Local two-qubit entanglement-annihilating channels

Sergey N. Filippov,[1] Tomáš Rybár,[2] and Mário Ziman[2,3]
[1]*Moscow Institute of Physics and Technology, Moscow Region, Russia*
[2]*Institute of Physics, Slovak Academy of Sciences, Bratislava, Slovakia*
[3]*Institute for Theoretical Physics, ETH Zurich, CH-8093 Zurich, Switzerland*
(Received 17 October 2011; published 4 January 2012)

We address the problem of the robustness of entanglement of bipartite systems (qubits) interacting with dynamically independent environments. In particular, we focus on the characterization of so-called local entanglement-annihilating (EA) two-qubit channels, which set the maximum permissible noise level allowing us to perform entanglement-enabled experiments. The differences, but also the subtle relations, between entanglement-breaking and local EA channels are emphasized. A detailed characterization of the latter ones is provided for a variety of channels including depolarizing, unital, (generalized) amplitude-damping, and extremal channels. We also consider the convexity structure of local EA qubit channels and introduce a concept of EA duality.

DOI: 10.1103/PhysRevA.85.012303 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

The flourishing field of quantum-information theory is obliged to the phenomenon of quantum entanglement [1] exhibited by multipartite quantum systems. In the past decades, many entanglement-enabled applications of quantum states have been developed and experimentally have been realized, such as quantum-key distribution, dense coding, quantum teleportation, etc. (see the detailed review in Ref. [2]). These quantum-information protocols operate efficiently providing that the entanglement between the involved parties (Alice and Bob) is preserved. However, during the protocols, the involved systems interact with an environment, which introduces (practically) unavoidable noise. As a result of these influences, Alice and Bob manipulate modified states whose entanglement can differ substantially from the original one. It even may happen that the systems become disentangled whatever state they start with. Under such circumstances, no entanglement-enabled application is implementable.

For purposes of quantum-communication protocols, it is reasonable to assume that the influence of environments of Alice and Bob are independent. That is, the joint noise applied on the shared state $\rho_{\rm in}$ is of the form $\mathcal{E}_1 \otimes \mathcal{E}_2$, where $\mathcal{E}_1,\mathcal{E}_2$ are local channels describing the interaction of Alice's and Bob's subsystem, respectively, with their environments. The question of our interest is the robustness of the initial entanglement with respect to local noises, i.e., to characterize the entanglement properties of states $\rho_{\rm out} = (\mathcal{E}_1 \otimes \mathcal{E}_2)[\rho_{\rm in}]$. Different variations in this problem of so-called entanglement dynamics were addressed in a number of papers (see, e.g., Refs. [3–12]) where the time evolution of two-qubit entanglement was studied for different physical systems (initial state, types of interqubit interactions, and environments). Many researchers have also paid attention to the phenomena known in the literature as sudden death and sudden birth of the entanglement (see, e.g., Refs. [13,14] and references therein). In contrast to the papers where the time evolution of the entanglement is deduced from the time evolution of the state, an attempt to find a direct relation (inequality) involving the initial and final entanglements of an arbitrary bipartite two-qubit state in the presence of local noises was undertaken in the papers [15–18].

Recently, a related concept of entanglement-annihilating (EA) channels was introduced [19]. These channels destroy any quantum entanglement completely within the system they act on. Following the paper [19], we refer to a local two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ as EA if the output state $(\mathcal{E}_1 \otimes \mathcal{E}_2)[\rho_{\rm in}]$ is separable for *all* input states $\rho_{\rm in}$. Therefore, the question of whether the channel is EA or not is a question of whether the noise level is acceptable for entanglement-enabled quantum applications or not.

It is worth emphasizing the contrast between EA and entanglement-breaking (EB) channels. Let us be reminded that a channel $\mathcal{E}$ (acting on some system) is called EB if, for all its extensions $\mathcal{E} \otimes \mathcal{I}_{\rm anc}$, it annihilates the entanglement between the system and the ancilla. In particular, a local two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EB if the output state $\rho_{\rm out}^{+\rm anc} = (\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{I}_{\rm anc})[\rho_{\rm in}^{+\rm anc}]$ is disentangled with respect to partitioning $1+2|{\rm anc}$ for any input state and any dimension of the ancillary system. The EB channels and their properties have been discussed widely in the literature (see, e.g., Refs. [20–25]). As shown in Ref. [19], even if channels $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \mathcal{E}$ are not EB, channel $\mathcal{E} \otimes \mathcal{E}$ can cancel any entanglement between quantum subsystems in interest. It means that, in order to fulfill an entanglement-enabled protocol, it does not suffice to know whether the individual local influences are described by an EB channel or not, one has to resort to the concept of the EA channel.

Our goal in this paper is to investigate, in detail, the properties of EA channels for the simplest case of the two-qubit system. In Sec. II, we briefly review some known properties of EA channels and add a new one for EA channels of the form $\mathcal{E} \otimes \mathcal{E}$. Such channels naturally occur in physical experiments where two parties experience the same influence from the environment and, thus, undergo the same local transformation $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Although such channels do not describe the general case, they are of significant physical relevance. For the sake of convenience, if $\mathcal{E} \otimes \mathcal{E}$ is EA, we will refer to a single-qubit channel $\mathcal{E}$ as a *two-locally EA (2-LEA) channel* .

In order to demonstrate the difference between the EA and the EB local two-qubit channels, in Sec. III we consider the simplest and the most widely used model of quantum noise, depolarizing channels. We find the overall noise level under which the entanglement is destroyed for all input states or can be preserved for some quantum states. Since depolarizing channels belong to the class of unital channels, in Sec. IV, we further focus our attention on this class. These channels describe important physical processes that do not increase the purity of the states.

In Sec. V, we move on to the EA behavior of nonunital channels. At first, we address the question of which extremal single-qubit channels $\mathcal{E}_1$ and $\mathcal{E}_2$ result in EA channels $\mathcal{E}_1 \otimes \mathcal{E}_2$? Then, we consider amplitude-damping and generalized amplitude-damping channels as the most prominent representatives of nonunital channels. In Sec. VI, we are reminded that the set of all two-qubit EA channels (including nonlocal ones) is convex [19]. This fact motivates us to find EA-extremal channels and to determine their positions with respect to the set of all two-qubit channels and its extreme points. In Sec. VII, we find it quite interesting to reveal and to briefly outline an EA duality between subsets of local channels. Finally, we summarize the obtained results in Sec. VIII.

II. PROPERTIES OF EA CHANNELS

To begin with, we epitomize some basic properties of general EA channels found in Ref. [19]:

(1) The set $\mathsf{T}_{\rm EA}$ of all EA channels (including nonlocal ones) is convex.

(2) The channel is EA if and only if it destroys entanglement of all pure input states.

(3) If $\mathcal{G}_{12}$ is EA (local or nonlocal), then $\mathcal{G}_{12} \cdot \mathcal{F}_{12}$ is EA for all two-qubit channels $\mathcal{F}_{12}$.

(4) Channels $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\mathcal{E}_2 \otimes \mathcal{E}_1$ exhibit the same EA behavior.

(5) If $\mathcal{E}_1$ or $\mathcal{E}_2$ is EB, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA. This follows immediately from the definitions of EA and EB channels.

(6) Channel $\mathcal{E} \otimes \mathcal{I}$ is EA if and only if $\mathcal{E}$ is EB.

(7) If $\mathcal{E}$ is 2-LEA and $\mathcal{F}$ is EB, then the convex combination $\mu\mathcal{E} + (1-\mu)\mathcal{F}$ is 2-LEA, $\mu \in [0,1]$.

The last property was not shown in Ref. [19]. In order to prove it, recall the definition of 2-LEA channels from the previous section. Now, suppose the composite channel $\mu^2\mathcal{E}\otimes\mathcal{E} + \mu(1-\mu)\mathcal{E}\otimes\mathcal{F} + \mu(1-\mu)\mathcal{F}\otimes\mathcal{E} + (1-\mu)^2\mathcal{F}\otimes\mathcal{F}$. Then, channel $\mathcal{E}\otimes\mathcal{E}$ is EA by definition of the 2-LEA channel $\mathcal{E}$, the rest of the channels $\mathcal{E}\otimes\mathcal{F}$, $\mathcal{F}\otimes\mathcal{E}$, and $\mathcal{F}\otimes\mathcal{F}$ are EA in view of property (5). The convexity property (1) concludes the proof of property (7).

