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Dissociation and annihilation of multipartite entanglement structure in dissipative quantum dynamics

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We study the dynamics of the entanglement structure of a multipartite system experiencing a dissipative evolution. We characterize the processes leading to a particular form of output-system entanglement and provide a recipe for their identification via concatenations of particular linear maps with entanglement-breaking operations. We illustrate the applicability of our approach by considering local and global depolarizing noises acting on general multiqubit states. A difference in the typical entanglement behavior of systems subjected to these noises is observed: the originally genuine entanglement dissociates by splitting off particles one by one in the case of local noise, whereas intermediate stages of entanglement clustering are present in the case of global noise. We also analyze the definitive phase of evolution when the annihilation of the entanglement compound finally takes place.

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I. INTRODUCTION

The physical phenomenon of entanglement naturally appears in composite quantum systems via interactions among constituents. Simple collision models already teach us that different interaction types lead to various types of multipartite entanglement [1]. Systems with local Hamiltonians exhibit correlations between the degree of entanglement and eigenenergies [2,3], phase transitions [4–6], and the number of interacting bodies [7]. Multipartite entanglement finds uses in quantum networking applications such as secret sharing [8], secret voting [9], open-destination teleportation [10], etc. For the latter purposes, entanglement can be created within the system not only by interaction among constituent bodies but also by a properly engineered interaction with the environment [11–13].

Suppose the prepared multipartite entangled state is intended for use in an entanglement-enabled quantum protocol involving remote clients. While transferring the quantum information to recipients, the state will be modified by inevitable noise processes. It can happen that the type of multipartite entanglement received by the clients differs significantly from the original one, and the realization of the desired protocol becomes impossible. Similarly, uncontrollable noise processes in quantum memory devices can result in destroying particular correlations within the stored multipartite system and make the released state ineffective [14,15]. Degradation of entanglement also imposes limitations on the benefit of advanced quantum metrology relying on genuinely multipartite entangled states [16]. These examples demonstrate the necessity of tracking the multipartite entanglement dynamics and finding noise levels corresponding to the change of entanglement type.

Previous efforts in this direction relied on specific entanglement measures. Negativity [17]—a measure detecting negativity of the density matrix under partial transpose (NPT) [18]—was originally used by Simon and Kempe [19] and Dür and Briegel [20] to analyze Greenberger-Horne-Zeilinger (GHZ), *W*, and cluster states under local depolarizing noise. Then Bandyopadhyay and Lidar [21] and Hein *et al.* [22] utilized it to study the behavior of GHZ states and graph states, respectively, under general local homogeneous noise. Generalized GHZ-type states under a local amplitude-damping channel were considered with the help of negativity by Man *et al.* [23]. Aolita *et al.* exploited negativity to study effects of local depolarizing, dephasing, and generalized amplitude-damping channels in GHZ states [24] and graph states [25]. Those results were obtained for an arbitrary number of qubits (except for some graph states [25] and randomly sampled states [26]) due to the ultimate simplicity of negativity computation. Depolarization and dephasing of qudit GHZ states were considered via negativity in [27]. Similarly, concatenated GHZ states (where blocks of a small number of qubits are GHZ states themselves) were considered in [28]. However, the negativity does not provide comprehensive information about the entanglement structure because it can be sensitive to the entanglement with respect to a particular bipartition only [remember, e.g., bound-entangled positive partial transpose states [29] and biseparable but nontriseparable states [30]].

The absence of full separability can also be detected by some other measures. For instance, Carvalho *et al.* used the lower bound for a specific generalization of the concurrence and applied it to the dynamics of several-qubit GHZ and *W* states under amplitude-damping and dephasing local channels [31]. Gühne *et al.* used the geometric measure of entanglement [32,33] to study the global dephasing process of four-qubit GHZ, cluster, *W*, and Dicke states [34]. Grimsmo *et al.* used the entropic measure for average n-partite entanglement over quantum trajectories [35]. Gheorghiu and Gour developed the evolution of an averaged SL-invariant entanglement measure for local decoherence [36]. A similar approach with a lower bound of the concurrence was exploited in [37]. A nonzero value of these quantities indicates the presence of some entanglement within the quantum system, but gives little information about its particular form and, therefore, the benefit of this entanglement for some applications remains questionable. Moreover, vanishing values of the above measures cannot guarantee the full separability of the state, and thus, the problem of fundamental noise limits eliminating any form of entanglement (resulting in fully separable states) is still open.

Genuine multipartite entanglement is the exact opposite of full separability: this form of entanglement is intrinsically multiparticle and cannot be attributed to the entanglement distributed among smaller subsystems. The detection of genuine entanglement for specific quantum states has been a subject of intensive recent research (see, e.g., [38–43] and references therein). Dissipative evolution of genuine multipartite entanglement has been analyzed with the help of some measures. The mean value of a projectorlike witness [44] was used by Bodoky *et al.* to study several qubits within a heuristic model of decoherence based on local relaxation and dephasing times [45]. Campbell *et al.* used fidelity- and collective-spin-based entanglement witnesses to analyze the dynamics of genuine multipartite entanglement of Dicke states under local amplitude-damping, phase-damping, and depolarizing channels [46]. Tripartite negativity and generalized concurrence were also applied to the dissipative dynamics of GHZ and W three-qubit entangled states [47–52]. Let us recall that the above measures are not precise, i.e., their zero values do not imply in general that the genuine entanglement is lost. On the other hand, precise measures (based on convex roof definitions) are quite hard to compute. This is the main reason why the research in entanglement dynamics is usually restricted to particular initial states (GHZ, W, X, Dicke, etc.) and the use of relatively simple measures.

Despite existing results for noises preserving genuine entanglement and entanglement on the whole (absence of full separability), the evolution of entanglement structure still remains unexplored. The aim of this paper is to track the transformations of entanglement structure during dissipative processes. By "structure" we understand the number of separate components and the number of particles within each of them (with allowance for convex mixtures) [7,53,54]. This structure resembles a Russian nested doll, and dissipative evolution maps states from the outer to the inner dolls. The evolution of entanglement structure can be seen as a dissociation of the entanglement compound due to interaction with the "solvent" (particles of the environment). Note that the "entanglement compound" refers to a genuinely entangled multipartite component and differs from the concept of an "entanglement molecule" whose bonds depict entanglement of reduced two-particle states [55]. The idea of tracking the entanglement structure was realized for three-qubit GHZ states under global depolarization in [56] and for the restricted Hilbert space of single-excitation states in [57]. We do not restrict ourselves to particular input states and develop a theory of transformations that map *any* initial state into a chosen doll. Note that mainstream research is focused on showing that a particular state is outside a given doll (mostly that of biseparable states) [7,38–43,54], whereas ours ensures the opposite and matches the recent approach of Ref. [58]. Our methodology relies on a neat decomposition of the physical map into simpler (but not necessary physical) processes involving entanglement-breaking operations [59]. The criteria obtained are formulated for general quantum channels.

To illustrate our approach, we discuss examples of local and global depolarizing noises modeling individual and common baths, respectively. Local depolarizing noises are relevant in quantum communication tasks (exploiting, e.g., optical fibers) as well as in purely physical systems such as nuclear spins in molecules [60]. Global depolarizing noise is an appropriate model in experiments where full-rank quantum states are detected [61,62] and is argued to be the worst-case scenario of system-environment interactions [63]. We find the noise levels of corresponding entanglement structure dissociations and reveal differences in the typical dissociation behavior between local and global noises.

The paper is organized as follows. In Sec. II, we precisely describe the multipartite entanglement formalism used, with attention being paid to higher-order partitions (tripartitions, tetrapartitions, etc.) which are often omitted from consideration. In Sec. III, we recall the necessary information about general and local quantum channels. In Sec. IV, the problem under investigation (the dynamics of entanglement structure) is precisely formulated. In Sec. V, we accomplish the development of the methodology and derive the criteria of entanglement dissociation and annihilation. In Sec. VI, we provide a recipe for applying the obtained criteria to the above-mentioned noises. In Sec. VII, the physical meaning of the results is discussed. In Sec. VIII, we concisely summarize the ideas, methods, and achieved results.

II. MULTIPARTITE ENTANGLEMENT FORMALISM

To express the idea of entanglement structure quantitatively, one can make use of the following formalism. Whenever we speak about entanglement, we imply a particular partition of the composite system. In general, an N-body system $ABC\ldots$ can be partitioned into k subsystems, where k ranges from 2 to N. If the system is not partitioned at all, we will reckon $k=1$. One can divide the N-body system $ABC\ldots$ into k subsystems (also referred to as parties) in $\{{N \atop k}\}$ different ways, where $\{{N \atop k}\} = \frac{1}{k!}\sum_{m=0}^{k}(-1)^m\binom{k}{m}(k-m)^N$ is the Stirling number of the second kind. Denote by $\mathcal{P}^k$ a set of possible partitions into k parties. Partitions are ordered in such a way that the parts with fewer bodies go first. Then, for a three-body system ABC we have $\mathcal{P}^1(ABC)=\{ABC\}$, $\mathcal{P}^2(ABC)=\{A|BC, B|AC, C|AB\}$, and $\mathcal{P}^3(ABC)=\{A|B|C\}$. In the case of a four-body system $ABCD$, the sets of possible partitions are $\mathcal{P}^1(ABCD)=\{ABCD\}$, $\mathcal{P}^2(ABCD)=\{A|BCD, B|ACD, C|ABD, D|ABC, AB|CD, AC|BD, AD|BC\}$, $\mathcal{P}^3(ABCD)=\{A|B|CD, A|C|BD, A|D|BC, B|C|AD, B|D|AC, C|D|AB\}$, and $\mathcal{P}^4(ABCD)=\{A|B|C|D\}$. Denote by $\mathcal{P}_j^k$ the jth partition of the set $\mathcal{P}^k$, e.g., $\mathcal{P}_5^3(ABCD)=B|D|AC$. In order to address the mth subsystem of the partition $\mathcal{P}_j^k$, we will use the notation $[\mathcal{P}_j^k]_m$, e.g., $[\mathcal{P}_5^3(ABCD)]_2=D$.

