{k=1}^{N} \left( \frac{\Delta}{2}\mathbb{I} - \frac{\Delta}{2} Z_k - 2\beta_k X_k \right) = \sum_{k=1}^{N} S_k, \tag{30}$$

such that

$$\min_{|\psi\rangle \in \mathcal{L}_2} \langle\psi|\tilde{H}|\psi\rangle \geq \min_{|\psi\rangle \in \mathcal{L}_2} \langle\psi|\tilde{H}'_0|\psi\rangle. \tag{31}$$

We now show that the energy of any direct ancilla state orthogonal to $\mathcal{K}_- = \text{span}\{|0\ldots 0\rangle_w\}$ is strictly lower bounded by $\lambda_* = \Delta/2$. Since the core ancilla state will always be $|0\cdots 0\rangle_{\mathcal{C}}$, we will exclude it from our discussion and thus omit the $w$ subscript for the direct ancilla. All quantum states in the proof from here on refer to the state of the direct ancillas.

To show the energy lower bound we use induction on the number of direct ancillas, $n$. Let

$$E_n = \min_{|\phi\rangle \perp |0\rangle^{\otimes n}} \langle\phi| \sum_{k=1}^{n} S_k |\phi\rangle. \tag{32}$$

Specifically, we prove the following statement:

$$E_n \geq \frac{3\Delta}{4} - \delta_n, \qquad \text{with} \qquad \delta_n = \frac{40 n \beta^2}{9\Delta}, \qquad n = 1, \cdots, N. \tag{33}$$

We start with the initial case $n = 1$. There the only state orthogonal to $|0\rangle$ is $|1\rangle$. Hence $E_1 = \Delta$, which satisfies (33). Now assume (33) holds for some $n$. An $(n+1)$-qubit state that is orthogonal to $|0\cdots 0\rangle$ (denoted by the superscript $\oslash$) must have the form

$$|\psi^{\oslash}_{n+1}\rangle = a|\xi^{\oslash}_n\rangle|0\rangle + b|\phi^{\oslash}_n\rangle|1\rangle + c|0\cdots 0\rangle|1\rangle, \tag{34}$$

where $|\xi^{\oslash}_n\rangle$ and $|\phi^{\oslash}_n\rangle$ are some states that are orthogonal to $|0\cdots 0\rangle$. Let us calculate the energy of the state (34).

$$E_{n+1} = \sum_{i=1}^{n} \langle\psi^{\oslash}_{n+1}|S_i|\psi^{\oslash}_{n+1}\rangle + \langle\psi^{\oslash}_{n+1}|S_{n+1}|\psi^{\oslash}_{n+1}\rangle \tag{35}$$

$$\begin{aligned} &= |a|^2 \sum_{i=1}^{n} \langle\xi^{\oslash}_n|S_i|\xi^{\oslash}_n\rangle\langle 0|0\rangle + |b|^2 \sum_{i=1}^{n} \langle\phi^{\oslash}_n|S_i|\phi^{\oslash}_n\rangle\langle 1|1\rangle + |c|^2 \sum_{i=1}^{n} \langle 0\cdots 0|S_i|0\cdots 0\rangle\langle 0|0\rangle \\ &+ 2\text{Re}\left( ab^* \sum_{i=1}^{n} \langle\xi^{\oslash}_n|S_i|\phi^{\oslash}_n\rangle\langle 0|1\rangle + ac^* \sum_{i=1}^{n} \langle\xi^{\oslash}_n|S_i|0\cdots 0\rangle\langle 0|1\rangle + bc^* \sum_{i=1}^{n} \langle\phi^{\oslash}_n|S_i|0\cdots 0\rangle\langle 1|1\rangle \right) \\ &+ |a|^2\langle 0|S|0\rangle_{n+1} + |b|^2\langle 1|S|1\rangle_{n+1} + |c|^2\langle 1|S|1\rangle_{n+1} \\ &+ 2\text{Re}\left( ab^*\langle\xi^{\oslash}_n|\phi^{\oslash}_n\rangle\langle 0|S|1\rangle + ac^*\langle\xi^{\oslash}_n|0\cdots 0\rangle\langle 0|S|1\rangle + bc^*\langle\phi^{\oslash}_n|0\cdots 0\rangle\langle 1|S|1\rangle \right). \end{aligned} \tag{36}$$