Is is worth mentioning that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EB if and only if both $\mathcal{E}_1$ and $\mathcal{E}_2$ are EB. Thus, if the local channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EB, then it also is EA by property (5).

III. CASE STUDY: DEPOLARIZING CHANNELS

The action of a depolarizing channel on the jth qubit ($j = 1,2$) is defined as follows:

$$\mathcal{E}_j[X] = q_j X + (1-q_j)\mathrm{tr}[X]\tfrac{1}{2}I, \tag{1}$$

where I denotes the identity operator and $q_j \in [-\frac{1}{3},1]$ specifies the range of parameters q_j for which the map is completely positive. As a result of such noise, the Bloch spheres of individual qubits symmetrically shrink (in all directions). The class of these channels was used in Ref. [19] to show the existence of not-EB 2-LEA channels. Entanglement dynamics under the action of the locally depolarizing two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ was considered in Ref. [26]. In this section, we analyze the EB and EA properties of such channels.

Each channel $\mathcal{E}_j$ is known to be EB if and only if $q_j \leqslant \frac{1}{3}$ (see, e.g., Ref. [23]). If $q_j \leqslant \frac{1}{3}$, then the jth qubit becomes disentangled from the arbitrary environment including the rest of the qubits. Therefore, the two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EB if and only if $q_1 \leqslant \frac{1}{3}$ and $q_2 \leqslant \frac{1}{3}$ simultaneously.

Let us find out when $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA and destroys any entanglement between qubits. We resort to property (2) and consider pure input states $\omega = |\psi\rangle\langle\psi|$. We use the Schmidt decomposition of the state vector $|\psi\rangle = \sqrt{p}|\varphi\otimes\chi\rangle + \sqrt{p_\perp}|\varphi_\perp\otimes\chi_\perp\rangle$ where $\{|\varphi\rangle,|\varphi_\perp\rangle\}$ and $\{|\chi\rangle,|\chi_\perp\rangle\}$ are suitable orthonormal bases of the first and second qubits, respectively, and p and $p_\perp$ are real non-negative numbers such that $p + p_\perp = 1$.

Action of the two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ on state ω yields

$$\begin{aligned}\omega_{\rm out} = (\mathcal{E}_1\otimes\mathcal{E}_2)[\omega] &= q_1q_2\omega + \tfrac{1}{2}(1-q_1)q_2 I\otimes\omega_2 \\ &\quad + \tfrac{1}{2}q_1(1-q_2)\omega_1\otimes I + \tfrac{1}{4}(1-q_1)(1-q_2)I\otimes I,\end{aligned} \tag{2}$$

with the reduced states $\omega_1 = p|\varphi\rangle\langle\varphi| + p_\perp|\varphi_\perp\rangle\langle\varphi_\perp|$ and $\omega_2 = p|\chi\rangle\langle\chi| + p_\perp|\chi_\perp\rangle\langle\chi_\perp|$. According to the Peres-Horodecki criterion [27,28], the output state $\omega_{\rm out}$ is separable if the partially transposed operator $\omega_{\rm out}^\Gamma$ is positive semidefinite. The condition $\omega_{\rm out}^\Gamma \geqslant 0$ reduces to

$$\begin{vmatrix} A+B & C \\ C & A-B \end{vmatrix} = A^2 - B^2 - C^2 \geqslant 0, \tag{3}$$

where $A = 1 - q_1q_2$, $B = (2p-1)(q_1-q_2)$, and $C = 4q_1q_2\sqrt{pp_\perp} = 4q_1q_2\sqrt{p(1-p)}$. After simplification, we obtain

$$(1+q_1q_2)(1-3q_1q_2) + 4\big(p-\tfrac{1}{2}\big)^2\big[4q_1^2q_2^2 - (q_1-q_2)^2\big] \geqslant 0.$$

The channel in question is EA if this inequality holds for all $p \in [0,1]$. If $2|q_1q_2| \geqslant |q_1-q_2|$, then the minimum in p is achieved for $p = \frac{1}{2}$, and we end up with the inequality $(1+q_1q_2)(1-3q_1q_2) \geqslant 0$. Taking that $-\frac{1}{3} \leqslant q_1, q_2 \leqslant 1$ into account, this inequality holds whenever $q_1q_2 \leqslant \frac{1}{3}$. If $2|q_1q_2| < |q_1-q_2|$, then the minimum is achieved for $p = 0$ or $p = 1$, and for this case, the inequality takes the form $(1-q_1^2)(1-q_2^2) \geqslant 0$, which is always satisfied for all allowed values of q_1,q_2.

To summarize, channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA if and only if $q_1q_2 \leqslant \frac{1}{3}$. On the contrary, such a two-qubit channel is EB if and only if $q_1 \leqslant \frac{1}{3}$ and $q_2 \leqslant \frac{1}{3}$ simultaneously (see Fig. 1).

IV. UNITAL CHANNELS

In this section, we focus our investigation on the class of qubit unital channels. By definition, a channel is unital if it preserves the identity operator, i.e., $\mathcal{E}[I] = I$. As shown in Ref. [29], any such channel can be expressed as a diagonal matrix (acting on Bloch vectors) in a properly chosen basis of self-adjoint operators $\{\sigma_0 \equiv I,\sigma_1,\sigma_2,\sigma_3\}$ where $\mathrm{tr}[\sigma_j\sigma_k] = 2\delta_{jk}$ and δ_{jk} is the conventional Kronecker δ symbol. Channels of this (diagonal) form also are known as Pauli channels.

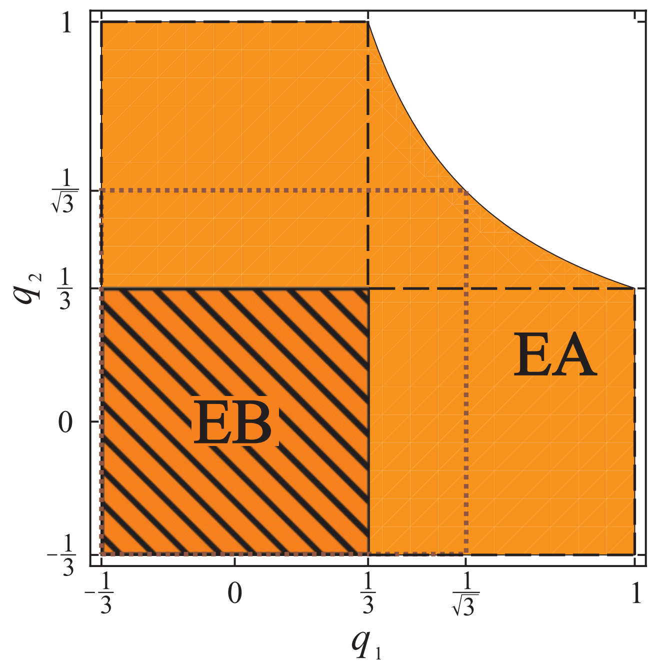

FIG. 1. (Color online) EA and EB local two-qubit channels $\mathcal{E}_1 \otimes \mathcal{E}_2$ composed of depolarizing channels $\mathcal{E}_j[X] = q_j X + (1 - q_j)\mathrm{tr}[X]\frac{1}{2}I$ and $j = 1,2$. Dashed lines depict regions where, at least, one of the channels $\mathcal{E}_1$ or $\mathcal{E}_2$ is EB and $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA [see property (5)]. The dotted line depicts the region where both $\mathcal{E}_1^2$ and $\mathcal{E}_2^2$ are EB, i.e., $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA (see Corollary 1).