Quantum states of the system $ABC\ldots$ are described by density operators $\varrho^{ABC\cdots}$ (positive and with unit trace) acting on the Hilbert space $\mathcal{H}^{ABC\cdots} \equiv \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \otimes \cdots$ and all together forming the convex set $\mathcal{S}(\mathcal{H}^{ABC\cdots})$. A state ϱ is called separable with respect to a particular partition $\mathcal{P}_j^k$ if the resolution $\varrho = \sum_i \mu_i \varrho_i^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes \varrho_i^{[\mathcal{P}_j^k]_k}$ holds true for some probability distribution $\{\mu_i\}$ and density operators $\varrho_i^{[\mathcal{P}_j^k]_m}$, $m = 1,\ldots,k$. We will denote such a separable state as σ_j^k for brevity. If $\varrho \neq \sigma_j^k$ for any σ_j^k, then ϱ is said to be entangled with respect to the partition $\mathcal{P}_j^k$.

The above consideration of partitions is important because the physics of multipartite entanglement can be quite counterintuitive. For instance, the three-qubit state of Refs. [30,64] is separable with respect to any bipartition $\mathcal{P}_j^2$ but is entangled with respect to tripartition $\mathcal{P}^3$. Another example is a four-qubit Smolin state [65] which is separable with respect to bipartitions $\mathcal{P}_5^2, \mathcal{P}_6^2, \mathcal{P}_7^2$ and is entangled with respect to bipartitions $\mathcal{P}_1^2, \mathcal{P}_2^2, \mathcal{P}_3^2, \mathcal{P}_4^2$, any tripartition $\mathcal{P}_j^3$, and quartering $\mathcal{P}^4$.

Now we can define the concept of k-separability of a quantum state, which indicates that the state can accommodate components each of which has k separate parties. Namely, the state ϱ is called k-separable and denoted $\varrho_{k\text{-sep}}$ if it adopts the resolution $\sum_{j=1}^{\{ {N \atop k} \}} p_j^k \sigma_j^k$ for some probability distribution $\{p_j^k\}_j$ and separable density operators σ_j^k. Note that $\varrho_{k\text{-sep}}$ can still be entangled with respect to partitions $\mathcal{P}_j^k$ if $\{ {N \atop k} \} > 1$. Clearly, if the state is k-separable, then it is also $(k-1)$-separable, which implies the inclusion relation $\mathcal{S}_{N\text{-sep}} \subset \cdots \subset \mathcal{S}_{2\text{-sep}} \subset \mathcal{S}_{1\text{-sep}}$ for convex sets of k-separable states. A natural measure of separability appears:

$$K_{\text{sep}}[\varrho] := \max_{\varrho = \varrho_{k\text{-sep}}} k. \tag{1}$$

If $K_{\text{sep}}[\varrho] = 1$, then the state ϱ is called genuinely entangled. If $K_{\text{sep}}[\varrho] = N$, then the state ϱ is fully separable.

One can quantify multipartite entanglement in an alternative way by counting the number of bodies that are actually entangled [7,54,66,67]. This number would indicate the resources needed to create the state. For instance, the state $\varrho^{AB} \otimes \varrho^{CDE}$ of a five-body system $ABCDE$ is 2-separable but comprises a party CDE which can be genuinely entangled ($K_{\text{sep}}[\varrho^{CDE}] = 1$), i.e., requires three bodies to be entangled. To embody this idea in a precise manner, we introduce the following definition of resource intensiveness (compatible with the concepts of entanglement depth [68] and producibility [69]):

$$R_{\text{ent}}[\varrho] := \min_{\varrho = \sum_{k=1}^{N} \sum_{j=1}^{\{ {N \atop k} \}} p_j^k \sigma_j^k} \; \max_{m=1,\ldots,k} \#\left\{\left[\mathcal{P}_j^k\right]_m\right\}, \tag{2}$$

where $\#\{[\mathcal{P}_j^k]_m\}$ is the number of bodies within the mth subsystem of the partition $\mathcal{P}_j^k$.

Denote by $\mathcal{S}_{r\text{-ent}} = \{\varrho : R_{\text{ent}}[\varrho] \leqslant r\}$ the convex set of r-entangled states. Obviously, $\mathcal{S}_{1\text{-ent}} \subset \mathcal{S}_{2\text{-ent}} \subset \cdots \subset \mathcal{S}_{N\text{-ent}}$. Importantly, $\mathcal{S}_{1\text{-ent}} = \mathcal{S}_{N\text{-sep}}$, $\mathcal{S}_{(N-1)\text{-ent}} = \mathcal{S}_{2\text{-sep}}$, and $\mathcal{S}_{N\text{-ent}} = \mathcal{S}_{1\text{-sep}} = \mathcal{S}(\mathcal{H}^{ABC\cdots})$. Depending on the quantum state, the range of R_{ent} can be $\lceil \frac{N}{K_{\text{sep}}}, N - K_{\text{sep}} + 1]$ for a fixed K_{sep}, and the range of K_{sep} can be $\lceil \frac{N}{R_{\text{ent}}}, N - R_{\text{ent}} + 1]$ for a fixed R_{ent}.[1] The relations between two families of sets $\{\mathcal{S}_{k\text{-sep}}\}$ and $\{\mathcal{S}_{r\text{-ent}}\}$ for a four-body system are shown in Fig. 1.

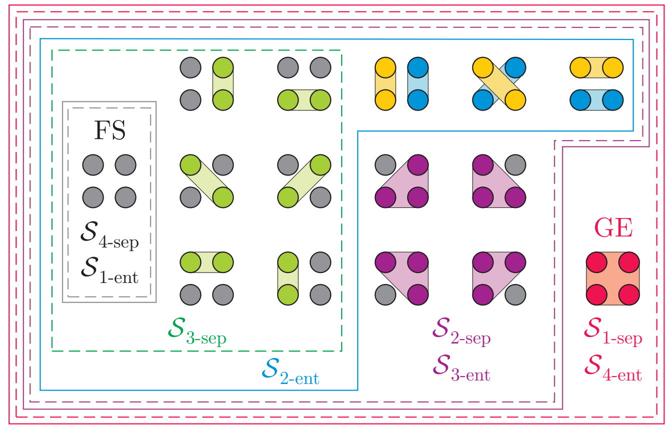

FIG. 1. (Color online) Schematic of sets $\mathcal{S}_{k\text{-sep}}$ (dashed) and $\mathcal{S}_{r\text{-ent}}$ (solid) for a four-body system.

III. QUANTUM DYNAMICS

We describe the physical evolution of open quantum systems by the input-output formalism of quantum channels: $\varrho_{\text{out}} = \Phi[\varrho_{\text{in}}]$, where $\Phi : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$ is a completely positive trace-preserving (CPT) linear map on trace-class operators $\mathcal{T}(\mathcal{H}_{\text{in}})$. The physical meaning of the evolution via a CPT map Φ can be readily seen from the Stinespring dilation [70]: $\Phi[\varrho_{\text{in}}] \equiv \text{tr}_{\text{env}}[U(\varrho_{\text{in}} \otimes \xi_{\text{env}})U^\dagger]$ for some state of the environment ξ_{env} and some unitary operator $U \in \mathcal{T}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{env}})$. Complete positivity (CP) of the map Φ acting on a system S guarantees that $(\Phi^S \otimes \text{Id}^{\text{anc}})[\varrho^{S+\text{anc}}] \geqslant 0$ for all composite states $\varrho^{S+\text{anc}} \in \mathcal{S}(\mathcal{H}^{S+\text{anc}})$ of the system S and an arbitrary ancilla, with Id being the identity transformation. Equivalently, the map Φ is CP if it adopts the diagonal sum representation $\Phi[X] = \sum_k A_k X A_k^\dagger$. If the Kraus operators $A_k : \mathcal{H}_{\text{in}} \mapsto \mathcal{H}_{\text{out}}$ satisfy $\sum_k A_k^\dagger A_k = I_{\text{in}}$ (identity operator), then Φ is CPT.