Note that $\langle 0|1\rangle = 0$ and $\langle \psi_n^{\oslash}|0\cdots 0\rangle = \langle \phi_n^{\oslash}|0\cdots 0\rangle = 0$. Also recall that $\langle 0|S_i|0\rangle = 0$, $\langle 1|S_i|1\rangle = \Delta$ and $\langle 0|S_i|1\rangle = -2\beta_i$. Hence,

$$E_{n+1} = |a|^2 \sum_{i=1}^{n} \langle \xi_n^{\oslash}|S_i|\xi_n^{\oslash}\rangle + |b|^2 \sum_{i=1}^{n} \langle \phi_n^{\oslash}|S_i|\phi_n^{\oslash}\rangle + 0 + 0 + 0 + 2\mathrm{Re}\left(bc^* \sum_{i=1}^{n} \langle \phi_n^{\oslash}|S_i|0\cdots 0\rangle\right) \tag{37}$$

$$+ 0 + |b|^2\Delta + |c|^2\Delta + 2\mathrm{Re}\left(-ab^*\langle \xi_n^{\oslash}|\phi_n^{\oslash}\rangle 2\beta_{n+1}\right) + 0 + 0$$

$$\geq |a|^2 E_n + |b|^2 E_n + 2\mathrm{Re}\left(bc^* \sum_{i=1}^{n} \langle \phi_n^{\oslash}|S_i|0\cdots 0\rangle\right) + |b|^2\Delta + |c|^2\Delta - 4|a||b|\beta_{\max}, \tag{38}$$

where we lower bounded the last term using absolute values, a maximum magnitude of the $\beta$'s, and $|\langle \psi_n^{\oslash}|\phi_n^{\oslash}\rangle| \leq 1$. Next, we observe that the term $\langle \phi_n^{\oslash}|S_i|0\cdots 0\rangle_n$ is nonzero only for parts of $|\phi_n^{\oslash}\rangle$ with a single $|1\rangle$. The largest magnitude it could possibly have is when the state $|\phi_n^{\oslash}\rangle$ is made *only* from states with a single $|1\rangle$ as $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}|0\cdots 1_i\cdots 0\rangle$. We then get $\sum_{i=1}^{n}\langle \phi_n^{\oslash}|S_i|0\cdots 0\rangle \geq -2\beta_{\max}\sqrt{n}$. Putting this in, recalling (33) and using absolute values, we get

$$E_{n+1} \geq \left(|a|^2 + |b|^2\right)\left(\frac{3\Delta}{4} - \delta_n\right) - |b||c|\cdot 4\beta_{\max}\sqrt{n} + \left(|b|^2 + |c|^2\right)\Delta - |a||b|\cdot 4\beta_{\max} \tag{39}$$

$$= \frac{3\Delta}{4}\left(|a|^2 + |b|^2 + |c|^2\right) - \left(|a|^2 + |b|^2\right)\delta_n + |b|^2\Delta - |a||b|\cdot 4\beta_{\max} - |b||c|\cdot 4\beta_{\max}\sqrt{n} + |c|^2\cdot\frac{\Delta}{4}$$

$$\geq \frac{3\Delta}{4} - \delta_n + |b|^2\Delta - |b|\cdot 4\beta_{\max} + \underbrace{|c|\left(|c|\frac{\Delta}{4} - |b|\cdot 4\beta_{\max}\sqrt{n}\right)}_{f(|c|)}, \tag{40}$$

where we have used (33), and then $|a|^2+|b|^2 \leq 1$ and $|a| \leq 1$. Independent of $|b|$, let us look at $f(|c|)$, a quadratic function of $|c|$. Its minimum is at $|c| = \frac{|b|\cdot 8\beta_{\max}\sqrt{n}}{\Delta}$, with the value $f_{\min} = -\frac{|b|^2\cdot 16n\beta_{\max}^2}{\Delta}$. In (40) it means

$$E_{n+1} \geq \frac{3\Delta}{4} - \delta_n + |b|^2\Delta - |b|\cdot 4\beta_{\max} - \frac{16|b|^2 n\beta_{\max}^2}{\Delta} \tag{41}$$

$$= \frac{3\Delta}{4} - \delta_n + |b|^2\underbrace{\left(\Delta - \frac{16n\beta_{\max}^2}{\Delta}\right)}_{\geq 9\Delta/10} - |b|\cdot 4\beta_{\max} \tag{42}$$

