Coexistence does not imply joint measurability

2013 J. Phys. A: Math. Theor. 46 462002

(http://iopscience.iop.org/1751-8121/46/46/462002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **46** (2013) 462002 (4pp) doi:10.1088/1751-8113/46/46/462002

FAST TRACK COMMUNICATION

Coexistence does not imply joint measurability

David Reeb, Daniel Reitzner and Michael M Wolf

Department of Mathematics, Technische Universität München, 85748 Garching, Germany

E-mail: david.reeb@tum.de, danreitzner@gmail.com and m.wolf@tum.de

Received 7 October 2013
Published 1 November 2013
Online at stacks.iop.org/JPhysA/46/462002

Abstract

One of the hallmarks of quantum theory is the realization that distinct measurements cannot in general be performed simultaneously, in stark contrast to classical physics. In this context the notions of *coexistence* and *joint measurability* are employed to analyze the possibility of measuring together two general quantum observables, characterizing different degrees of compatibility between measurements. It is known that two jointly measurable observables are always coexistent, and that the converse holds for various classes of observables, including the case of observables with two outcomes. Here we resolve, in the negative, the open question of whether this equivalence holds in general. Our resolution strengthens the notions of coexistence and joint measurability by showing that both are robust against small imperfections in the measurement setups.

PACS numbers: 03.65.Ta, 03.65.Ca, 03.67.−a

It is well known that two quantum observables can in general not be measured together [1]. In describing the relation between two or more quantum observables, several related notions are in use. The most prominent ones are: commutativity (COM), non-disturbance (ND), joint measurability (JM), and coexistence (COEX) [2–6]. Whereas, as the names suggest, joint measurability and non-disturbance can easily be understood in operational terms, commutativity and coexistence at first glance rely more on the underlying mathematical representation of quantum observables.

The connections between all these properties are well studied for pairs of general quantum observables, which are given in terms of *positive operator-valued measures (POVMs)*. If the POVMs are projection-valued—the case considered in most undergraduate quantum physics textbooks—then all four notions turn out to coincide, which may explain why they are sometimes used interchangeably. In general, we know that

$$\mathsf{COM} \Rightarrow \mathsf{ND} \Rightarrow \mathsf{JM} \Rightarrow \mathsf{COEX}$$

holds, and that the first two implications are strict in the sense that the reverse implications do not hold in general [2]. The last implication, however, appears to be more subtle: while joint measurability is known to imply coexistence, it is a persistent open problem whether the converse holds as well [6–8]. The present communication resolves this problem.

We begin by recalling the basic definitions and setting the notation. On a complex Hilbert space $\mathcal{H}$, a linear operator E with $0 \leqslant E \leqslant \mathbb{1}$ is called an *effect*. The set of effects is denoted by $\mathcal{E}(\mathcal{H})$. A general quantum *observable* (or *measurement*) is described by a POVM A, which is a countably additive mapping $A : \mathcal{A} \to \mathcal{E}(\mathcal{H})$ on a σ-algebra $\mathcal{A}$ of subsets of Ω_A satisfying $A(\Omega_A) = \mathbb{1}$. The set Ω_A represents the possible outcomes of the measurement. For any input state ρ describing the initial preparation of the quantum system and for any $X \in \mathcal{A}$, the expression $\mathrm{tr}[\rho A(X)]$ then gives the probability of obtaining a measurement outcome $x \in X$ [9]. We denote by $\mathrm{ran}(A) := \{A(X)|X \in \mathcal{A}\}$ the set of effects corresponding to A.

For the results below it will be sufficient to consider finite outcome sets $\Omega_A = \{1, \ldots, n\}$ equipped with the discrete σ-algebra $\mathcal{A}$ that contains all subsets of Ω_A; we call A an n-outcome observable. In this case, A is fully determined by the effects $A_k := A(\{k\})$ for $k \in \{1, \ldots, n\}$, and abusing notation we then write $A = (A_1, \ldots, A_n)$.

We now define the two notions whose relationship we want to clarify.

Definition 1 (Coexistence)**.** *Two POVMs A and B are called coexistent if there exists a POVM M such that* $\mathrm{ran}(A) \cup \mathrm{ran}(B) \subseteq \mathrm{ran}(M)$.

The notion of *coexistence* was introduced for effects and for observables by Ludwig [3] and refined to the present definitions by Busch, Lahti and Mittelstaedt [10]. Coexistence of the two observables A and B ensures that each effect of A or B can be simulated by the measurement M, and even that all binary observables that can be formed from A and B can be measured simultaneously, but it does not directly provide a way to measure the entire observables A and B simultaneously.