In order to define a linear map Φ acting on a system S, we will use the Choi-Jamiołkowski isomorphism [71,72]:

$$\Omega_\Phi^{SS'} := (\Phi^S \otimes \text{Id}^{S'})[|\Psi_+^{SS'}\rangle\langle\Psi_+^{SS'}|], \tag{3}$$

$$\Phi[X] = d^S \, \text{tr}_{S'}\big[\Omega_\Phi^{SS'}\big(I_{\text{out}}^S \otimes X^{\text{T}}\big)\big], \tag{4}$$

where $d = \dim\mathcal{H}$, $|\Psi_+^{SS'}\rangle = (d^S)^{-1/2} \sum_{i=1}^{d^S} |i \otimes i'\rangle$ is a maximally entangled state shared by system S and its clone S', $X^{\text{T}} = \sum_{i,j} \langle j|X|i\rangle |i'\rangle\langle j'| \in \mathcal{T}(\mathcal{H}_{\text{in}}^{S'})$ is the transposition in some orthonormal basis, and $\text{tr}_{S'}$ denotes the partial trace over S'. The linear map Φ^S is CP if and only if $\Omega_\Phi^{SS'} \geqslant 0$.

Since our main interest is focused on many-body systems, let us consider a composite system $S = ABC\ldots$ acted upon by some channel Φ^S. To begin with, $|\Psi_+^{SS'}\rangle = (d^A d^B d^C \cdots)^{-1/2} \sum_{i=1}^{d^A} \sum_{j=1}^{d^B} \sum_{k=1}^{d^C} \sum_{\ldots} |ijk\cdots\rangle$

[1] Hereafter, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

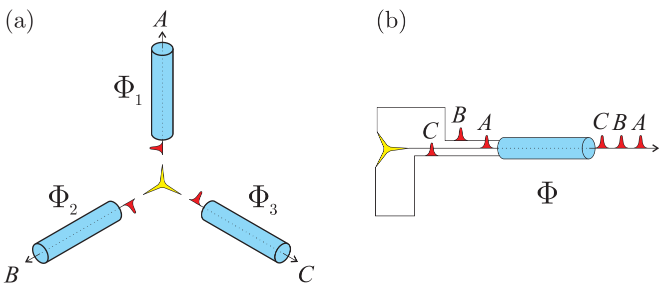

FIG. 2. (Color online) Local channels: (a) general; (b) homogeneous.

$\otimes |i'j'k'\cdots\rangle = |\Psi_+^{AA'}\rangle \otimes |\Psi_+^{BB'}\rangle \otimes |\Psi_+^{CC'}\rangle \otimes \cdots$, which explicitly shows the separability of the maximally entangled state with respect to the partition $AA'|BB'|CC'|\ldots$. While constructing the Choi operator (3), the map $\Phi^{ABC\ldots}$ can in general entangle these subsystems.

Suppose a local channel $\Phi_1^A \otimes \Phi_2^B \otimes \Phi_3^C \otimes \cdots$ which serves as an adequate model in situations when each particle is sent to a corresponding receiver through an individual quantum cable [Fig. 2(a)]. In this case, $\Omega^{ABC\ldots A'B'C'\ldots}_{\Phi_1\otimes\Phi_2\otimes\Phi_3\otimes\cdots} = \Omega^{AA'}_{\Phi_1} \otimes \Omega^{BB'}_{\Phi_2} \otimes \Omega^{CC'}_{\Phi_3} \otimes \cdots$. Clearly, $\Phi_1^A \otimes \Phi_2^B \otimes \Phi_3^C \otimes \cdots$ is CP if and only if each of the maps Φ_1^A, Φ_2^B, $\Phi_3^C,\ldots$ is CP.

In quantum communication, the typical scenario is to use a single quantum cable to transmit time-separated parties of a multipartite state from the encoder to the decoder [Fig. 2(b)]. Neglecting the memory effects, the evolution of a multipartite system is governed by the homogeneous local channel $\Phi^{\otimes N}$, which also appears in the definition of channel capacities (see, e.g., the review [73]).

IV. PROBLEM FORMULATION

Consider a composite N-body system $S = ABC\ldots$ that undergoes the physical evolution $\varrho_{\rm out} = \Phi[\varrho_{\rm in}]$ determined by some CPT map Φ (we also assume $\mathcal{H}^S_{\rm in} = \mathcal{H}^S_{\rm out}$). If $\varrho_{\rm out}$ is separable with respect to the partition $\mathcal{P}_j^k$ (i.e., $\varrho_{\rm out} = \sigma_j^k$), then we say that the channel Φ dissociates the entanglement compound of a given $\varrho_{\rm in}$ into smaller compounds of $[\mathcal{P}_j^k]_1,\ldots,[\mathcal{P}_j^k]_k$ and denote by $\mathcal{D}_j^k(\varrho_{\rm in})$ the set of such channels. If the channel Φ dissociates the entanglement of all input states $\varrho_{\rm in} \in \mathcal{S}(\mathcal{H}^S)$ in this way, then we will refer to Φ as dissociating entanglement with respect to the partition $\mathcal{P}_j^k$ and denote $\Phi \in \mathcal{D}_j^k \equiv \bigcap_{\varrho_{\rm in}\in\mathcal{S}(\mathcal{H}^S)} \mathcal{D}_j^k(\varrho_{\rm in})$.

Using entanglement measures (1) and (2), we can quantitatively describe the processes of entanglement structure dynamics. Namely, denote $k{\rm Sep}(\varrho_{\rm in})$ a set of channels Φ such that $K_{\rm sep}[\Phi[\varrho_{\rm in}]] \geqslant k$. By construction, $k{\rm Sep}(\varrho_{\rm in})$ is a convex hull of the sets $\mathcal{D}_j^k(\varrho_{\rm in})$. Similarly, $r{\rm Ent}(\varrho_{\rm in})$ is a set of channels Φ such that $R_{\rm ent}[\Phi[\varrho_{\rm in}]] \leqslant r$. Regarding state-independent properties, we straightforwardly introduce the sets of channels $k{\rm Sep} := \bigcap_{\varrho_{\rm in}\in\mathcal{S}(\mathcal{H}^S)} k{\rm Sep}(\varrho_{\rm in})$ and $r{\rm Ent} := \bigcap_{\varrho_{\rm in}\in\mathcal{S}(\mathcal{H}^S)} r{\rm Ent}(\varrho_{\rm in})$. The developed formalism of Sec. II immediately results in the following inclusion diagram for the above sets:

$$
\begin{array}{ccccccc}
N{\rm Sep} \subset & (N-1){\rm Sep} & \subset\cdots\subset & 2{\rm Sep} & \subset & 1{\rm Sep} \\
\| & & & \| & & \| \\
1{\rm Ent} \subset & 2{\rm Ent} & \subset\cdots\subset & (N-1){\rm Ent} & \subset & N{\rm Ent} \\
\| & & & \| & & \| \\
{\rm EA} & & & {\rm DGE} & & {\rm CPT}.
\end{array}
$$

We have used a special notation for two distinctive classes of channels:

(a) entanglement annihilating (EA) channels transforming any input state into a fully separable one [74];

(b) channels that dissociate genuine entanglement (DGE), thus transforming genuinely entangled states into nongenuinely entangled ones.

The problem under investigation is twofold: (i) to characterize the sets of channels $k{\rm Sep}(\varrho_{\rm in})$ and $r{\rm Ent}(\varrho_{\rm in})$ as well as state-independent sets from the above diagram, and (ii) to track how exactly the multiparticle entanglement structure dissociates under particular noises. Our special attention is paid to EA and DGE channels.

Before proceeding to the derivation of criteria, we need to clarify the relation between the problem involved and the well-known approaches developed so far. Consider a (not necessarily composite) system S acted upon by a channel $\Phi : \mathcal{T}(\mathcal{H}^S_{\rm in}) \mapsto \mathcal{T}(\mathcal{H}^S_{\rm out})$. If the Choi operator $\Omega^{SS'}_\Phi$ is separable with respect to the partition $S|S'$, then Φ is a so-called entanglement-breaking (EB) map [59,75], whose peculiarity is that $(\Phi^S \otimes {\rm Id}^{\rm anc})[\varrho^{S+{\rm anc}}]$ is separable with respect to the partition $S|{\rm anc}$ for all density operators $\varrho^{S+{\rm anc}} \in \mathcal{S}(\mathcal{H}^{S+{\rm anc}})$. In fact, separability of $\Omega^{SS'}_\Phi$ implies that Φ has the Holevo form $\Phi[X] = \sum_k {\rm tr}[F_k X]\omega_k$, where $\{F_k\}$ is a positive operator-valued measure and $\omega_k \in \mathcal{S}(\mathcal{H}_{\rm out})$, i.e., Φ is a measure-and-prepare procedure. The latter representation, in its turn, implies [59] that there exists a diagonal sum representation with rank-1 Kraus operators $A_k \propto |\varphi_k\rangle\langle\psi_k|$ with $|\psi_k\rangle \in \mathcal{H}^S_{\rm in}$ and $|\varphi_k\rangle \in \mathcal{H}^S_{\rm out}$.