$$\geq \frac{3\Delta}{4} - \delta_n + \underbrace{|b|\left(|b|\frac{9\Delta}{10} - 4\beta_{\max}\right)}_{g(|b|)}, \tag{43}$$

where in the second line we have used $\Delta \geq 160M\gamma_{\max}$ to guarantee $\Delta - \frac{16\beta_{\max}^2 n}{\Delta} \geq \Delta - \frac{16\beta_{\max}^2 N}{\Delta} = \Delta - 16M\gamma_{\max} \geq \frac{9\Delta}{10}$. The expression $g(|b|)$ is quadratic in $|b|$, minimized at $|b| = \frac{20\beta_{\max}}{9\Delta}$, giving the value $g_{\min} = -\frac{40\beta_{\max}^2}{9\Delta}$. Putting it into (43), we get

$$E_{n+1} \geq \frac{3\Delta}{4} - \delta_n - \frac{40\beta_{\max}^2}{9\Delta} = \frac{3\Delta}{4} - \delta_{n+1}, \tag{44}$$

which proves our induction step, as $\delta_n = \frac{40n\beta_{\max}^2}{9\Delta}$. Therefore, (33) holds. Let $n = N$ and we have for any $|\psi\rangle \in \mathcal{L}_2$,

$$\begin{aligned}\langle\psi|\tilde{H}_0'|\psi\rangle \geq E_N \geq \frac{3\Delta}{4} - \frac{40N\beta_{\max}^2}{9\Delta} = \frac{3\Delta}{4} - \frac{40MR\beta_{\max}^2}{9\Delta} \\ = \frac{3\Delta}{4} - \frac{20M\gamma_{\max}}{9} \geq \left(\frac{3}{4} - \frac{1}{72}\right)\Delta = \frac{53}{72}\Delta > \frac{\Delta}{2} = \lambda_*, \end{aligned} \tag{45}$$

where in the last line we have used (11) and $\Delta \geq 160M\gamma_{\max}$. Combining the above statement with (31), we have $\langle\psi|\tilde{H}|\psi\rangle > \lambda_*$ for any $|\psi\rangle \in \mathcal{L}_2$.

This concludes the proof of Claim 1.

### *3.2 The perturbation series converges.*

Let us now state and prove our second claim – the convergence of the perturbation series for our gadget construction.

**Claim 2** *Consider the 2-body gadget Hamiltonian $\tilde{H} = H + V$ defined in* (13) *with spectral gap $\Delta$ between the ground and the first excited subspace of $H$, and a target Hamiltonian $H_{targ} = H_{else} + \sum_{j=1}^{M} \gamma_j A_j \otimes B_j$ with $\gamma_j = O(1)$ and $H_{else}$ positive semi-definite. Choose a constant parameter $d \in (0,1)$ and an error tolerance $\epsilon$. If we set $\Delta = M^3R^d$ and choose the number of direct ancillas per target term $R$ and the core size $C$ according to*

$$R = \Omega\left(\max\left\{\epsilon^{-\frac{2}{d}}, \left(\frac{\|H_{else}\|^2}{2M^4\gamma_{max}}\right)^{\frac{1}{d}}, \left(M^3\epsilon^{-2}\right)^{\frac{1}{1-d}}\right\}\right), \qquad C = \Omega\left(M^3R^d\,\epsilon^{-1}\right), \tag{46}$$

*then the strengths of the interaction terms in the gadget Hamiltonian are small, i.e. $\beta_j, J = O(\epsilon)$.*

*Furthermore, the self energy expansion* (2) *satisfies*

$$\|\Sigma_-(z) - H_{targ} \otimes \Pi_-\| = O(\epsilon), \tag{47}$$

*where $\Pi_-$ is the projector onto $\mathcal{L}_-$, and $z$ obeys $|z| \leq \epsilon + \|H_{else}\| + \sum_{j=1}^{M} |\gamma_j|$.*

This claim is one of the central results of this work – it shows that our gadget Hamiltonian (for a 2-local target Hamiltonian) uses only interactions of strength $O(\epsilon)$, *i.e.* no strong interactions. This is qualitatively different from previous constructions which require interactions of strength poly$(\epsilon^{-1})$. However, the price we pay for avoiding strong interactions is that the number of ancillas scales as poly$(\epsilon^{-1})$, as shown in (46), while previous constructions require some number of ancillas independent of $\epsilon$. Hence we present a tradeoff between interaction strength and ancilla number in a gadget Hamiltonian.