A simultaneous measurement is possible when A and B are both marginals of a single observable. This is captured by the following notion:

Definition 2 (Joint measurability)**.** *Two POVMs* $A : \mathcal{A} \to \mathcal{E}(\mathcal{H})$ *and* $B : \mathcal{B} \to \mathcal{E}(\mathcal{H})$ *are jointly measurable if there exists a POVM* $J : \mathcal{J} \to \mathcal{E}(\mathcal{H})$ *on the* σ*-algebra* $\mathcal{J}$ *generated by* $\mathcal{A} \times \mathcal{B}$*, such that for all* $X \in \mathcal{A}$ *and* $Y \in \mathcal{B}$*,*

$$A(X) = J(X \times \Omega_B), \qquad B(Y) = J(\Omega_A \times Y).$$

Joint measurability of two observables immediately implies their coexistence. The converse also follows easily for binary observables: the two-outcome POVMs $A = (A_1, \mathbb{1} - A_1)$ and $B = (B_1, \mathbb{1} - B_1)$ are coexistent if and only if they are jointly measurable [6, 11, 12].

Beyond this case of two outcomes, several broad classes of observables have been identified for which coexistence and joint measurability are equivalent [5, 6, 12–14]: for example projection-valued POVMs [5]; all cases in which one of the POVMs is determined by a discrete set of linearly independent rank-1 effects, as noticed very recently [14][1,2] or POVMs with effects contained in a regular effect algebra [12]. So far, however, it has been an open question whether the equivalence holds for all pairs of observables [6–8].

We answer this question by providing an instance of coexistent observables that are not jointly measurable. The example has $|\Omega_A| = 3$ and $|\Omega_B| = 2$ and is thus minimal in terms of the number of outcomes beyond the known two-outcome case.

[1] Note that this work followed the initial submission of the present work as a preprint at arXiv:1307.6986.

[2] The results of [14] have recently been extended to the case where one of the POVMs is discrete and extreme, but not necessarily with rank-1 effects (T Heinosaari, private communication).

Let $\{|1\rangle, |2\rangle, |3\rangle\}$ be an orthonormal basis in $\mathcal{H} = \mathbb{C}^3$ and $|\psi\rangle := (|1\rangle + |2\rangle + |3\rangle)/\sqrt{3}$. Consider the following effects:

$$A_i := \tfrac{1}{2}\big(\mathbb{1} - |i\rangle\langle i|\big), \quad i \in \{1, 2, 3\},$$
$$B_1 := \tfrac{1}{2}\, |\psi\rangle\langle\psi|, \quad B_2 := \mathbb{1} - B_1.$$

Proposition 1. *The POVMs* $A := (A_1, A_2, A_3)$ *and* $B := (B_1, B_2)$ *are coexistent, but not jointly measurable.*

Proof. To prove coexistence of A and B, each of which has at most three outcomes, we have to construct a POVM whose range contains each A_i and B_j. The five-outcome observable

$$M := \big(\tfrac{1}{2}|1\rangle\langle 1|, \tfrac{1}{2}|2\rangle\langle 2|, \tfrac{1}{2}|3\rangle\langle 3|, B_1, \tfrac{1}{2}\mathbb{1} - B_1\big)$$

clearly does the job. Concerning joint measurability we argue by contradiction. Suppose the observables A and B were jointly measurable. Then, by definition 2, there exist effects $J_{ij} \geqslant 0$ such that

$$\forall i: \sum_{j=1}^{2} J_{ij} = A_i, \quad \text{and} \quad \forall j: \sum_{i=1}^{3} J_{ij} = B_j\,. \tag{1}$$

Since by equation (1) the positive-semidefinite operators J_{i1} sum to the rank-1 operator B_1, we must necessarily have $J_{i1} = c_i B_1$ for some numbers $c_i \geqslant 0$. Hence, again by equation (1), $A_i = c_i B_1 + J_{i2}$, which, after taking the overlap $\langle i| \cdot |i\rangle$, becomes

$$0 = \frac{c_i}{2}|\langle i|\psi\rangle|^2 + \langle i|J_{i2}|i\rangle \qquad \forall i \in \{1, 2, 3\}.$$

This implies $c_i = 0$ for all i due to $|\langle i|\psi\rangle|^2 = 1/3$ and $\langle i|J_{i2}|i\rangle \geqslant 0$. Then, however, $J_{i1} = 0$ for all i, and equation 1 leads to the desired contradiction $B_1 = \sum_i J_{i1} = 0$. □

By padding both POVMs from proposition 1 with effects $0 \in \mathcal{E}(\mathcal{H})$, one sees that for every $n \geqslant 3$, $m \geqslant 2$ there exist n- resp. m-outcome POVMs A and B that are coexistent but not jointly measurable.