As concerns a composite system $S = ABC\ldots$, the EB channel Φ^S disentangles S from any other system but can in principle result in any entanglement dynamics within S (among A, B, C, ...). For instance, the output state can be genuinely entangled or fully separable depending on the entanglement of vectors $|\varphi_k\rangle$ constituting Kraus operators. However, the local channel $\Phi^S = \Phi_1^A \otimes \Phi_2^B \otimes \Phi_3^C \otimes \cdots$ is entanglement breaking if and only if each of the channels Φ_1^A, Φ_2^B, Φ_3^C, ... is entanglement breaking. This can be readily seen from the requirement of separability of the Choi operator $\Omega^{ABC\ldots A'B'C'\ldots}_{\Phi_1\otimes\Phi_2\otimes\Phi_3\otimes\cdots} = \Omega^{AA'}_{\Phi_1} \otimes \Omega^{BB'}_{\Phi_2} \otimes \Omega^{CC'}_{\Phi_3} \otimes \cdots$ with respect to the partition $ABC\ldots|A'B'C'\ldots$. Thus, the local entanglement-breaking channel is automatically entanglement annihilating but the converse is not true. These and other differences between entanglement-breaking and entanglement-annihilating channels are discussed in [74,76,77].

V. METHODOLOGY AND CRITERIA

In this section, we provide criteria to detect the different kinds of entanglement dissociation discussed above. We start with a description of our methodology which is based on an extensive use of various convex sets of operators and maps.

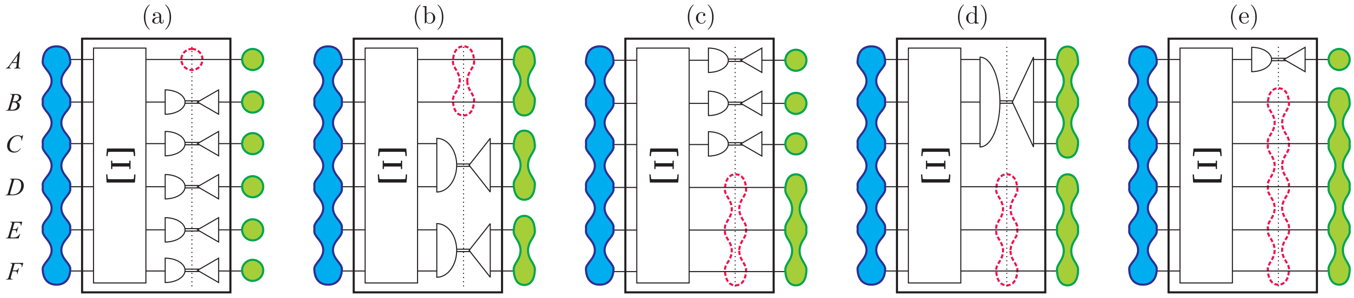

FIG. 3. (Color online) Elementary blocks of entanglement dissociation for the six-body system $ABCDEF$ constructed via concatenation of a linear Hermitian map Ξ and measure-and-prepare (EB) operations. Semicircles and triangles depict projections onto $|\psi_n\rangle$ and preparations of $|\varphi_n\rangle$ of Kraus operators $A_n \propto |\varphi_n\rangle\langle\psi_n|$, respectively, and double lines depict the classical information transfer. Only one restriction is imposed: $\Xi[\varrho_{\rm in}]$ becomes positive semidefinite after performing the "measure" part of EB operations (red dotted compound). Partitions: (a) $A|B|C|D|E|F$, (b) $AB|CD|EF$, (c) $A|B|C|DEF$, (d) $ABC|DEF$, and (e) $A|BCDEF$.

In addition to quantum states described by positive semidefinite unit-trace operators $\varrho \in \mathcal{S}(\mathcal{H}^{ABC\cdots})$, an important role will be played by block-positive operators [72]. The operator ξ_j^k is called block positive with respect to the partition $\mathcal{P}_j^k$ if it satisfies

$$\left\langle \psi_1^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes \psi_k^{[\mathcal{P}_j^k]_k} \middle| \xi_j^k \middle| \psi_1^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes \psi_k^{[\mathcal{P}_j^k]_k} \right\rangle \geqslant 0$$

for all vectors $\psi_1,\ldots,\psi_k$. Block-positive operators are closely related to entanglement witnesses [78,79] and can be used to determine separability: a state $\varrho \in \mathcal{S}(\mathcal{H}^{ABC\cdots})$ is separable with respect to the partition $\mathcal{P}_j^k$ if and only if ${\rm tr}[\varrho\xi_j^k] \geqslant 0$ for all block-positive operators ξ_j^k.

We must emphasize that the concepts of entanglement dissociation and annihilation from Sec. IV do not imply any ancillary system besides the multipartite system $S = ABC\ldots$ itself. This allows the CPT condition of the physical transformation Φ to be relaxed. We consider an extended set $\mathcal{E}[\Phi]$ of (mathematical) linear maps Υ having the same entanglement behavior as Φ on the corresponding domain of input states. For example, the extended set $\mathcal{E}[\mathcal{D}_j^k(\varrho_{\rm in})]$ consists of linear maps Υ satisfying the only restriction that $\Upsilon[\varrho_{\rm in}]$ is equal to some σ_j^k. Similarly, $\mathcal{E}[k{\rm Sep}(\varrho_{\rm in})]$ and $\mathcal{E}[r{\rm Ent}(\varrho_{\rm in})]$ denote the extensions of sets $k{\rm Sep}(\varrho_{\rm in})$ and $r{\rm Ent}(\varrho_{\rm in})$, respectively. As we show later, the extensions turn out to be useful because they adopt a good characterization. The original set of maps can be found by intersecting with CPT maps, e.g., $\mathcal{D}_j^k(\varrho_{\rm in}) = {\rm CPT} \cap \mathcal{E}[\mathcal{D}_j^k(\varrho_{\rm in})]$.

Proposition 1. Suppose a linear map Υ acting on a system $ABC\ldots$. Then $\Upsilon \in \mathcal{E}[\mathcal{D}_j^k(\varrho_{\rm in})]$ if and only if ${\rm tr}\{\Omega_\Upsilon^{ABC\ldots A'B'C'\ldots}[(\xi_j^k)^{ABC\ldots} \otimes (\varrho_{\rm in}^{\rm T})^{A'B'C'\ldots}]\} \geqslant 0$ for all ξ_j^k.

Proof. Separability of $\Upsilon[\varrho_{\rm in}]$ with respect to the partition $\mathcal{P}_j^k$ is equivalent to the inequality ${\rm tr}(\Upsilon[\varrho_{\rm in}]\xi_j^k) \geqslant 0$ for all ξ_j^k. Substituting (4) for $\Upsilon[\varrho_{\rm in}]$ concludes the proof. ■

As a result, the cone $\mathcal{E}[\mathcal{D}_j^k(\varrho_{\rm in})]$ is dual to the cone of maps $\Upsilon^\circ[X] = \xi_j^k {\rm tr}[\varrho_{\rm in}X]$. As concerns the state-independent property $\mathcal{D}_j^k$, the map Υ belongs to the set $\mathcal{D}_j^k$ if its Choi matrix satisfies ${\rm tr}\{\Omega_\Upsilon^{ABC\ldots A'B'C'\ldots}[(\xi_j^k)^{ABC\ldots} \otimes \varrho^{A'B'C'\ldots}]\} \geqslant 0$ for all ξ_j^k and $\varrho^{A'B'C'\ldots}$.

The criterion provided by Proposition 1 is not quite operational. To overcome this obstacle we derive sufficient criteria of entanglement dissociation.

Consider a particular partition $\mathcal{P}_j^k$. Suppose a linear map $\Xi : \mathcal{T}(\mathcal{H}_{\rm in}) \mapsto \mathcal{T}(\mathcal{H}_{\rm in})$ which transforms the density operator $\varrho_{\rm in}$ into some Hermitian (but not necessarily positive) operator $\Xi[\varrho_{\rm in}]$ such that

$$\begin{aligned}\Big\langle \psi_1^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes \psi_{m-1}^{[\mathcal{P}_j^k]_{m-1}} \otimes I \otimes \psi_{m+1}^{[\mathcal{P}_j^k]_{m+1}} \otimes \cdots \otimes \psi_k^{[\mathcal{P}_j^k]_k} \Big| \\ \times\, \Xi[\varrho_{\rm in}] \Big| \psi_1^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes \psi_{m-1}^{[\mathcal{P}_j^k]_{m-1}} \otimes I \otimes \psi_{m+1}^{[\mathcal{P}_j^k]_{m+1}} \\ \otimes \cdots \otimes \psi_k^{[\mathcal{P}_j^k]_k} \Big\rangle \geqslant 0 \qquad (5)\end{aligned}$$

is fulfilled for some vectors $\psi_1,\ldots,\psi_{m-1},\psi_{m+1},\ldots,\psi_k$, i.e., $\Xi[\varrho_{\rm in}]$ after projection onto these vectors becomes a positive operator from the cone $\mathcal{S}(\mathcal{H}^{[\mathcal{P}_j^k]_m})$. If this is the case, then for rank-1 Kraus operators $A_n \propto |\varphi_n\rangle\langle\psi_n|$ with arbitrary $|\varphi_n\rangle$, the operator $(A_1 \otimes \cdots \otimes A_{m-1} \otimes I \otimes A_{m+1} \otimes \cdots \otimes A_k)\Xi[\varrho_{\rm in}](A_1^\dagger \otimes \cdots \otimes A_{m-1}^\dagger \otimes I \otimes A_{m+1}^\dagger \otimes \cdots \otimes A_k^\dagger)$ belongs to a cone of separable states σ_j^k.