Let us prove Claim 2. First we show that $\tilde{H}$ consists of only weak interaction terms. When we choose $\Delta = M^3R^d$ for some $d \in (0,1)$ and substitute it into (11), we find that the interaction strength between the target spins and direct ancillas will be $\beta_j = \sqrt{\frac{\gamma_j\Delta}{2R}} = O(\epsilon)$, if we choose

$$R \gg \left(M^3\epsilon^{-2}\right)^{\frac{1}{1-d}}. \tag{48}$$

Next, recalling $\Delta = CJ$, the strength of the interaction $J$ between the core ancillas will be $O(\epsilon)$ if we choose $C \gg M^3R^d\,\epsilon^{-1}$.

Furthermore, once we set $\Delta = M^3 R^d$, we can easily satisfy the requirement $\Delta \geq 160 M \gamma_{\max}$ in Claim 1 for reasonable $R$ – more specifically, we need $R \gg \left(160\gamma_{\max}/M^2\right)^{1/d}$.

We will now analyze the higher order terms in the self energy expansion $\Sigma_-(z)$ according to (2) and show that the error term in Eq. 2 scales as $O(\epsilon)$. The perturbative expansion of $\Sigma_-(z)$ for the construction in (13) yields

$$\Sigma_-(z) = H_{\text{else}} + \frac{1}{z-\Delta}\sum_{j=1}^{M} R\beta_j^2 (A_{a_j} - B_{b_j})^2 + \underbrace{\sum_{k=1}^{\infty} V_{-+}\left(G_+ V_+\right)^k G_+ V_{+-}}_{\text{error}}. \tag{49}$$

We can associate every term in the perturbation series with a *path* starting in the ancilla state $|0\rangle_w|0\rangle_{\mathcal{C}}$ (i.e. belonging to $\mathcal{L}_-$) to states in $\mathcal{L}_+$ and back to $\mathcal{L}_-$. Each path consists of a sequence of virtual *transition steps* between states of the ancillas, denoted $x \to x'$ with $R$-bit strings $x, x'$. The number of steps for a path is dependent on the order of the perturbation term. A path for the $m$-th order is

$$\mathcal{L}_- \xrightarrow{V_{-+}} |y\rangle \underbrace{\xrightarrow{V_+} |y_1\rangle \xrightarrow{V_+} |y_2\rangle \xrightarrow{V_+} \cdots \xrightarrow{V_+} |y_{m-2}\rangle \xrightarrow{V_+}}_{(m-2)\text{ steps}} |y'\rangle \xrightarrow{V_{+-}} \mathcal{L}_-, \tag{50}$$

where $y$ and $y'$ are $R$-bit strings with Hamming weight 1, and $|y_i\rangle \in \mathcal{L}_+$. In particular, these states belong to the subspace $\mathcal{L}_2 = \mathcal{K}_-^{\perp} \otimes \mathcal{S}_0$ in (19). Observe that each term in $T_m = V_{-+}(G_+V_+)^{m-2}G_+V_{+-}$ is composed from transitions of the following three types

1. a $|0\rangle \to |1\rangle$ flip of some direct ancilla qubit $w$,
2. a $|1\rangle \to |0\rangle$ flip of some $w$,
3. the state of the ancillas stays the same.

In the first two cases, $V_+$ (also $V_{-+}$ or $V_{+-}$) contributes the term from $V$ that flips the direct ancilla $w$ via $\cdots \otimes X_w$. In the third case, the ancilla state stays the same, and $V_+$ contributes a term that contains interaction with $w$ via $\cdots \otimes |1\rangle\langle 1|_w$. This type of term contains the factor $H_{\text{else}}$. Note that for the $k-2$ transitions, the number of flips $k_f$ cannot exceed $k$. Furthermore, it must be even for the transition to terminate in $\mathcal{L}_-$. Finally, every transition step $y_i \to y_{i+1}$ also contributes a factor $\frac{1}{z-h(y_i)\Delta}$ coming from $G_+$, with $h(y_i)$ the Hamming weight of the string $y_i$.