It follows that channel $\mathcal{E}_1 \otimes \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are unital single-qubit channels, has a diagonal matrix representation in the basis of individual Pauli operators $\{\sigma_m \otimes \sigma_n'\}_{m,n=0}^3$. In particular, the entries of the matrix representation of $\mathcal{E}_1 \otimes \mathcal{E}_2$ read

$$\tfrac{1}{4}\mathrm{tr}[\sigma_k \otimes \sigma_l'(\mathcal{E}_1 \otimes \mathcal{E}_2)[\sigma_m \otimes \sigma_n']] = \lambda_m \lambda_n' \delta_{km}\delta_{ln},$$

where $\lambda_0 = \lambda_0' = 1$ in view of the trace-preserving property and $\{\lambda_m\}_{m=1}^3$ and $\{\lambda_n'\}_{n=1}^3$ are singular values of channels $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. The output state of the local two-qubit unital channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ takes the form

$$\rho_{\mathrm{out}} = \frac{1}{4}\sum_{m,n=0}^{3} \lambda_m \lambda_n' \mathrm{tr}[\rho_{\mathrm{in}}\sigma_m \otimes \sigma_n']\sigma_m \otimes \sigma_n'. \quad (4)$$

For each unital qubit channel $\mathcal{E}$, we introduce a unital map $\overline{\mathcal{E}}$ with $\lambda_0 = 1$ and singular values $\{-\lambda_m\}_{m=1}^3$, where $\{\lambda_m\}_{m=1}^3$ are singular values of the original channel $\mathcal{E}$. Note that the map $\overline{\mathcal{E}}$ is positive and trace preserving but not necessarily completely positive. This means that $\overline{\mathcal{E}}[\rho] \geqslant 0$ for all qubit density operators ρ, whereas, $(\overline{\mathcal{E}} \otimes \mathcal{I})[\rho_{\mathrm{in}}]$ can, in principle, have negative eigenvalues for some two-qubit density operators ρ_{in}.

The EA behavior of local two-qubit unital channels $\mathcal{E}_1 \otimes \mathcal{E}_2$ is governed by the following lemma.

Lemma 1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be unital qubit channels. The two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA if and only if the maps $\overline{\mathcal{E}_1} \otimes \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \overline{\mathcal{E}_2}$ are positive.

Proof. Separability of the output state (4) of channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ can be checked by the reduction criterion [30], which turns out to be a necessary and sufficient separability condition for two-qubit systems. Assuming that channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA, the two-qubit state ρ_{out} is separable, thus, in accordance with the reduction criterion, the following conditions hold:

$$\mathrm{tr}_2[\rho_{\mathrm{out}}] \otimes I - \rho_{\mathrm{out}} \geqslant 0 \quad \text{and} \quad I \otimes \mathrm{tr}_1[\rho_{\mathrm{out}}] - \rho_{\mathrm{out}} \geqslant 0, \quad (5)$$

where $\mathrm{tr}_1[\cdot]$ and $\mathrm{tr}_2[\cdot]$ denote partial traces over the first and second qubits, respectively. Since

$$(\mathcal{E}_1 \otimes \overline{\mathcal{E}_2})[\varrho_{\mathrm{in}}] = \mathrm{tr}_2[\rho_{\mathrm{out}}] \otimes I - \rho_{\mathrm{out}},$$
$$(\overline{\mathcal{E}_1} \otimes \mathcal{E}_2)[\varrho_{\mathrm{in}}] = I \otimes \mathrm{tr}_1[\rho_{\mathrm{out}}] - \rho_{\mathrm{out}},$$

the above separability conditions are equivalent with the positivity of maps $\overline{\mathcal{E}_1} \otimes \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \overline{\mathcal{E}_2}$, respectively. ∎

Before we explore the consequences of Lemma 1 and derive some properties of local two-qubit unital channels $\mathcal{E}_1 \otimes \mathcal{E}_2$, let us make some remarks about single-qubit unital channels. A qubit unital map $\mathcal{E}$ with singular values $\{\lambda_m\}_{m=1}^3$ is indeed a channel (i.e., completely positive trace-preserving map) if $1 + \lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$, $1 + \lambda_1 - \lambda_2 - \lambda_3 \geqslant 0$, $1 - \lambda_1 + \lambda_2 - \lambda_3 \geqslant 0$, and $1 - \lambda_1 - \lambda_2 + \lambda_3 \geqslant 0$. These four inequalities define a tetrahedron in the conventional reference frame $(\lambda_1,\lambda_2,\lambda_3)$ in $\mathbb{R}^3$ (see Fig. 2). Channel $\mathcal{E}$ is known to be EB if and only if $|\lambda_1| + |\lambda_2| + |\lambda_3| \leqslant 1$ [29]. This inequality corresponds to the octahedron in Fig. 2.

Let us note that $\overline{\mathcal{E}}$ is a quantum channel if and only if $\mathcal{E}$ is EB. That is, Lemma 1 guarantees that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA if $\mathcal{E}_1$ or $\mathcal{E}_2$ is EB, which is in agreement with property (5). Nevertheless, channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ can be EA even if neither $\mathcal{E}_1$ nor $\mathcal{E}_2$ is EB. The following proposition gives a sufficient condition for $\mathcal{E}_1 \otimes \mathcal{E}_2$ being EA.

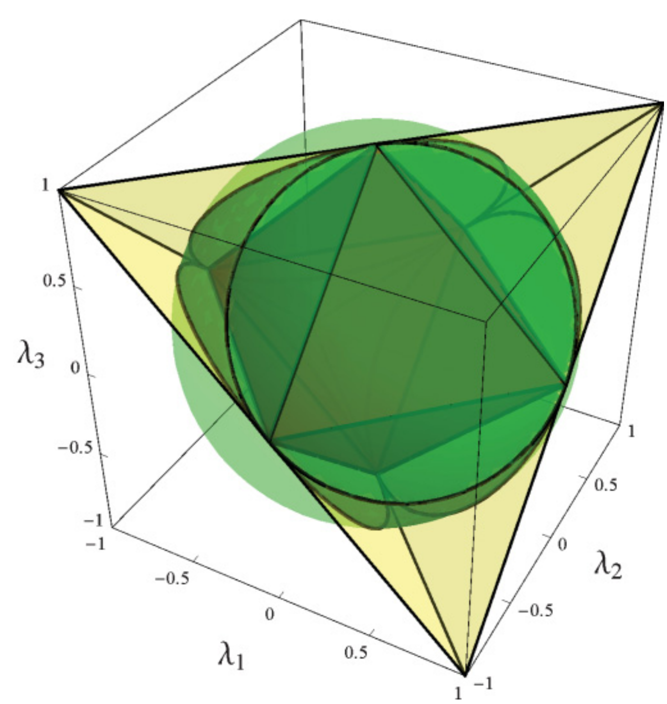

FIG. 2. (Color online) Intersection of the tetrahedron and the ball is a set of 2-LEA unital qubit channels $\mathcal{E}$, i.e., such channels $\mathcal{E}$ that $\mathcal{E} \otimes \mathcal{E}$ is EA. The octahedron inside the sphere represents EB qubit channels.

Proposition 1. Suppose $\mathcal{E}_1,\mathcal{E}_2$ are unital qubit channels such that $\mathcal{E}_1^2$ and $\mathcal{E}_2^2$ are EB channels, i.e., $\sum_{m=1}^3 \lambda_m^2 \leqslant 1$ and $\sum_{n=1}^3 {\lambda'_n}^2 \leqslant 1$. Then, $\mathcal{E}_1 \otimes \mathcal{E}_2$ is an EA channel.

Proof. Consider map $\overline{\mathcal{E}_1} \otimes \mathcal{E}_2$. Let us demonstrate that $(\overline{\mathcal{E}_1} \otimes \mathcal{E}_2)[\rho_{\rm in}] \geqslant 0$ for all two-qubit input states $\rho_{\rm in}$. In view of the convexity of the state space, it suffices to show that $(\overline{\mathcal{E}_1} \otimes \mathcal{E}_2)[|\psi\rangle\langle\psi|] \geqslant 0$ for all pure two-qubit states $|\psi\rangle$.