Thus, we obtain the following sufficient criterion of entanglement dissociation.

Proposition 2. Concatenation of a linear Hermitian map Ξ and a $(k-1)$-partite EB operation $(\mathcal{O}_{\rm EB}^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes {\rm Id}^{[\mathcal{P}_j^k]_m} \otimes \cdots \otimes \mathcal{O}_{\rm EB}^{[\mathcal{P}_j^k]_k})$ belongs to $\mathcal{E}[\mathcal{D}_j^k(\varrho_{\rm in})]$ if $\Xi[\varrho_{\rm in}]$ becomes positive after projection on right-singular vectors of the rank-1 Kraus operators of the EB operation.

The idea of Proposition 2 is shown for a six-body system in Fig. 3. The benefit of the constructed concatenation is that the map Ξ does not have to be positive[2] (in contrast to Ref. [77]), which makes the set $\mathcal{E}[\mathcal{D}_j^k(\varrho_{\rm in})]$ even larger.

When all possible states $\varrho_{\rm in}$ are considered, the satisfaction of requirement (5) becomes equivalent to the positivity of the map $(\mathcal{O}_{\rm EB}^{[\mathcal{P}_j^k]_1} \otimes \cdots \otimes {\rm Id}^{[\mathcal{P}_j^k]_m} \otimes \cdots \otimes \mathcal{O}_{\rm EB}^{[\mathcal{P}_j^k]_k}) \circ \Xi$. This map is automatically positive if Ξ transforms density operators into block-positive operators ξ_j^k, which in turn is equivalent to the fact that its Choi operator is block positive of the form $\Omega_\Xi^{ABC\ldots A'B'C'\ldots} = \xi^{\mathcal{P}_j^k(ABC\ldots)|A'B'C'\ldots}$.

[2]A linear map is called positive if it maps positive operators into positive ones.

TABLE I. Local depolarizing N-qubit channel Φ_q^{local}: ranges of parameter q, for which the various entanglement-dissociative behaviors are detected (within the interval $[-1/3,1]$).

N	ϱ_{in}	EA	$\frac{N}{2}$Sep$\bigcap$2Ent	$(\frac{N}{2}+1)$Sep$\bigcap\frac{N}{2}$Ent	2Sep$\bigcap\frac{N}{2}$Ent	$(N-1)$Ent=DGE	Not DGE	NPT$(1,N-1)$	NPT$(\frac{N}{2},\frac{N}{2})$
3	$\|\text{GHZ}\rangle$	$\leqslant 0.490$				$\leqslant 0.713$	>0.716[a,b]	>0.557	
	$\|W\rangle$	$\leqslant 0.485$				$\leqslant 0.686$	>0.772[a]	>0.576	
	ϱ_{UPB}	$\leqslant 0.698$				$\leqslant 0.852$	$\varnothing$	$\varnothing$	
	All	$\leqslant 0.477$				$\leqslant 0.650$			
4	$\|\text{GHZ}\rangle$	$\leqslant 0.453$	$\leqslant 0.548$	$\leqslant 0.553$	$\leqslant 0.548$	$\leqslant 0.751$	>0.781[a,b]	>0.578	>0.512
	$\|W\rangle$	$\leqslant 0.447$	$\leqslant 0.473$	$\leqslant 0.581$	$\leqslant 0.473$	$\leqslant 0.756$	>0.842[a]	>0.585	>0.548
	$\|\text{Cl}\rangle$	$\leqslant 0.444$	$\leqslant 0.478$	$\leqslant 0.574$	$\leqslant 0.478$	$\leqslant 0.742$	>0.774[a]	>0.532	>0.550
	All	$\leqslant 0.444$	$\leqslant 0.472$	$\leqslant 0.550$	$\leqslant 0.472$	$\leqslant 0.715$			
6	$\|\text{GHZ}\rangle$	$\leqslant 0.414$	$\leqslant 0.433$	$\leqslant 0.591$	$\leqslant 0.530$	$\leqslant 0.826$	>0.850[b]	>0.638	>0.490

[a]Computation via the method of Ref. [41].
[b]Computation via the method of Ref. [54].

Corollary 1. If $\Omega_{\Xi}^{ABC\ldots A'B'C'\ldots}$ is block positive with respect to the partition $\mathcal{P}_j^k(ABC\ldots)|A'B'C'\ldots$, then $(\mathcal{O}_{\text{EB}}^{[\mathcal{P}_j^k]_1}\otimes\cdots\otimes\text{Id}^{[\mathcal{P}_j^k]_m}\otimes\cdots\otimes\mathcal{O}_{\text{EB}}^{[\mathcal{P}_j^k]_k})\circ\Xi\in\mathcal{D}_j^k$ for arbitrary EB operations.

The sets $\mathcal{E}[k\text{Sep}(\varrho_{\text{in}})]$ and $\mathcal{E}[r\text{Ent}(\varrho_{\text{in}})]$ are nothing else but appropriate convex hulls of sets $\mathcal{E}[\mathcal{D}_j^k(\varrho_{\text{in}})]$ which can be detected by Proposition 2. Let us remember, however, that we are interested in characterizing sets $k\text{Sep}(\varrho_{\text{in}})$ and $r\text{Ent}(\varrho_{\text{in}})$ of physical (CPT) maps. Since the map Φ under investigation is originally CPT, its decomposition into mathematical maps of the above propositions does not change this fact but ensures that it belongs to a desired set of maps. Therefore, we have the following statement.

Proposition 3. Suppose a quantum channel Φ can be decomposed into the sum $\Phi=\sum_{\mathcal{P}_j^k\in\mathsf{P}}\mathcal{M}_j^k$, where each elementary map $\mathcal{M}_j^k\in\mathcal{E}[\mathcal{D}_j^k(\varrho_{\text{in}})]$ is constructed via Proposition 2. If P is a subset of partitions contributing to k-separable or r-entangled states, then Φ belongs to $k\text{Sep}(\varrho_{\text{in}})$ or $r\text{Ent}(\varrho_{\text{in}})$, respectively.

Similarly, to detect maps from the state-independent sets kSep and rEnt one can use Corollary 1 instead of Proposition 2 in the statement of Proposition 3.

VI. APPLICABILITY OF CRITERIA TO DEPOLARIZING CHANNELS

The sufficient criterion to detect $k\text{Sep}(\varrho_{\text{in}})$ and $r\text{Ent}(\varrho_{\text{in}})$ channels, Proposition 3, implies the existence of the specific decomposition of the channel of interest, Φ. In this section, we provide a recipe for construction of such a decomposition for relatively simple one-parametric families of channels Φ. Although we do not raise the question of optimality, our findings enable us to reveal features of the entanglement structure dynamics.

A general depolarizing map $\Phi:\mathcal{T}(\mathcal{H}_d)\mapsto\mathcal{T}(\mathcal{H}_d)$ is given by the formula $\Phi=q\text{Id}+(1-q)\text{Tr}$, where $\text{Tr}[X]=\text{tr}[X]\frac{1}{d}I_d$ is the tracing map. The map Φ represents a valid channel (CPT map) if $q\in[-(d^2-1)^{-1},1]$. Let us consider two one-parametric families of channels acting on N qubits: the local depolarizing noise $\Phi_q^{\text{local}}\equiv\Phi_q^{\otimes N}$, where Φ_q is a single-qubit map ($d=2$), and the global depolarizing noise Φ_q^{global} ($d=2^N$). Our goal is the following: for fixed k and r, find the region of parameter q such that the channel Φ_q^{local} (or Φ_q^{global}) surely adopts the decomposition into elementary blocks constituting $k\text{Sep}\cap r\text{Ent}$.

In what follows, we do not restrict the number of qubits N but, in view of the enormous number of possible partitions, we consider the most interesting cases. All of them represent channels dissociating genuine entanglement but correspond to various structures of output states:

(a) $k=N$ and $r=1$; the output state is fully separable (EA channels).

(b) $k=\frac{N}{2}$ and $r=2$; the output-state entanglement mixture is composed of pairs of entangled particles.

(c) $k=\frac{N}{2}+1$ and $r=\frac{N}{2}$; the biggest clusters in the output-state entanglement mixture cannot contain more than $\frac{N}{2}$ particles, with the remaining $\frac{N}{2}$ particles being disentangled.

(d) $k=2$ and $r=\frac{N}{2}$; the output-state entanglement contains mixtures of two or more clusters of maximum size $\frac{N}{2}$.

(e) $k=2$ and $r=N-1$; at least one particle is separated from entanglement compounds in the output-state entanglement mixture (the biggest subset of DGE channels).

For $N=6$ the elementary blocks of these kinds of channels are illustrated in Fig. 3.

Since the depolarizing channels under investigation are permutationally invariant, we also consider all possible permutations of elementary blocks. This is equivalent to relabeling of particles and, therefore, leads to a simplification of the analysis of permutationally invariant input states.