We can find the norms of the perturbation terms at a given order by enumerating all possible paths and adding up their contributions. For this, we introduce a graphical representation of the paths in Fig. 5. Each grid point in the lower-right triangle, including the diagonal points, corresponds to a state with a particular number of the direct ancillas flipped. We start from the lower-leftmost point, which corresponds to the all-zero subspace $\mathcal{L}_-$. Each transition (ancilla flip) maps to a rightwards or upwards movement on the graph, while remaining in the high-energy subspace is depicted by a diagonal step. A valid path ends at the top-rightmost point, which again belongs to the ground state subspace $\mathcal{L}_-$. Furthermore, a valid path can touch the diagonal line only at the last step of the transition (otherwise, it would be a composition of paths at lower orders).

Suppose at a certain point the direct ancillas are in a state $|y\rangle$ with $h(y)$ ancillas in $|1\rangle$ and the rest in $|0\rangle$, with $h(y) \in \{1, 2, \cdots, N\}$ being the Hamming weight of $y$. Let us first look at transition steps that flip an ancilla takes $|y\rangle$ to a new state $|y'\rangle$ where $y$ and $y'$ differ by one bit.

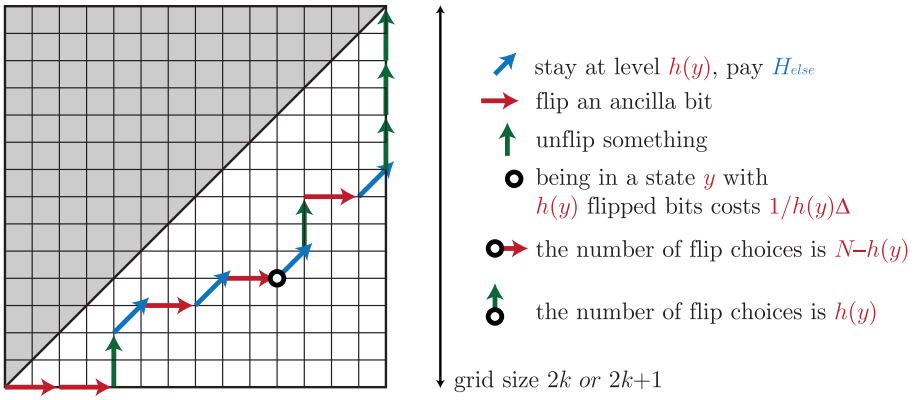


Fig. 5. A graphical representation of the contributions to the error term of order $m = 2k$ or $m = 2k+1$. An up- and right-moving, sub-diagonal path corresponds to a sequence of transitions. A bit flip moves 2 squares horizontally/vertically, while "staying" moves across one square diagonally. The distance from the diagonal corresponds to the number $h(y)$ of flipped ancillas in a given state $y$.

1. If an ancilla in $|y\rangle$ is flipped from $|0\rangle$ to $|1\rangle$, we move to the right in Figure 5. There are $N-h(y)$ ways to flip a 0 to 1 at this point, and we simply overestimate it by $N$. Furthermore, we get a contribution from $G_+$, and we overestimate it by $\|G_+\| \leq \frac{1}{|h(y)\Delta - z|} \leq \frac{1}{\Delta}$. Thus, we find that the norm of a contribution from this first type of transition step is upper bounded by $\frac{N}{\Delta}$.

2. Second, when an ancilla is flipped from $|1\rangle$ to $|0\rangle$, we move up in Figure 5. There are $h(y)$ ways to unflip a spin now. The resolvent $G_+$ again contributes a factor $\frac{1}{z-h(y)\Delta}$. Taken together, the factor $h(y)$ "cancels"[k] the $h(y)$ in the denominator from the resolvent. The contribution from this process is less than $\frac{1}{\Delta}$.

3. Third, for a step that keeps the ancilla state, we remain in the same state $y$, and move one step diagonally on the graph, getting a contribution $H_{\text{else}}$. We can do this in $h(y)$ ways, because there are $h(y)$ ancillas in the state $|1\rangle$ that terms like $\cdots \otimes |1\rangle\langle 1|$ apply to. The resolvent $G_+$ again contributes a factor $\frac{1}{z-h(y)\Delta}$. Similarly to what we did above, we "cancel" the factor $h(y)$, and conclude that a contribution from this type of step is upper bounded by $\frac{\|H_{\text{else}}\|}{\Delta}$.

Altogether, at order $m$, our paths have length $m$, out of which we have $f$ flips, $f$ unflips, and $m-2f$ diagonal steps. The contribution of each such path is upper bounded by

$$c_{\text{path}} \leq \frac{N^f (2\beta_{\max})^{2f} \|H_{\text{else}}\|^{m-2f}}{\Delta^{m-1}}. \tag{51}$$

We now need to find an upper bound on the number of valid paths such as the one shown in Fig. 5.