Proposition 1 enables a geometric picture of joint measurability and coexistence for pairs of observables on a fixed Hilbert space $\mathcal{H}$ of finite dimension at least 3. First consider the pair (I_n, I_m), where $I_k := (\mathbb{1}/k, \ldots, \mathbb{1}/k)$ denotes the k-outcome POVM corresponding to the toss of an unbiased k-sided coin. Obviously, I_n and I_m are jointly measurable. Since n- resp. m-outcome observables A and B are jointly measurable whenever all their effects satisfy $A_i \geqslant \mathbb{1}/2n$ and $B_j \geqslant \mathbb{1}/2m$ [15], any pair (A, B) sufficiently close to (I_n, I_m) is jointly measurable as well. Within the set ALL of all pairs of POVMs, the set JM of jointly measurable pairs has thus a non-empty open interior, see figure 1. By definition 2, JM is furthermore convex and closed in the direct sum space of all pairs. Closedness of JM is ensured for pairs of finite-outcome observables on the finite-dimensional $\mathcal{H}$ since the joint observables J from definition 2 then live in a compact set.

The set COEX of coexistent pairs of n- resp. m-outcome POVMs is convex as well since coexistence of A and B is equivalent to the joint measurability of the collection of all binary POVMs that can be formed from A and B. The preceding observations thus lead to the conclusion depicted in figure 1: since proposition 1 guarantees the existence of a pair $(A, B) \in \text{COEX}\setminus\text{JM}$ for $n \geqslant 3$, $m \geqslant 2$, the intersection of COEX\JM with the convex set spanned by (A, B) and JM has a non-empty open interior. Therefore, the set COEX\JM itself has a non-empty open interior, i.e. positive volume.

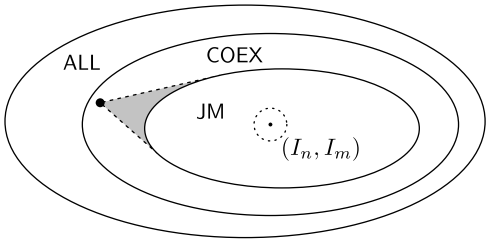

Figure 1. An illustration of the compact convex sets of all pairs (A, B) of n- resp. m-outcome observables (ALL), of coexistent pairs (COEX), and of jointly measurable pairs (JM). The set JM contains an open neighborhood around the pair (I_n, I_m) of uniformly random observables and thus has positive volume. For $n \geqslant 3$, $m \geqslant 2$, the existence of the pair (solid dot) from proposition 1 implies that the set difference COEX $\setminus$ JM has positive volume (see shaded area), whereas COEX = JM whenever $n, m \leqslant 2$ [6, 11, 12]. By similar reasoning, ALL $\setminus$ COEX has positive volume iff $n, m \geqslant 2$.

We emphasize that the latter conclusion resolves the question answered by this communication in a strong sense: whereas it was previously unknown whether there exists even one coexistent but not jointly measurable pair of observables, our proposition 1 implies that both notions are different and that this difference is not merely an exceptional or spurious effect. Rather, a positive fraction of all pairs of observables are jointly measurable, and another positive fraction are coexistent but not jointly measurable. This ensures stability features against small perturbations in the distinction between coexistence and joint measurability, which makes both notions more meaningful and robust in experimental setups.

Acknowledgments

The authors thank Toby Cubitt, Teiko Heinosaari, and Jukka Kiukas for valuable comments.

References

[1] Heisenberg W 1927 *Z. Phys.* **43** 172
[2] Heinosaari T and Wolf M M 2010 *J. Math. Phys.* **51** 092201
[3] Ludwig G 1983 *Foundations of Quantum Mechanics I* (Berlin: Springer)
[4] Busch P, Grabowski M and Lahti P 1995 *Operational Quantum Physics* (Berlin: Springer)
[5] Lahti P and Pulmannová S 1997 *Rep. Math. Phys.* **39** 339
[6] Lahti P 2003 *Int. J. Theor. Phys.* **42** 893
[7] Carmeli C, Heinonen T and Toigo A 2005 *J. Phys. A: Math. Gen.* **38** 5253
[8] Busch P, Kiukas J and Lahti P 2010 *Math. Slovaca* **60** 665
[9] Holevo A 2011 *Probabilistic and Statistical Aspects of Quantum Theory* (Pisa: Edizioni della Normale)
[10] Busch P, Lahti P and Mittelstaedt P 1991 *The Quantum Theory of Measurement* (Berlin: Springer)
[11] Kraus K 1983 *States, Effects, and Operations* (Berlin: Springer)
[12] Lahti P and Pulmannová S 2001 *Rep. Math. Phys.* **47** 199
[13] Dvurečenskij A, Lahti P, Pulmannová S and Ylinen K 2005 *Rep. Math. Phys.* **55** 241
[14] Pellonpää J-P 2013 arXiv:1308.3154 [quant-ph]
[15] Busch P, Heinosaari T, Schultz J and Stevens N 2013 *Europhys. Lett.* **103** 10002