To anticipate the results, in Tables I and II we present the ranges of parameter q for which the depolarizing channels Φ_q^{local} and Φ_q^{global}, respectively, fall into one of the classes (a)–(e). Within these ranges, the existence of a corresponding decomposition in the statement of Proposition 3 can be shown [we sum up technical details for each class (a)–(e) in the forthcoming sections of the same label]. The column "Not DGE" in Tables I and II is based on detection of geunine entanglement according to Refs. [41,54]. The last two columns in Tables I and II are based on the conventional negativity under partial transpose (NPT) entanglement criterion for most asymmetric bipartitions (one body vs $N-1$ bodies) and symmetric bipartition ($\frac{N}{2}$ bodies vs $\frac{N}{2}$ bodies). In the following Secs. VI A–VI E, we present algebra leading

TABLE II. Global depolarizing N-qubit channel $\Phi_q^{\rm global}$: ranges of parameter q for which the various entanglement-dissociative behaviors are detected (within the interval $[-(2^{2N}-1)^{-1},1]$).

N	$\varrho_{\rm in}$	EA	$\frac{N}{2}$Sep $\bigcap$ 2Ent	$(\frac{N}{2}+1)$Sep $\bigcap$ $\frac{N}{2}$Ent	2Sep $\bigcap$ $\frac{N}{2}$Ent	$(N-1)$Ent=DGE	Not DGE	NPT$(1,N-1)$	NPT$(\frac{N}{2},\frac{N}{2})$
3	$\lvert\mathrm{GHZ}\rangle$	$\leqslant$0.147				$\leqslant$0.402	$>$0.429[a,b]	$>$0.200	
	$\lvert W\rangle$	$\leqslant$0.125				$\leqslant$0.317	$>$0.479[a]	$>$0.210	
	$\varrho_{\rm UPB}$	$\leqslant$0.400				$\leqslant$0.690	$\varnothing$	$\varnothing$	
	All	$\leqslant$0.111				$\leqslant$0.289			
4	$\lvert\mathrm{GHZ}\rangle$	$\leqslant$0.062	$\leqslant$0.202	$\leqslant$0.111	$\leqslant$0.202	$\leqslant$0.262	$>$0.467[a,b]	$>$0.112	$>$0.112
	$\lvert W\rangle$	$\leqslant$0.048	$\leqslant$0.123	$\leqslant$0.124	$\leqslant$0.123	$\leqslant$0.256	$>$0.474[a]	$>$0.127	$>$0.112
	$\lvert\mathrm{Cl}\rangle$	$\leqslant$0.052	$\leqslant$0.123	$\leqslant$0.109	$\leqslant$0.123	$\leqslant$0.229	$>$0.385[a]	$>$0.112	$>$0.112
	All	$\leqslant$0.047	$\leqslant$0.121	$\leqslant$0.107	$\leqslant$0.121	$\leqslant$0.184			
6	$\lvert\mathrm{GHZ}\rangle$	$\leqslant$0.011	$\leqslant$0.034	$\leqslant$0.032	$\leqslant$0.046	$\leqslant$0.131	$>$0.493[b]	$>$0.031	$>$0.031

[a]Computation via the method of Ref. [41].
[b]Computation via the method of Ref. [54].

to the parameters q for the classes of channels (a)–(e) above.

A. Entanglement-annihilating channels

The elementary block of EA channels is obtained by applying entanglement-breaking operations on $N-1$ particles [see Fig. 3(a)]. The exact form of EB operations chosen reads $\mathcal{O}_{\mathrm{EB}\psi_i}[X]=\frac{1}{2}|\psi_i\rangle\langle\psi_i|X|\psi_i\rangle\langle\psi_i|$, where $\{\frac{1}{2}|\psi_i\rangle\langle\psi_i|\}_{i=1}^4$ form a symmetric informationally complete positive operator-valued measure (SIC-POVM) for qubits (see the explicit analytical form of the vectors $\{|\psi_i\rangle\}_{i=1}^4$ in [80]). This choice is justified by the fact that $\sum_{i=1}^4\mathcal{O}_{\mathrm{EB}\psi_i}=\Phi_{q=1/3}$. (The same result would be obtained by using projectors on mutually unbiased bases [81] instead of SIC-POVM elements; however, this approach leads to worse results for some input states $\varrho_{\rm in}$.) The suggested decomposition reads

$$\Phi=\frac{1}{N}\sum_{m=1}^{N}\Big(\Phi_{q=1/3}^{[\mathcal{P}^N]_1}\otimes\cdots\otimes\mathrm{Id}^{[\mathcal{P}^N]_m}\otimes\cdots\otimes\Phi_{q=1/3}^{[\mathcal{P}^N]_N}\Big)\circ\Xi_a(m), \tag{6}$$

where m is the index of a particle not subjected to EB operations. We have taken into account that each $\Phi_{q=1/3}$ is composed of EB operations $\mathcal{O}_{\mathrm{EB}\psi_i}$ and therefore it is convenient to parametrize the map Ξ_a in such a way that the vectors $|\psi_i\rangle$ are not included in the parametrization directly. However, the linear map $\Xi_a(m)$ should satisfy the requirement (5) for all choices of vectors $|\psi_{i_t}\rangle_{t=1}^{N-1}$ from the set $\{|\psi_i\rangle\}_{i=1}^4$. To parametrize the map $\Xi_a(m)$ we resort to a so-called diagonal map of the form

$$\Xi[X]=\frac{1}{2^N}\sum_{i_1,\ldots,i_N=0,\ldots,3}x_{i_1\cdots i_N}\mathrm{tr}[(\varsigma_{i_1}\otimes\cdots\otimes\varsigma_{i_N})X]\times\varsigma_{i_1}\otimes\cdots\otimes\varsigma_{i_N}, \tag{7}$$

where $\varsigma_0=I_2$ and $\varsigma_1,\varsigma_2,\varsigma_3$ are conventional Pauli matrices. Let $\#_0[i_1\cdots i_N]$ denote the number of zeros in the sequence $i_1,\ldots,i_N$. Consider diagonal maps $\Xi_a(m)$ such that the coefficients $\{x_{i_1\cdots i_N}\}$ depend on $\#_0[i_m]$ and $\#_0[i_1\cdots i_{m-1}i_{m+1}\cdots i_N]$ only, i.e., $x_{i_1\cdots i_N}=f_a(\#_0[i_m],\#_0[i_1\cdots i_{m-1}i_{m+1}\cdots i_N])$, with restrictions on the parameters $\{f_a\}$ being imposed by (5). Then the relation (6) becomes valid if

$$\frac{n}{3^{n-1}N}f_a(0,N-n)+\frac{N-n}{3^nN}f_a(1,N-n-1)=\begin{cases}q^n, & \Phi=\Phi_q^{\rm local},\\ q^{1-\delta_{n,0}}, & \Phi=\Phi_q^{\rm global},\end{cases}\quad n=0,\ldots,N, \tag{8}$$

where $\delta_{s,t}$ is the conventional Kronecker delta.

For a fixed input state $\varrho_{\rm in}$, we find the restrictions on the parameters $\{f_a\}$ given by (5) and then solve the system of equations (8) numerically. If the system has a solution for some $\tilde{q}$, then it also has a solution for $q<\tilde{q}$. Solutions (max $\tilde{q}$) are presented for some interesting states[3] $\varrho_{\rm in}$ of $N=3,4,6$ qubits in Tables I and II for local and global noises, respectively. We also consider the case of all possible input states as follows: since Ξ_a linearly depends on the parameters $\{f_a\}$, we check the corresponding block positivity of Ω_{Ξ_a} (see Corollary 1) for some number of parameters $\{f_a\}$ and construct a convex hull of satisfactory parameters; then we solve the system of equations (8) for $\{f_a\}$ from the convex hull; the maximum value of q for which the system has a solution is presented in Tables I and II in the rows "All."

B. $\frac{N}{2}$Sep $\bigcap$ 2Ent channels

The output state will be $\frac{N}{2}$-separable and 2-entangled if the channel can be decomposed into elementary transformations $\mathcal{E}[\mathcal{D}_j^{N/2}(\varrho_{\rm in})]$ from Proposition 2, each containing $(\frac{N}{2}-1)$ EB operations $\mathcal{O}_{\rm EB}$ on two qubits [see Fig. 3(b)]. As in the previous section, we choose EB operations of the form $\mathcal{O}_{\mathrm{EB}\psi_i}[X]=\frac{1}{4}|\psi_i\rangle\langle\psi_i|X|\psi_i\rangle\langle\psi_i|$, where $\{\frac{1}{4}|\psi_i\rangle\langle\psi_i|\}_{i=1}^{16}$ form a SIC-POVM in $\mathcal{T}(\mathcal{H}_4)$ (see the explicit analytical form of the vectors $\{|\psi_i\rangle\}_{i=1}^{16}$ in [80]). Then $\sum_{i=1}^{16}\mathcal{O}_{\mathrm{EB}\psi_i}^{AB}=\Phi_{q=1/5}^{AB}$

[3]The states of interest are $|\mathrm{GHZ}\rangle=(1/\sqrt{2})(|0\rangle^{\otimes N}+|1\rangle^{\otimes N})$; $|W\rangle=(1/\sqrt{N})(|10\ldots0\rangle+|01\ldots0\rangle+\cdots+|00\ldots1\rangle)$; $\varrho_{\rm UPB}=(1/4)(I_8-P_{\rm UPB})$, where $P_{\rm UPB}$ is a projector on unextendable product bases for three qubits [30,64]; $|\mathrm{Cl}\rangle=(1/2)(|0000\rangle+|0011\rangle+|1100\rangle-|1111\rangle)$.