If we did not have the diagonal steps, for even $m = 2k$, this would be the $k^{\text{th}}$ Catalan number – the number of up- & right-moving paths between corners of a square of size $2k$ that don't pass above the diagonal. In our case, the situation is just a bit more difficult. The number of $2k$-step (resp. $(2k+1)$-step) paths is upper bounded by the *Motzkin number* of order $2k$ (resp. $2k+1$). These

[k] Here "cancel" means that the product $\frac{h(y)}{|h(y)\Delta - z|}$ is $O(\Delta^{-1})$.

numbers correspond to a number of up-, diagonal-, and right-moving paths across a square, remaining below the diagonal. It suffices for our purposes to use a crude upper bound on the Motzkin numbers: $M_{2k} \leq 3^{2k}$ and $M_{2k+1} \leq 3 \cdot 3^{2k}$, basically saying we have $\leq 3$ ways to go at each step. This is grossly over-counting (e.g. going above the diagonal, going farther from the diagonal than $N$, etc.), but we do not mind, as it will suffice for our argument. Let us finish it first for even $m = 2k$ and then for odd $m = 2k + 1$.

*Upper bounds on the $(2k)^{th}$ order.* In estimating the error, we here consider only the $4^{\text{th}}$ order and onward, i.e. $k \geq 2$, as the second order is the actual term that we want to generate (for details of the 2nd order, see Appendix 1).

In order to make sure that the sequence of transitions finishes at $\mathcal{L}_-$, the number of flips $k_f = 2f$ must be even ($f \in \mathbb{N}$, $f \leq k$). Hence, the number of steps where the ancilla state stays the same is $2(k - f)$, an even number. A contribution from some path to $\Sigma_-(z)$ is a term whose norm is upper bounded by

$$\leq \frac{\overbrace{N(2\beta_{\max})^2 \times \cdots \times N(2\beta_{\max})^2}^{f} \times \overbrace{\|H_{\text{else}}\|^2 \times \cdots \times \|H_{\text{else}}\|^2}^{k-f}}{\Delta^{2k-1}}. \tag{52}$$

The condition $R \geq \left(\frac{\|H_{\text{else}}\|^2}{2M^4\gamma_{\max}}\right)^{\frac{1}{d}}$ from (46), combined with $N = MR$ and (11) implies

$$\|H_{\text{else}}\|^2 \leq 2M^4R^d\gamma_{\max} = 4MR\frac{\Delta\gamma_{\max}}{2R} = N\left(2\beta_{\max}\right)^2. \tag{53}$$

Using this and (11), we conclude that the overall contribution of a single path (52) is bounded from above by

$$\leq \left(\frac{N(2\beta_{\max})^2}{\Delta}\right)^k \frac{1}{\Delta^{k-1}} = 2^k\left(M\gamma_{\max}\right)^k \frac{1}{\Delta^{k-1}} = \Delta\left(\frac{2M\gamma_{\max}}{\Delta}\right)^k. \tag{54}$$

The total number of legal paths is less than $9^k$. Thus, the norm of the $(2k)^{\text{th}}$ order is bounded from above by

$$\|T_{2k}\| \leq 9^k\Delta\left(\frac{2M\gamma_{\max}}{\Delta}\right)^k = \Delta\left(\frac{18M\gamma_{\max}}{\Delta}\right)^k. \tag{55}$$

We have chosen $\Delta \gg M$, which makes it a small contribution, as we wanted.

*Upper bounds on the $(2k+1)^{th}$ order.* Finding a bound on the $3^{\text{rd}}$ order is straightforward:

$$\|T_3\| = N \cdot (2\beta_{\max}) \cdot \frac{1}{\Delta} \cdot \|H_{\text{else}}\| \cdot \frac{1}{\Delta} \cdot (2\beta_{\max}) \leq \frac{\left(4N\beta_{\max}^2\right)^{\frac{3}{2}}}{\Delta^2} = \sqrt{\frac{\left(2M\gamma_{\max}\right)^3}{\Delta}}, \tag{56}$$

using (11). Recalling $\Delta = M^3R^d$, we get $\|T_3\| \leq (2\gamma_{\max})^{3/2}R^{-d/2} = O\left(R^{-d/2}\right)$, a small contribution.