Any state $|\psi\rangle$ is given by its Schmidt decomposition $\sqrt{p}|\varphi \otimes \chi\rangle + \sqrt{p_\perp}|\varphi_\perp \otimes \chi_\perp\rangle$, where $p,p_\perp \geqslant 0$ and $p + p_\perp = 1$, the orthonormal basis $\{|\varphi\rangle,|\varphi_\perp\rangle\}$ can be parametrized by the angles $\theta \in [0,\pi]$, and $\phi \in [0,2\pi]$ as follows:

$$|\varphi\rangle = \begin{pmatrix} \cos(\theta/2)\exp(-i\phi/2) \\ \sin(\theta/2)\exp(i\phi/2) \end{pmatrix}, \tag{6}$$

$$|\varphi_\perp\rangle = \begin{pmatrix} -\sin(\theta/2)\exp(-i\phi/2) \\ \cos(\theta/2)\exp(i\phi/2) \end{pmatrix}, \tag{7}$$

and the basis $\{|\chi\rangle,|\chi_\perp\rangle\}$ is obtained from formulas (6) and (7) by replacing $|\varphi\rangle \to |\chi\rangle$, $|\varphi_\perp\rangle \to |\chi_\perp\rangle$, $\theta \to \theta'$, and $\phi \to \phi'$.

Map $\overline{\mathcal{E}_1} \otimes \mathcal{E}_2$ transforms $|\psi\rangle\langle\psi|$ into the operator,

$$\begin{aligned} &\tfrac{1}{4}\{I \otimes I - (\mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{n}'\cdot\boldsymbol{\sigma}') \\ &\quad -(p - p_\perp)[(\mathbf{n}\cdot\boldsymbol{\sigma}) \otimes I - I \otimes (\mathbf{n}'\cdot\boldsymbol{\sigma}')] \\ &\quad -2\sqrt{pp_\perp}[(\mathbf{k}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{k}'\cdot\boldsymbol{\sigma}') - (\mathbf{l}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{l}'\cdot\boldsymbol{\sigma}')]\}, \end{aligned} \tag{8}$$

where $(\mathbf{n}\cdot\boldsymbol{\sigma}) = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ and vectors $\mathbf{n},\mathbf{k},\mathbf{l} \in \mathbb{R}^3$ are expressed through singular values $\{\lambda_m\}_{m=1}^3$ of channel $\mathcal{E}_1$ by formulas,

$$\mathbf{n} = (\lambda_1 \cos\phi \sin\theta, \lambda_2 \sin\phi \sin\theta, \lambda_3 \cos\theta), \tag{9}$$

$$\mathbf{k} = (-\lambda_1 \cos\phi \cos\theta, -\lambda_2 \sin\phi \cos\theta, \lambda_3 \sin\theta), \tag{10}$$

$$\mathbf{l} = (\lambda_1 \sin\phi, -\lambda_2 \cos\phi, 0). \tag{11}$$

Vectors $\mathbf{n}',\mathbf{k}'$ and $\mathbf{l}'$ are obtained from Eqs. (9)–(11), respectively, by replacing $\boldsymbol{\lambda} \to \boldsymbol{\lambda}'$, $\theta \to \theta'$, and $\phi \to \phi'$.

Both sets of vectors $\{\mathbf{n},\mathbf{k},\mathbf{l}\}$ and $\{\mathbf{n}',\mathbf{k}',\mathbf{l}'\}$ have a particular property,

$$|\mathbf{n}|^2 + |\mathbf{k}|^2 + |\mathbf{l}|^2 = \sum_{m=1}^{3} \lambda_m^2 \leqslant 1, \tag{12}$$

$$|\mathbf{n}'|^2 + |\mathbf{k}'|^2 + |\mathbf{l}'|^2 = \sum_{n=1}^{3} {\lambda'_n}^2 \leqslant 1 \tag{13}$$

thanks to the statement of the proposition.

The output state (8) is positive semidefinite if and only if its average is $\langle\rho_{\rm out}\rangle \geqslant 0$ for all two-qubit states. In other words, we want to show that the inequality,

$$\begin{aligned} &\langle I \otimes I - (\mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{n}'\cdot\boldsymbol{\sigma}')\rangle \\ &\quad \geqslant (p - p_\perp)\langle(\mathbf{n}\cdot\boldsymbol{\sigma}) \otimes I - I \otimes (\mathbf{n}'\cdot\boldsymbol{\sigma}')\rangle \\ &\quad +2\sqrt{pp_\perp}\langle(\mathbf{k}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{k}'\cdot\boldsymbol{\sigma}') - (\mathbf{l}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{l}'\cdot\boldsymbol{\sigma}')\rangle \end{aligned} \tag{14}$$

holds true for any averaging state. Since $(p - p_\perp)^2 + (2\sqrt{pp_\perp})^2 = (p + p_\perp)^2 = 1$, one can treat $(p - p_\perp)$ as $\cos\alpha$ and $2\sqrt{pp_\perp}$ as $\sin\alpha$. Due to the fact that $\max_\alpha(A\cos\alpha + B\sin\alpha) = \sqrt{A^2 + B^2}$, the inequality (14) if fulfilled whenever

$$\begin{aligned} &\langle I \otimes I - (\mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{n}'\cdot\boldsymbol{\sigma}')\rangle^2 \geqslant \langle(\mathbf{n}\cdot\boldsymbol{\sigma}) \otimes I - I \otimes (\mathbf{n}'\cdot\boldsymbol{\sigma}')\rangle^2 \\ &\quad + \langle(\mathbf{k}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{k}'\cdot\boldsymbol{\sigma}') - (\mathbf{l}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{l}'\cdot\boldsymbol{\sigma}')\rangle^2, \end{aligned} \tag{15}$$

or, equivalently,

$$\begin{aligned} &\langle[I + (\mathbf{n}\cdot\boldsymbol{\sigma})] \otimes [I - (\mathbf{n}'\cdot\boldsymbol{\sigma}')]\rangle \\ &\quad \times \langle[I - (\mathbf{n}\cdot\boldsymbol{\sigma})] \otimes [I + (\mathbf{n}'\cdot\boldsymbol{\sigma}')]\rangle \\ &\quad \geqslant \langle(\mathbf{k}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{k}'\cdot\boldsymbol{\sigma}') - (\mathbf{l}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{l}'\cdot\boldsymbol{\sigma}')\rangle^2. \end{aligned} \tag{16}$$

Taking into account that $|\varphi \otimes \chi\rangle$, $|\varphi \otimes \chi_\perp\rangle$, $|\varphi_\perp \otimes \chi\rangle$, and $|\varphi_\perp \otimes \chi_\perp\rangle$ are all eigenvectors of the operators in the left-hand side of Eq. (16) and using the Cauchy-Schwarz inequality in the form $\langle X^\dagger X\rangle\langle Y^\dagger Y\rangle \geqslant |\langle X^\dagger Y\rangle|^2$, one readily can see that

$$\begin{aligned} &\langle[I + (\mathbf{n}\cdot\boldsymbol{\sigma})] \otimes [I - (\mathbf{n}'\cdot\boldsymbol{\sigma}')]\rangle\langle[I - (\mathbf{n}\cdot\boldsymbol{\sigma})] \otimes [I + (\mathbf{n}'\cdot\boldsymbol{\sigma}')]\rangle \\ &\quad \underset{\rm CS}{\geqslant} (1 - |\mathbf{n}|^2)(1 - |\mathbf{n}'|^2) \\ &\quad \underset{(12),(13)}{\geqslant} (|\mathbf{k}|^2 + |\mathbf{l}|^2)(|\mathbf{k}'|^2 + |\mathbf{l}'|^2) \\ &\quad \geqslant (|\mathbf{k}||\mathbf{k}'| + |\mathbf{l}||\mathbf{l}'|)^2 \geqslant \langle(\mathbf{k}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{k}'\cdot\boldsymbol{\sigma}') - (\mathbf{l}\cdot\boldsymbol{\sigma}) \otimes (\mathbf{l}'\cdot\boldsymbol{\sigma}')\rangle^2. \end{aligned} \tag{17}$$