is a depolarizing map acting on two qubits (A and B) simultaneously. The decomposition of channel Φ reads

$$\Phi = \binom{N}{2}^{-1} \sum_{\mathcal{P}_j^{N/2} \in \mathsf{P}} \sum_{m=1}^{N/2} \left(\Phi_{q=1/5}^{[\mathcal{P}_j^{N/2}]_1} \otimes \cdots \otimes \mathrm{Id}^{[\mathcal{P}_j^{N/2}]_m} \otimes \cdots \otimes \Phi_{q=1/5}^{[\mathcal{P}_j^{N/2}]_{N/2}} \right) \circ \Xi_b(j,m), \tag{9}$$

where P is a set of $\frac{N!}{2^{N/2}(N/2)!}$ partitions $\mathcal{P}_j^{N/2}$ such that $\#[\mathcal{P}_j^{N/2}]_1 = \cdots = \#[\mathcal{P}_j^{N/2}]_{N/2} = 2$ (two qubits in each party), and the map $\Xi_b(j,m)$ must meet the condition (5) for all choices of vectors $|\psi_{i_t}\rangle_{t=1}^{(N/2)-1}$ from the set $\{|\psi_i\rangle\}_{i=1}^{16}$. Using diagonal maps $\Xi_b(j,m)$ of the form (7) with the parametrization $x_{i_1\cdots i_N} = f_b(\#_0[i_l i_{l'}],\#_0[i_1 \cdots i_{l-1} i_{l+1} \cdots i_{l'-1} i_{l'+1} \cdots i_N])$, $(i_l i_{l'}) \in [\mathcal{P}_j^{N/2}]_m$, we obtain the following system of equations:

$$\binom{N}{2}^{-1} \left\{ \binom{n}{2} \frac{f_b(0,N-n)}{5^{\lceil n/2 \rceil - 1}} + n(N-n) \frac{f_b(1,N-n-1)}{5^{\lfloor n/2 \rfloor}} + \binom{N-n}{2} \frac{f_b(2,N-n-2)}{5^{\lceil n/2 \rceil}} \right\}$$
$$= \begin{cases} q^n, & \Phi = \Phi_q^{\mathrm{local}}, \\ q^{1-\delta_{n,0}}, & \Phi = \Phi_q^{\mathrm{global}}, \end{cases} \quad n = 0,\ldots,N. \tag{10}$$

The maximal values of q for which the system has a solution compatible with (5) are presented for various input states in Tables I and II.

C. $(\frac{N}{2}+1)$Sep $\bigcap$ $\frac{N}{2}$Ent channels

The output state will be $(\frac{N}{2}+1)$-separable and $\frac{N}{2}$-entangled if the channel can be decomposed into elementary transformations $\mathcal{E}[\mathcal{D}_j^{N/2+1}(\varrho_{\mathrm{in}})]$, $j = 1,\ldots,\binom{N}{N/2}$ [for such j's, the N-body system is divided into $\frac{N}{2}$ single-body parts plus one part comprising $\frac{N}{2}$ bodies; see Fig. 3(c)]. To find the decomposition for Proposition 3, we use the single-qubit EB operations $\mathcal{O}_{\mathrm{EB}\psi_i}$, $|\psi_i\rangle \in \mathcal{H}_2$, $i = 1,\ldots,4$, as in Sec. VI A. This yields the following decomposition:

$$\Phi = \binom{N}{N/2}^{-1} \sum_{j=1}^{\binom{N}{N/2}} \left(\Phi_{q=1/3}^{[\mathcal{P}_j^{N/2+1}]_1} \otimes \cdots \otimes \Phi_{q=1/3}^{[\mathcal{P}_j^{N/2+1}]_{N/2}} \otimes \mathrm{Id}^{[\mathcal{P}_j^{N/2+1}]_{N/2+1}} \right) \circ \Xi_c(j), \tag{11}$$

where the map $\Xi_c(j)$ must satisfy the requirement (5) for all choices of vectors $|\psi_{i_t}\rangle_{t=1}^{N/2}$ from the set $\{|\psi_i\rangle\}_{i=1}^{4}$ (vectors corresponding to a SIC-POVM for a qubit). Using diagonal maps $\Xi_c(j)$ of the form (7) with the parametrization $x_{i_1\cdots i_N} = f_c(\#_0[\{i_1 \cdots i_N\} \setminus \{i_{l_1} \cdots i_{l_{N/2}}\}],\#_0[i_{l_1} \cdots i_{l_{N/2}}])$, $(i_{l_1} \cdots i_{l_{N/2}}) \in [\mathcal{P}_j^{N/2+1}]_{N/2+1}$, we obtain the following system of equations:

$$\binom{N}{N/2}^{-1} \sum_{l=0}^{N/2} \binom{n}{N/2-l} \binom{N-n}{l} \frac{f_c(l,N-n-l)}{3^{N/2-l}} = \begin{cases} q^n, & \Phi = \Phi_q^{\mathrm{local}}, \\ q^{1-\delta_{n,0}}, & \Phi = \Phi_q^{\mathrm{global}}, \end{cases} \quad n = 0,\ldots,N. \tag{12}$$

The maximal values of q for which the system has a solution compatible with (5) are presented for various input states in Tables I and II.

D. 2Sep $\bigcap$ $\frac{N}{2}$Ent channels

The output state will be 2-separable and $\frac{N}{2}$-entangled if the channel can be decomposed into elementary transformations $\mathcal{E}[\mathcal{D}_j^2(\varrho_{\mathrm{in}})]$, $j = \{{N \atop 2}\} - \frac{1}{2}\binom{N}{N)/2} + 1,\ldots,\{{N \atop 2}\}$ [this choice of j's corresponds to bipartitions of an N-body system into equal $\frac{N}{2}$-body parts; see Fig. 3(d)]. We use the following EB operations $\mathcal{O}_{\mathrm{EB}}$ on $\frac{N}{2}$ qubits: $\mathcal{O}_{\mathrm{EB}\psi_i}[X] = \frac{1}{2^{N/2}} |\psi_i\rangle\langle\psi_i|X|\psi_i\rangle\langle\psi_i|$, where $\{|\psi_i\rangle\}_{i=1}^{2^N}$ is a set of normalized vectors such that $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^{2^N}$ is a set of SIC projectors (see the explicit forms of vectors $\{|\psi_i\rangle\}_{i=1}^{2^N}$ up to $N = 12$ in [80]). [Let us recall that a particular form of vectors $|\psi_i\rangle$ is important only for a particular input state ϱ_{in}. If the input state is arbitrary, i.e., the domain is $\mathcal{S}(\mathcal{H}_2^{\otimes N})$, then one should not care about the specific form of EB operations.] The important fact is that $\sum_{i=1,\ldots,2^N} \mathcal{O}_{\mathrm{EB}\psi_i} = \Phi_{q=(2^{N/2}+1)^{-1}}$ is a depolarizing map acting on $\frac{N}{2}$ qubits. The possible decomposition reads

$$\Phi = \binom{N}{N/2}^{-1} \sum_{j=\{{N \atop 2}\}-\frac{1}{2}\binom{N}{N/2}+1}^{\{{N \atop 2}\}} \sum_{m=1}^{2}$$
$$\times \left(\Phi_{q=(2^{N/2}+1)^{-1}}^{[\mathcal{P}_j^2]_m} \otimes \mathrm{Id}^{[\mathcal{P}_j^2]_{\{1,2\}\setminus m}} \right) \circ \Xi_d(j,m), \tag{13}$$

where $\Xi_d(j,m)$ must satisfy condition (5) for all vectors $\{|\psi_i\rangle\}_{i=1}^{2^N}$ and for the corresponding domain of density operators ϱ_{in}. For computational reasons let us note that checking the validity of (5) is less time consuming when we justify the positivity of the Hermitian operator without revealing its eigenvalues. Namely, the eigenvalues of a $d \times d$ Hermitian matrix X are non-negative if and only if $C_k \geqslant 0$ for $k = 1,\ldots,d$, where C_k is given by the recurrence relation $C_k = \frac{1}{k} \sum_{l=1}^{k} (-1)^{l-1} C_{k-l} \mathrm{tr}[X^l]$ with initial condition $C_0 = 1$ [82]. We use this technique for $N = 6$.

Using diagonal maps $\Xi_d(j,m)$ of the form (7) with the parametrization $x_{i_1\cdots i_N} = f_d(\#_0[i_{l_1}\cdots i_{l_{N/2}}],\#_0[\{i_1\cdots i_N\}\setminus\{i_{l_1}\cdots i_{l_{N/2}}\}])$, $(i_{l_1}\cdots i_{l_{N/2}})\in[\mathcal{P}_j^2]_m$, we obtain the following system of equations:

$$\binom{N}{N/2}^{-1}\sum_{l=0}^{N/2}\binom{n}{N/2-l}\binom{N-n}{l}\frac{f_d(l,N-n-l)}{(2^{N/2}+1)^{1-\delta_{l,N/2}}} = \begin{cases} q^n, & \Phi=\Phi_q^{\text{local}},\\ q^{1-\delta_{n,0}}, & \Phi=\Phi_q^{\text{global}},\end{cases} \quad n=0,\ldots,N. \tag{14}$$

The maximal values of q for which the system has a solution are presented for various ϱ_{in} in Tables I and II.