Analogously, we do the calculation for the general $(2k+1)^{\text{th}}$ order, obtaining

$$\begin{aligned}\|T_{2k+1}\| &\leq 3\cdot 9^k \cdot \frac{[N(2\beta_{\max})^2]^f \cdot \|H_{\text{else}}\|^{2(k-f)+1}}{\Delta^{2k}} \leq 3\cdot 9^k \cdot \frac{[N(2\beta_{\max})^2]^{k+\frac{1}{2}}}{\Delta^{2k}} \\ &\leq 3\cdot 2^k \cdot 9^k \cdot \left(\frac{M\gamma_{\max}}{\Delta}\right)^k \sqrt{2M\gamma_{\max}\Delta} = \left(3\sqrt{2}\right)\Delta\left(\frac{18M\gamma_{\max}}{\Delta}\right)^{k+\frac{1}{2}}. \end{aligned} \tag{57}$$

Comparing with (56), we find that the last expression is also true for $k=1$. Therefore, together with (55), we can bound all of the terms in the error series by

$$\|T_m\| \leq 3\sqrt{2}\,\Delta\left(\frac{18M\gamma_{\max}}{\Delta}\right)^{\frac{m}{2}} = 3\sqrt{2}\Delta q^m, \tag{58}$$

for $m\geq 3$ with $q=\sqrt{18M\gamma_{\max}/\Delta} = O\left(M^{-1}R^{-\frac{d}{2}}\right)$. Thus, the whole series $\sum_{m=3}^{\infty}\|T_m\|$ is upper bounded by a geometric series that converges, implying

$$\sum_{m=3}^{\infty}\|T_m\| \leq const.\times \Delta q^3 = O\left(R^{-\frac{d}{2}}\right) \leq \epsilon, \tag{59}$$

for our choice of $\epsilon$ when we choose a suitably large $R \gg \epsilon^{-\frac{2}{d}}$. This concludes the proof of Claim 2. □

In conclusion, in Eq. 10 we have $\|\Sigma_-(z) - H_{\text{eff}}\| = O(\epsilon)$ where the effective Hamiltonian $H_{\text{eff}} = H_{\text{targ}}\otimes\Pi_- + \gamma\Pi_-$ (up to an overall shift) captures the target Hamiltonian. Therefore we have proven Theorem 1. Let us have a last look at the required resources:

**Remark 1** *If $H_{targ}$ acts on $n$ qubits, our gadget Hamiltonian $\tilde{H}$ acts on*

$$\begin{aligned} &n + MR + C \\ &\gg n + MR + M^3R^d\,\epsilon^{-1} \\ &\gg n + \max\left\{M\epsilon^{-\frac{2}{d}} + M^3\epsilon^{-3},\ \left(M^{4-d}\epsilon^{-2}\right)^{\frac{1}{1-d}},\ M\left(\frac{\|H_{else}\|^2}{2M^4\gamma_{max}}\right)^{\frac{1}{d}} + \epsilon^{-1}\,\frac{\|H_{else}\|^2}{2M\gamma_{max}}\right\} \end{aligned} \tag{60}$$

*qubits. If the interaction graph of $H_{targ}$ has degree $D$, then the interaction graph of the gadget Hamiltonian has total degree* $\max\{DR, RC\} = poly(D, \epsilon^{-1}, \|H_{else}\|, M)$.

This concludes the story of the 2-body gadgets with weak interactions. Let us now apply the construction to reducing $k$-local to 2-body with weak interaction ($k\geq 3$), and prove Corollary 1.

## 4 Reducing $k$-body to 2-body interactions ($k\geq 3$)

With the new 2-body construction in mind, is it possible to use the *core* idea and "parallelism" of the 2-body gadgets to construct a 3-body to 2-body gadget that also uses only weak interactions? There is a straightforward way to combine the usual 3-to-2-body gadgets with our strong-from-weak 2-body construction as sketched in Figure 6. This is what we claim in Corollary 1. We start from the usual 3-body to 2-body construction in [2] and replace the strong 1-local term of magnitude $\Delta$ by interactions with a core. Finally, we reduce the large-norm 2-body interactions in these gadgets with weak ones using the 2-body gadgets from Section 3.