Thus, $(12)\wedge(13) \Rightarrow (17) \Rightarrow (16) \Leftrightarrow (15) \Rightarrow (14) \Rightarrow \overline{\mathcal{E}_1} \otimes \mathcal{E}_2$ is a positive map. In the same way, $\mathcal{E}_1 \otimes \overline{\mathcal{E}_2}$ also is a positive map. According to Lemma 1, channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA. ■

Note that Proposition 1 provides the sufficient but not necessary condition for channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ to be EA. For instance, in the case of two depolarizing channels $\mathcal{E}_1$ and $\mathcal{E}_2$ from Sec. III, channels $\mathcal{E}_1^2$ and $\mathcal{E}_2^2$ are EB if $q_1 \leqslant \frac{1}{\sqrt{3}}$ and $q_2 \leqslant \frac{1}{\sqrt{3}}$, respectively. It means that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA if $q_1,q_2 \leqslant \frac{1}{\sqrt{3}}$. The corresponding area of parameters (q_1,q_2) is depicted in Fig. 1, which illustrates the power of Proposition 1. However, in the case of identical environments, i.e., $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, the use of Proposition 1 provides a sufficient and necessary condition for $\mathcal{E} \otimes \mathcal{E}$ to be EA.

Proposition 2. A unital qubit channel $\mathcal{E}$ is 2-LEA if and only if channel $\mathcal{E}^2$ is EB, i.e., $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leqslant 1$.

Proof. The condition $\{\mathcal{E}^2$ is EB$\}$ is sufficient due to Proposition 1. It turns out also to be necessary if we consider Bell states, e.g., $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It readily is seen that the operator $\{(\mathcal{E} \otimes \mathcal{E})[|\psi\rangle\langle\psi|]\}^\Gamma$ is positive semidefinite if and only if $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leqslant 1$. On the other hand, $\{\lambda_i^2\}_{i=1}^3$ are singular values of $\mathcal{E}^2$, and their sum is less than or equal to 1, i.e., $\mathcal{E}^2$ is EB [23]. ■

An illustration of 2-LEA unital channels $\mathcal{E}$ in the conventional reference frame $\{\lambda_1,\lambda_2,\lambda_3\}$ is presented in Fig. 2. Points outside the tetrahedron do not satisfy the complete positivity of $\mathcal{E}$. Points outside the sphere do not correspond to 2-LEA channels, which can easily be checked by Bell states. Both the sphere and the tetrahedron comprise an octahedron of EB single-qubit channels $\mathcal{E}$ with $|\lambda_1| + |\lambda_2| + |\lambda_3| \leqslant 1$. Since the intersection of the sphere and tetrahedron is a convex body, we have just revealed an interesting property:

Proposition 3. The set of 2-LEA unital qubit channels is convex, i.e., if $\mathcal{E}_1$ and $\mathcal{E}_2$ are 2-LEA, then $\mu\mathcal{E}_1 + (1 - \mu)\mathcal{E}_2$ also is 2-LEA for all $\mu \in [0,1]$.

Proof. Despite the fact that this fact is evident from geometrical consideration, it also follows from the relation $[\mu\mathcal{E}_1 + (1-\mu)\mathcal{E}_2] \otimes [\mu\mathcal{E}_1 + (1-\mu)\mathcal{E}_2] = \mu^2\mathcal{E}_1 \otimes \mathcal{E}_1 + \mu(1-\mu)\mathcal{E}_1 \otimes \mathcal{E}_2 + (1-\mu)\mu\mathcal{E}_2 \otimes \mathcal{E}_1 + (1-\mu)^2\mathcal{E}_2 \otimes \mathcal{E}_2$. As channels $\mathcal{E}_1 \otimes \mathcal{E}_1$ and $\mathcal{E}_2 \otimes \mathcal{E}_2$ are EA by a statement of the involved proposition, channels $\mathcal{E}_1^2$ and $\mathcal{E}_2^2$ are EB by Proposition 2. Due to Proposition 1, both $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\mathcal{E}_2 \otimes \mathcal{E}_1$ are EA. Then, we use property (1) to conclude the proof. ■

One more property immediately follows from Propositions 1 and 2:

Corollary 1. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are unital 2-LEA qubit channels, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA.

We can make an interesting observation from Fig. 2, namely, if one takes a phase-damping channel $\mathcal{E}$ (belonging to an edge of the tetrahedron), then channel $\mathcal{E} \otimes \mathcal{E}$ preserves the entanglement of some input states unless $\mathcal{E}$ contracts the whole Bloch sphere into a line.

Let us now analyze unital channels $\mathcal{E}_1 \otimes \mathcal{E}_2$ such that neither $\mathcal{E}_1$ nor $\mathcal{E}_2$ is a 2-LEA channel (i.e., both $\mathcal{E}_1$ and $\mathcal{E}_2$ are outside the sphere in Fig. 2). Surprisingly, it turns out that $\mathcal{E}_1 \otimes \mathcal{E}_2$ can still be EA as demonstrated by the following example.

Example 1. Suppose $\mathcal{E}_1 = \mathrm{diag}\{1,\frac{1}{20},\frac{1}{20},1\}$ and $\mathcal{E}_2 = \mathrm{diag}\{1,\frac{2}{3},\frac{2}{3},\frac{2}{3}\}$. Let us demonstrate that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA although neither $\mathcal{E}_1$ nor $\mathcal{E}_2$ is 2-LEA. We note that the channel in question can be represented in the form $\mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{G} \cdot \mathcal{F}$, where $\mathcal{G} = \frac{3}{4}\mathcal{G}_1 \otimes \mathcal{I} + \frac{1}{4}\mathcal{I} \otimes \mathcal{G}_2$, $\mathcal{G}_1 = \mathrm{diag}\{1,0,0,1\}$, $\mathcal{G}_2 = \mathrm{diag}\{1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$, and the two-qubit map $\mathcal{F}$ is defined by its matrix representation $\frac{1}{4}\mathrm{tr}[\sigma_k \otimes \sigma_l' \mathcal{F}(\sigma_m \otimes \sigma_n')] = F_{mn}\delta_{km}\delta_{ln}$ with $F_{0n} = F_{3n} = (4+\delta_{n0})/5$ and $F_{1n} = F_{2n} = (2-\delta_{n0})/5$. By considering the Choi matrix [31,32] of map $\mathcal{F}$, it is not hard to see that $\mathcal{F}$ is a channel indeed, i.e., a completely positive trace-preserving map. As both $\mathcal{G}_1$ and $\mathcal{G}_2$ are EB, channels $\mathcal{G}_1 \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{G}_2$ are EA by property (5). Hence, $\mathcal{G}$ is EA by property (1), and $\mathcal{G} \cdot \mathcal{F}$ is EA by property (3). The equality $\mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{G} \cdot \mathcal{F}$ completes the proof.

A natural question to ask is under which conditions on $\mathcal{E}_1$ and $\mathcal{E}_2$ is channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ EA. The following proposition formulates a sufficient condition for the converse statement.

Proposition 4. Consider qubit unital channels $\mathcal{E}_1$ and $\mathcal{E}_2$ with singular values $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3)$ and $\boldsymbol{\lambda}' = (\lambda_1',\lambda_2',\lambda_3')$, respectively. If $\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}' > 1$, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is not an EA channel.