E. Channels dissociating genuine entanglement

The elementary blocks of these channels can be obtained by applying an EB operation on a single qubit [Fig. 3(e)]. We use the same EB operations as in Sec. VI A. This yields the decomposition

$$\Phi = \frac{1}{N}\sum_{m=1}^{N}\left(\Phi_{q=1/3}^{[\mathcal{P}_{j=m}^2]_1}\otimes \mathrm{Id}^{[\mathcal{P}_{j=m}^2]_2}\right)\circ\Xi_e(m), \tag{15}$$

where the map $\Xi_e(m)$ must satisfy (5) for all vectors $\{|\psi_i\rangle\}_{i=1}^4$ corresponding to a SIC-POVM for a qubit. Using diagonal maps $\Xi_e(m)$ of the form (7) with the parametrization $x_{i_1\cdots i_N} = f_e(\#_0[i_m],\#_0[i_1\cdots i_{m-1}i_{m+1}\cdots i_N])$, we find that (15) becomes a valid equality if

$$\frac{n}{3N}f_e(0,N-n)+\frac{N-n}{N}f_e(1,N-n-1) = \begin{cases} q^n, & \Phi=\Phi_q^{\text{local}},\\ q^{1-\delta_{n,0}}, & \Phi=\Phi_q^{\text{global}},\end{cases} \quad n=0,1,\ldots,N. \tag{16}$$

The maximal values of q for which the system has a solution and (5) is fulfilled are presented for various input states in Tables I and II.

VII. DISCUSSION

To begin with, the NPT criterion gives a little information about the multipartite entanglement structure. Indeed, one can observe in Tables I and II many situations when $\Phi[\varrho_{\text{in}}]$ is either negative under partial transpose but not genuinely entangled, or positive under partial transpose but not fully separable. The gap between states $\Phi[\varrho_{\text{in}}]$ that are surely not genuinely entangled and those that are definitely genuinely entangled is quite narrow for particular input states. This can be treated as an indication of the efficiency of the rather simple decomposition (15) involving single-qubit entanglement-breaking operations.

The remarkable fact is that our method enables us to consider *all* input states and find channels that transform any of them to a particular entanglement structure. This is what we mean by a "typical" behavior. For example, we can detect channels that annihilate entanglement, for which the output state is always fully separable whatever the input state is. Note that the bounds obtained on q for entanglement annihilation are higher than those that can be found via the condition of sufficiently small purity $\mathrm{tr}\{(\Phi[\varrho_{\text{in}}])^2\}$ [83]. Scaling of EA for the local depolarizing channel is shown in Fig. 4(a). When $N\to\infty$, the channel $\Phi_q^{\otimes N}$ cannot be EA if $q>\frac{1}{\sqrt{5}}$ [19]; however, the question of whether EA equals EB still remains an open problem. On the contrary, from formula (16) one can see that the genuine entanglement of any input state can be dissociated by a negligible noise in the limit $N\to\infty$ [Fig. 4(b)].

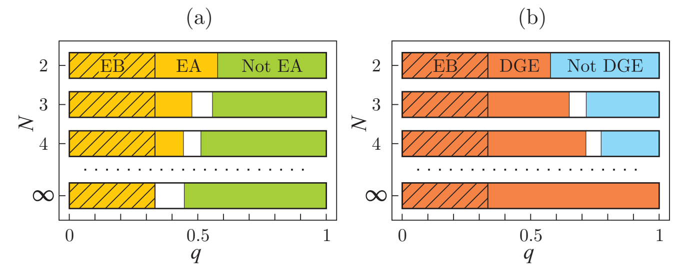

FIG. 4. (Color online) Scaling of the entanglement degradation properties of an N-qubit local depolarizing channel $\Phi_q^{\otimes N}$ with increasing N: (a) entanglement annihilation; (b) dissociation of genuine entanglement.

The dissipative dynamics under consideration can be described by gradually decreasing the parameter $q\sim e^{-\Gamma t}$, where the dissipation rate Γ takes, in principle, different values for local and global noises. For our purposes it is enough to know that q continuously diminishes from $q=1$ to $q=0$. For such types of dissipative dynamics, the state evolution through the nested sets of Sec. II is irreducible: once the state comes into a particular "doll" of the structure, it cannot escape it in the future.

Using the data from Table I, we may conclude that the dissociation of genuine multiparticle entanglement under local depolarizing noise starts by detaching a single random particle [i.e., the state becomes $(N-1)$-entangled]. Then the noise detaches particles one by one, resulting in k-separable $(N-k+1)$-entangled states (k increases with decreasing q). Indeed, since the noise is local, once a particle is detached from the entanglement compound, there is no way for it to rejoin it (Fig. 5). Finally, the noisy evolution makes the state fully separable.

The analysis of Table II shows that the entanglement dissociation progresses in a different way under global depolarizing noise: while the beginning stage also implies detaching of a single random particle from the entanglement compound, in further dynamics this particle can fuse with another one and form a two-particle entanglement cluster that is detached from the main compound (a convex combination of such states). The process continues until the point when the original compound is divided into two clusters (2-separable $\frac{N}{2}$-entangled state); then the detachment of particles and their successive fusion result in the formation of more entanglement clusters of smaller size (k-separable $\frac{N}{k}$-entangled state; k increases with decreasing q), and so on until full separability (see Fig. 5 for the case of six qubits).

VIII. SUMMARY

Our study was motivated by the necessity to know the multiparticle entanglement structure and its vulnerability to

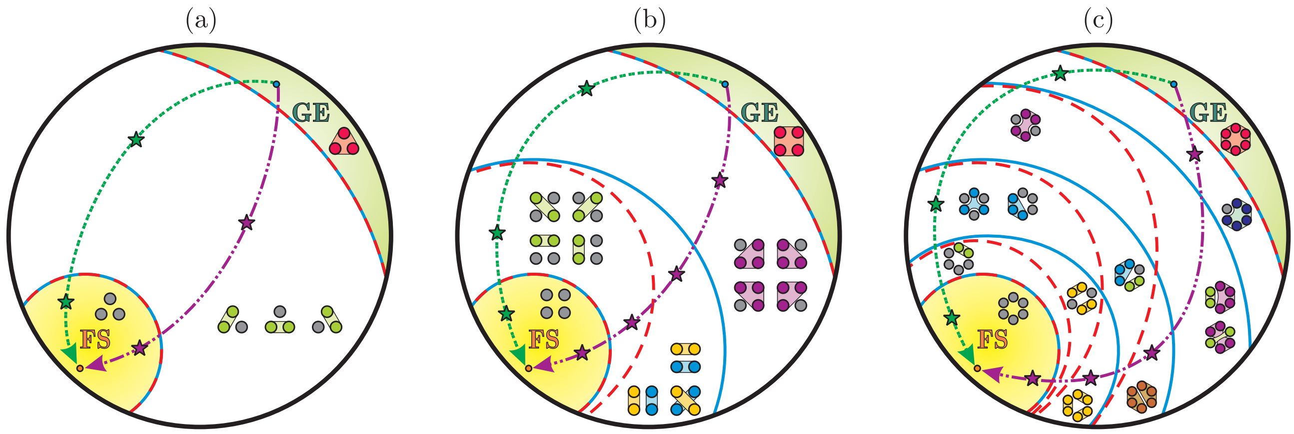

FIG. 5. (Color online) Tracks of the typical entanglement structure dynamics subject to local depolarizing noise (green dotted line) and global depolarizing noise (purple dash-dotted line) for N-qubit systems: (a) $N=3$, (b) $N=4$, and (c) $N=6$. The state space $\mathcal{S}(\mathcal{H}_2^{\otimes N})$ is divided into areas of k-separable states (red dashed lines) and r-entangled states (blue solid lines), with representatives of the states being depicted. Stars on the tracks denote points detected in Sec. VI and listed in Tables I and II.

noises in physical and quantum-informational applications. We did not restrict ourselves to specific input states and considered the set of all possible states as well. We found criteria for maps dissociating entanglement with respect to a particular partition and developed sufficient conditions for their reliable detection. Namely, the channel of interest should adopt a decomposition into (not necessarily completely positive) linear maps which give rise to the desired form of the output. One can draw a rough analogy between this decomposition and the path integral formulation of quantum mechanics, where the trajectories can be quite nonphysical but this does not affect the resulting physical evolution. For local and global depolarizing N-qubit channels we provided a simple strategy for constructing decompositions that allowed us to find noise levels guaranteeing the particular form of entanglement structure. Our decompositions are not optimal and can in principle be improved by applying modifications of semidefinite programming [41] and other algorithms [58] for Choi operators. Nevertheless, our toolbox allowed us to reveal differences in entanglement structure dynamics under local and global noises: the particles split one by one from the entanglement compound in the case of local noise, and tend to form clusters in the case of global noise. We believe that the obtained results may be extended to other noise models and provide additional information about the general rules of the dynamics of multiparticle entanglement structure.

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