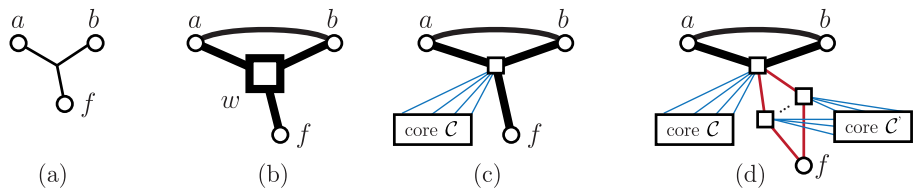


Fig. 6. 3-local interactions from weak interactions. (a) The 3-local interaction we want to approximate. (b) The standard construction by Oliveira and Terhal [2] with target term $A_a \otimes B_b \otimes F_f$ replaced by one (direct) ancilla $w$ in a large field $\Delta$, interacting with the target spins via strong interactions of order $\Delta^{2/3}$. In addition, $a$ and $b$ interact with strength of order $\Delta^{1/3}$ to compensate for the error term at 2$^{\text{nd}}$ order perturbation theory. (c) The local fields are replaced by interactions with a core. (d) Each strong 2-local interaction term can be reduced to many $O(1)$ terms by our 2-body gadget construction, using another common core.

For general $k$-body to 2-body reduction, we can resort to the construction from [5], where the gadget Hamiltonian consists of only 2-local interaction terms (*i.e.* no extra 1-local terms). This makes it easy to directly apply our new 2-body gadgets and reduce the gadget Hamiltonian to one with only weak interactions.

Let us conclude with an open question. Although it is possible to apply our construction to reduce any $k$-body target Hamiltonian to a 2-body one with arbitrarily weak interactions, the qubit overhead is likely exponential in $k$. With the original constructions proposed in [1, 2] it is possible to use $O(k)$ qubits with an exponential overhead in interaction strengths [8]. Perhaps a middle ground between the two constructions could be sought such that both the interaction strength and qubit overheads are polynomial in $k$.

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## Appendix A The second order perturbation term for 2-body gadgets

In this Appendix, for the purpose of illustration we calculate the upper bound on the norm of the second order in the perturbation series for the self-energy for our 2-body gadget construction from Section 3.2. The interested reader can find similar bounds for the fourth and sixth orders as well in the arXiv version of this paper.

The 2nd order is what contributes to the effective Hamiltonian, which has $M$ terms of norm $O(1)$ there. Let us see what we get here. From (49) we see that $T_2 = \frac{1}{z-\Delta}\sum_{j=1}^{M} R\beta_j^2 (A_{a_j} - B_{b_j})^2$. Every term at the second order corresponds to a transition of the form

$$\mathcal{L}_- \to |y\rangle \to \mathcal{L}_-. \tag{A.1}$$

Here $|y\rangle$ is a state where only one direct ancilla qubit $w$ is flipped to $|1\rangle$ while the others remain at $|0\rangle$. From our construction of $V$ in (13), observe that each term that involves a particular direct ancilla $w_i^{(j)}$ is associated with a corresponding coefficient $\beta_j$. Therefore all the transitions of the form (A.1) involving $w_i^{(j)}$ would contribute a term of the form

$$\underbrace{\beta_j(A_{a_j} - B_{b_j})}_{V_{-+}} \cdot \underbrace{\frac{1}{z-\Delta}}_{G_+} \cdot \underbrace{\beta_j(A_{a_j} - B_{b_j})}_{V_{+-}} \tag{A.2}$$

to the perturbative expansion $\Sigma_-(z)$. Note that because the Hamming weight of $y$ is $h(y) = 1$, the resolvent component $G_+$ contributes a factor $\frac{1}{z-h(y)\Delta} = \frac{1}{z-\Delta}$. Since $R$ direct ancillas are introduced for the target 2-local term involving $a_j$ and $b_j$, the total contribution of the direct ancillas used for generating the $j$-th target term would be multiplied by a factor of $R$. Summing over all the target terms from $j = 1$ to $M$, we get the current form of $T_2$. Assuming $A_{a_j}$ and $B_{b_j}$ are both unit-norm operators,

$$\|T_2\| \leq \frac{1}{\Delta} \cdot MR(2\beta_{\max})^2 = 2M\gamma_{\max}, \tag{A.3}$$

using the choice $\beta_i = \sqrt{\frac{\gamma_i \Delta}{2R}}$. This is just what we expected (because the norm of what we are generating should be something on the order of $M$).