Proof. Channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ preserves entanglement of the Bell state $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ when the matrix,

$$\rho_{\mathrm{out}}^{\Gamma} = \{(\mathcal{E}_1 \otimes \mathcal{E}_2)[|\psi_+\rangle\langle\psi_+|]\}^{\Gamma} = \frac{1}{4}\begin{pmatrix} 1+\lambda_3\lambda_3' & 0 & 0 & \lambda_1\lambda_1' - \lambda_2\lambda_2' \\ 0 & 1-\lambda_3\lambda_3' & \lambda_1\lambda_1' + \lambda_2\lambda_2' & 0 \\ 0 & \lambda_1\lambda_1' + \lambda_2\lambda_2' & 1-\lambda_3\lambda_3' & 0 \\ \lambda_1\lambda_1' - \lambda_2\lambda_2' & 0 & 0 & 1+\lambda_3\lambda_3' \end{pmatrix}$$

has negative eigenvalues. It takes place if $\lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3' \equiv \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}' > 1$. In view of the Peres-Horodecki criterion [27,28], the output state ρ_{out} remains entangled. ■

Let us note that this proposition is very efficient, e.g., for depolarizing channels from Sec. III, we immediately have the not-EA channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ if $q_1 q_2 > \frac{1}{3}$. Let us be reminded that $q_1 q_2 = \frac{1}{3}$ is a boundary between EA and not-EA behaviors of depolarizing channels.

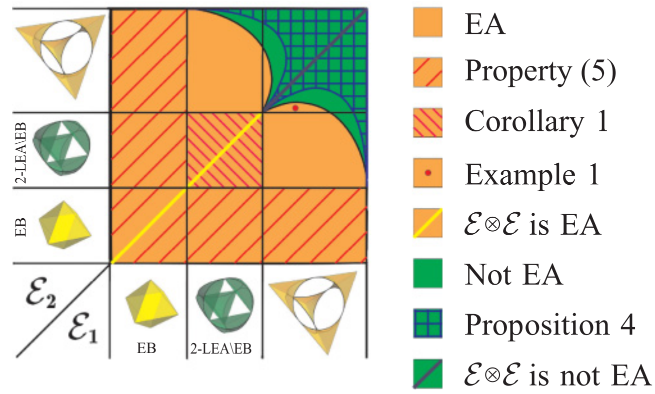

FIG. 3. (Color online) Schematic of unital two-qubit channels of the factorized form $\mathcal{E}_1 \otimes \mathcal{E}_2$ with respect to the EA behavior. Each axis is divided into three classes of unital single-qubit channels: EB, 2-LEA, and others.

Our goal was to give a complete picture of unital two-qubit channels of the form $\mathcal{E}_1 \otimes \mathcal{E}_2$. Our findings are graphically summarized in Fig. 3. Based on this figure, one could argue that the property of being EA is not something rare, and the noises should be relatively close to unitary ones (or, equivalently, sufficiently far from complete depolarization) in order to guarantee the conservation of some entanglement.

V. NONUNITAL CHANNELS

In this section, we present a case study of a specific class of nonunital qubit channels. In particular, we are interested in which of the statements, valid in the case of unital channels, can also be generalized to the nonunital case.

A. Factorized extremal channels

A channel is extremal if it cannot be expressed as a convex combination of some other channels. A typical example is unitary channels, which are the only extremal unital qubit channels. All others are nonunital and include, for example, the families of amplitude-damping channels $\mathcal{A}_{p,0}$ and $\mathcal{A}_{p,1}$ considered in the next subsection.

Suppose constituents of channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ can be written as nontrivial convex sums of other channels, i.e., $\mathcal{E}_j = \mu_j\mathcal{F}_j + (1-\mu_j)\mathcal{G}_j$, $j = 1,2$. Then, apparently, $\mathcal{E}_1 \otimes \mathcal{E}_2$ also is a convex combination of factorized channels $\mathcal{F}_j \otimes \mathcal{G}_k$ and $\mathcal{G}_j \otimes \mathcal{F}_k$, $j,k = 1,2$. If such factorized channels are EA, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA by property (1). This analysis stimulates us to consider the extreme case of channels $\mathcal{E}_1 \otimes \mathcal{E}_2$, where neither $\mathcal{E}_1$ nor $\mathcal{E}_2$ cannot be resolved into nontrivial convex combinations.

As shown in the seminal paper [29] by Ruskai *et al.*, any extremal qubit channel can be expressed in an appropriate basis as a matrix,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix}, \tag{18}$$

where $u \in [0,2\pi)$ and $v \in [0,\pi)$.

At first, we are reminded when an extremal qubit channel $\mathcal{E}$ of the form (18) is EB. Let $|\psi\rangle$ be a maximally entangled two-qubit state. Applying the partial transpose (PPT) criterion to the Choi-Jamiołkowski state [31,32] $(\mathcal{E}\otimes\mathcal{I})[|\psi_+\rangle\langle\psi_+|]$, we find that $\mathcal{E}$ is EB if and only if $\cos u = 0$ or $\cos v = 0$.

In what follows, we use the fact that, in formula (18), the entries $\cos v$ and $\sin u \sin v$ can simultaneously be made non-negative by an appropriate choice of basis operators $\{\sigma_i\}_{i=0}^3$. The singular values $\cos u$ and $\cos u \cos v$ have the same sign (either positive or negative).

Proposition 5. Suppose $\mathcal{E}_1,\mathcal{E}_2$ are extremal qubit channels. Then $\mathcal{E}_1\otimes\mathcal{E}_2$ is EA if and only if either $\mathcal{E}_1$ or $\mathcal{E}_2$ is EB.

Proof. If either $\mathcal{E}_1$ or $\mathcal{E}_2$ is EB, then $\mathcal{E}_1\otimes\mathcal{E}_2$ is EA in view of property (5). Let us now prove that channel $\mathcal{E}_1\otimes\mathcal{E}_2$ is not EA if neither $\mathcal{E}_1$ nor $\mathcal{E}_2$ is EB, i.e., $\cos u_1 \cos v_1 \cos u_2 \cos v_2 \neq 0$, where parameters (u_j,v_j) define channel $\mathcal{E}_j$ according to formula (18).

Without loss of generality, it can be assumed that $|\cos u_j| \geqslant |\cos v_j|$, $j=1,2$. Consider the input state $|\psi_{\rm in}\rangle = \frac{1}{\sqrt{2}}(|\varphi\otimes\chi\rangle + |\varphi_\perp\otimes\chi_\perp\rangle)$ where the orthogonal qubit states $\{|\varphi\rangle,|\varphi_\perp\rangle\}$ and $\{|\chi\rangle,|\chi_\perp\rangle\}$ are parametrized by angles (θ_1,ϕ_1) and (θ_1,ϕ_2), respectively, as in Proposition 1 [see formulas (6) and (7)]. We set $\phi_j=0$ and choose angles θ_j such that

$$\cos\theta_j = \frac{\sin u_j \cos v_j}{\cos u_j \sin v_j}, \qquad \sin\theta_j = \frac{(\cos^2 u_j - \cos^2 v_j)^{1/2}}{\cos u_j \sin v_j},$$

then $\mathcal{E}_1[|\varphi\rangle\langle\varphi|]$ and $\mathcal{E}_2[|\chi\rangle\langle\chi|]$ are known to be pure states [29].

The reduction criterion (5) guarantees that channel $\mathcal{E}_1\otimes\mathcal{E}_2$ is not EA if the operator,

$$M = {\rm tr}_2\{(\mathcal{E}_1\otimes\mathcal{E}_2)[|\psi_{\rm in}\rangle\langle\psi_{\rm in}|]\}\otimes I - (\mathcal{E}_1\otimes\mathcal{E}_2)[|\psi_{\rm in}\rangle\langle\psi_{\rm in}|] \quad (19)$$

is not positive semidefinite. This takes place if there exists a two-qubit state $|\psi_{\rm test}\rangle$ such that $\langle\psi_{\rm test}|M|\psi_{\rm test}\rangle < 0$.

Let us construct a one-parametric family of candidates $|\psi_p\rangle$, $p\in[0,1]$ for state $|\psi_{\rm test}\rangle$. To do that, we use the pure states $|\xi\rangle\langle\xi| = \mathcal{E}_1[|\varphi\rangle\langle\varphi|]$ and $|\zeta\rangle\langle\zeta| = \mathcal{E}_2[|\chi\rangle\langle\chi|]$ and write $|\psi_p\rangle = \sqrt{p}|\xi\otimes\zeta\rangle + \sqrt{1-p}|\xi_\perp\otimes\zeta_\perp\rangle$. Direct calculation yields

$$\begin{aligned}4\langle\psi_p|M|\psi_p\rangle = {} & 1 - \sin^2 u_1 \sin^2 u_2 - \cos^2 u_1 \cos^2 u_2 \\ & - (1-2p)(\sin^2 u_1 - \sin^2 u_2) \\ & - 4\sqrt{p(1-p)}\{\cos v_1 \cos v_2 \\ & + \sin u_1 \sin u_2[(\cos^2 u_1 - \cos^2 v_1) \\ & \times (\cos^2 u_2 - \cos^2 v_2)]^{1/2}\}. \end{aligned} \quad (20)$$

State $|\psi_{\rm test}\rangle$ then is equal to such $|\psi_p\rangle$ that minimizes the expression (20). By a remark before the proposition involved, $\sin u_j \geqslant 0$ and $\cos v_j \geqslant 0$, $j=1,2$. Since $(1-2p)^2 + [2\sqrt{p(1-p)}]^2 \equiv 1$, the minimum of Eq. (20) is achievable and reads

$$\begin{aligned}4\langle\psi_{\rm test}|M|\psi_{\rm test}\rangle = {} & 1 - \sin^2 u_1 \sin^2 u_2 - \cos^2 u_1 \cos^2 u_2 \\ & - ((\sin^2 u_1 - \sin^2 u_2)^2 + 4\{\cos v_1 \cos v_2 \\ & + \sin u_1 \sin u_2[(\cos^2 u_1 - \cos^2 v_1) \\ & \times (\cos^2 u_2 - \cos^2 v_2)]^{1/2}\}^2)^{1/2}.\end{aligned}$$

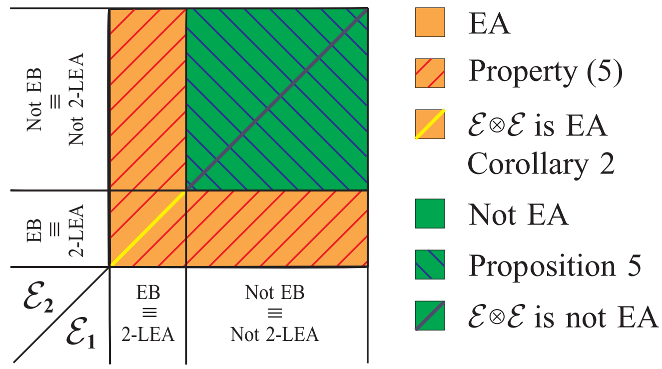

FIG. 4. (Color online) Schematic of factorized extremal two-qubit channels $\mathcal{E}_1\otimes\mathcal{E}_2$ with respect to the EA behavior. Each axis is divided into two classes of extremal single-qubit channels: EB that is equivalent to 2-LEA, and others.

It is not hard to see that inequality $\langle\psi_{\rm test}|M|\psi_{\rm test}\rangle < 0$ is equivalent to

$$\begin{aligned}&\sin u_1 \sin u_2\sqrt{(\cos^2 u_1 - \cos^2 v_1)(\cos^2 u_2 - \cos^2 v_2)} \\ &\quad + \cos v_1 \cos v_2 > |\sin u_1 \sin u_2 \cos u_1 \cos u_2|,\end{aligned}$$

which is fulfilled for non-negative $\sin u_j$ and $\cos v_j$ whenever $\cos u_1 \cos u_2 \cos v_1 \cos v_2 \neq 0$ because $\sqrt{1-t^2} > 1-t$ for all $t\in(0,1)$. Thus, the operator (19) is not positive semidefinite, and the channel $\mathcal{E}_1\otimes\mathcal{E}_2$ is not EA. ■

Corollary 2. Suppose $\mathcal{E}$ is an extremal qubit channel. Then, $\mathcal{E}\otimes\mathcal{E}$ is EA, hence, $\mathcal{E}$ is 2-LEA, if and only if $\mathcal{E}$ is EB.

Using the results of Proposition 5, we can present a complete picture (Fig. 4) of factorized extremal channels, the picture being analogous to that of unital channels (Fig. 3). Comparison of two figures gives a clear insight that the factorized extremal channels exhibit better entanglement-preserving properties than the unital factorized channels in the following sense. Whenever none of the extremal channels is EB, there exist some initial states for which the entanglement survives the action of $\mathcal{E}_1\otimes\mathcal{E}_2$. For local unital channels, such a property does not hold. Even if none of the channels is EB, it can happen that the output states are separable for all input states.

B. Generalized amplitude-damping channels

Amplitude-damping channels describe how a two-level system approaches the equilibrium due to coupling with its environment, e.g., thanks to a spontaneous emission process at zero temperature. If the environment has a finite temperature, then such a dissipation process is described by the action of a generalized amplitude-damping channel (see, e.g., Ref. [33]).

Kraus operators for a generalized amplitude-damping channel $\mathcal{A}_{p,\gamma}$ have the form

$$E_0 = \sqrt{\gamma}\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix}, \quad E_1 = \sqrt{\gamma}\begin{pmatrix}0 & \sqrt{p}\\ 0 & 0\end{pmatrix},$$
$$E_2 = \sqrt{1-\gamma}\begin{pmatrix}\sqrt{1-p} & 0\\ 0 & 1\end{pmatrix}, \quad E_3 = \sqrt{1-\gamma}\begin{pmatrix}0 & 0\\ \sqrt{p} & 0\end{pmatrix},$$

where $p \in [0,1]$ determines the amplitude-damping rate and $\gamma \in [0,1]$ is a parameter that depends on the temperature and defines a fixed (equilibrium) state of $\mathcal{A}_{p,\gamma}$,

$$\rho_\infty = \begin{pmatrix} \gamma & 0 \\ 0 & 1-\gamma \end{pmatrix}.$$

If $\gamma = 0$ or $\gamma = 1$, then $\mathcal{A}_{p,\gamma}$ simply is an amplitude-damping channel. As mentioned above in Sec. V A, amplitude-damping channels $\mathcal{A}_{p,0}$ and $\mathcal{A}_{p,1}$ are extremal, thus, obeying Proposition 5. A channel $\mathcal{A}_{p_1,0} \otimes \mathcal{A}_{p_2,0}$ is EA if and only if $p_1 = 1$ or $p_2 = 1$, i.e., when at least one of the constituent channels contracts the whole Bloch sphere into a single pure state.

Let us now move on to generalized amplitude-damping channels $\mathcal{A}_{p,\gamma}$. We note that $\mathcal{A}_{p,\gamma} = \gamma\mathcal{A}_{p,1} + (1-\gamma)\mathcal{A}_{p,0}$. Since $\mathcal{A}_{p,\gamma}$ is a convex combination of two amplitude-damping channels (extremal qubit channels), we expect $\mathcal{A}_{p_1,\gamma_1} \otimes \mathcal{A}_{p_2,\gamma_2}$ to exhibit worse entanglement-preserving properties than factorized extremal channels. We can expect that the closer weights of two channels (the closer γ is to $\frac{1}{2}$), the stronger the entanglement annihilation.

First, we note that channel $\mathcal{A}_{p,\gamma}$ is EB if $p \geqslant [\sqrt{1+4\gamma(1-\gamma)} - 1]/2\gamma(1-\gamma)$. Let us be reminded that a channel is EB if and only if its Choi-Jamiołkowski state is separable, and for two qubits, one can employ the PPT criterion to verify the separability. In particular, for $\gamma = 0$ (zero bath temperature), channel $\mathcal{A}_p$ is EB only if $p = 1$, hence, it contracts the whole Bloch sphere into the equilibrium state.

Second, if we focus on 2-LEA channels, then it turns out that $\mathcal{A}_{p,\gamma} \otimes \mathcal{A}_{p,\gamma}$ preserves the entanglement of state $|\psi_+\rangle$ if $p \leqslant [1 - \sqrt{2\gamma(1-\gamma)}]/[1-2\gamma(1-\gamma)]$. These results are shown in Fig. 5(a) where the dotted line depicts a symbolic boundary between EA and not-EA channels. The numerical analysis encourages assuming that this boundary coincides with the solid line.

Third, if $\gamma = 1/2$, then $\mathcal{A}_{p,\gamma}$ becomes unital. In this case, to conclude EA of channel $\mathcal{A}_{p,\gamma} \otimes \mathcal{A}_{p,\gamma}$ it suffices to consider its action of on the maximally entangled state, e.g., $|\psi_+\rangle$. An illustration of such unital channels with respect to other unital channels is presented in Fig. 5(b).

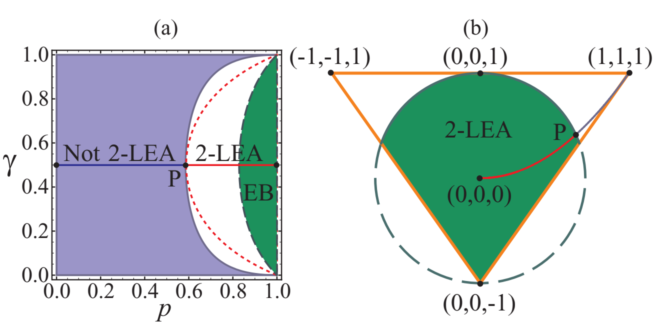

FIG. 5. (Color online) (a) Regions of parameters p and γ where the generalized amplitude-damping channel $\mathcal{A}_{p,\gamma}$ is definitely not 2-LEA (violet), or $\mathcal{A}_{p,\gamma}$ is 2-LEA for sure (EB, red solid line). The dotted curve presents a symbolic boundary between two regions and seems to coincide with the solid curve. (b) The 2-LEA characterization of the generalized amplitude-damping channels $\mathcal{A}_{p,\gamma}$ with $\gamma = 1/2$. Such channels are unital and correspond to infinite temperatures. The family of these channels ($p \in [0,1]$) is depicted among other unital channels in the cut $\lambda_1 = \lambda_2$ of the conventional reference frame $(\lambda_1,\lambda_2,\lambda_3)$. The transition from 2-LEA to not-2-LEA behavior happens at point P $= (\sqrt{\sqrt{2}-1}, \sqrt{\sqrt{2}-1}, \sqrt{2}-1)$.

Finally, we can draw the conclusion that a picture of the EA behavior of generalized amplitude-damping channels takes an intermediate form between the unitary two-qubit channels (Fig. 3) and the factorized extremal channel (Fig. 4).

VI. EA-EXTREMAL LOCAL CHANNELS

Convexity is an important property of many channel sets. Thus, the set of all qubit channels T_2 and the set of all EB qubit channels $\mathsf{T}_{\rm EB}$ are convex. Extreme points of these sets are studied in Refs. [29] and [24], respectively. Extreme points of the set of EB channels are referred to as EB extremal. In the paper [24], it is shown that, in the qubit case, all EB-extremal single-qubit channels can be represented in the form

$$\mathcal{F}[\rho] = \langle\psi|\rho|\psi\rangle|\varphi_1\rangle\langle\varphi_1| + \langle\psi_\perp|\rho|\psi_\perp\rangle|\varphi_2\rangle\langle\varphi_2|, \quad (21)$$

where $|\psi\rangle, |\psi_\perp\rangle, |\varphi_1\rangle, |\varphi_2\rangle$ are pure qubit states and $\langle\psi|\psi_\perp\rangle = 0$.

If we consider general two-qubit channels (not necessarily local), then along with the set of all channels $\mathsf{T}_{\rm chan}$ and the set of EB two-qubit channels, there is a set of all EA channels $\mathsf{T}_{\rm EA}$, which is also convex [property (1)]. The question itself arises to find extreme points of this set, i.e., channels that are extreme for two-qubit EA channels. We will refer to such channels as EA extremal. The following proposition provides EA-extremal channels of factorized form.

Proposition 6. Let $\mathcal{F}$ be an EB-extremal one-qubit channel, then channels $\mathcal{F} \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{F}$ are EA extremal.

Proof. Let us now prove the statement of Proposition 6 by *reductio ad absurdum*. Suppose $\mathcal{F} \otimes \mathcal{I}$ is not-EA extremal, i.e., there exist two noncoincident EA channels $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{F} \otimes \mathcal{I} = \mu\mathcal{G}_1 + (1-\mu)\mathcal{G}_2$, where $\mu \in [0,1]$. Using the Choi-Jamiołkowski isomorphism [31,32], the latter equation can be rewritten as $\rho_{\mathcal{F}} \otimes P_+ = \mu\omega_1 + (1-\mu)\omega_2$, where P_+ is a 1-rank projector onto maximally entangled state $|\psi_+\rangle$. Taking a partial trace over the first subsystem A, we get $P_+ = \mu\, \mathrm{tr}_A[\omega_1] + (1-\mu)\mathrm{tr}_A[\omega_2]$, from which it follows that $\mathrm{tr}_A[\omega_1] = \mathrm{tr}_A[\omega_2] = P_+$. Consequently, $\omega_j = \rho_j \otimes P_+$, $j = 1,2$, because, if a subsystem is in a pure state, then it necessarily is factorized from any other system. As a result, we may conclude that $\mathcal{F} \otimes \mathcal{I} = [\mu\mathcal{E}_1 + (1-\mu)\mathcal{E}_2] \otimes \mathcal{I}$, where $\mathcal{E}_j \otimes \mathcal{I}$ is EA, i.e., $\mathcal{E}_j$ is EB, $j = 1,2$ [property (6)]. Since $\mathcal{F}$ is EB extremal, then the relation $\mathcal{F} = \mu\mathcal{E}_1 + (1-\mu)\mathcal{E}_2$ can only be fulfilled if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{F}$, i.e., $\mathcal{G}_1 = \mathcal{G}_2$. This contradiction concludes the proof. ■

Let us stress two points. First, the identity channel in the proposition can be replaced by any unitary channel. Second, this proposition is not a special case of Proposition 5 because there, we assume channels $\mathcal{E}_1$ and $\mathcal{E}_2$ to be extremal in the set of all qubit channels T_2, whereas, in Proposition 6, the local channel $\mathcal{F}$ is assumed to be EB extremal. In particular, if both $\mathcal{E}_1, \mathcal{E}_2$ are extremal and EB, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is simultaneously extremal, EB extremal, and EA extremal. The following example demonstrates the existence of EA-extremal channels that are not extreme points of the set of all two-qubit channels $\mathsf{T}_{\rm chan}$.

Example 2. Consider a phase-damping channel $\mathcal{F}[\rho] = \frac{1}{2}(\rho + \sigma_z\rho\sigma_z)$. Clearly, this channel is unital and not extremal in the set of all channels. Nevertheless, its action can also be expressed as

$$\mathcal{F}[\rho] = \langle 0|\rho|0\rangle|0\rangle\langle 0| + \langle 1|\rho|1\rangle|1\rangle\langle 1|.$$

This means the phase-damping channel is an extreme point of the set of EB channels, hence, it is EB extremal [24]. Then, by Proposition 6, the two-qubit channels $\mathcal{F}\otimes\mathcal{I}$ and $\mathcal{I}\otimes\mathcal{F}$ are EA extremal.

VII. EA DUALITY

In this section, we employ the idea of EA to introduce the concept of EA duality for local channels. Let us denote, by T_d, the set of channels on a d-dimensional (noncomposite) quantum system (qudit). Let $\mathsf{Q}\subset\mathsf{T}_d$ be an arbitrary subset of qudit channels. A subset $\widetilde{\mathsf{Q}}_n\subset\mathsf{T}_n$ is called n-EA dual to Q if $\mathcal{E}_1\otimes\mathcal{E}_2$ is EA for all $\mathcal{E}_1\in\mathsf{Q}$ and $\mathcal{E}_2\in\widetilde{\mathsf{Q}}_n$. To be more precise,

$$\widetilde{\mathsf{Q}}_n = \{\mathcal{E}_2\in\mathsf{T}_n|\mathcal{E}_1\otimes\mathcal{E}_2 \text{ is EA for all } \mathcal{E}_1\in\mathsf{Q}\}. \tag{22}$$

When it is clear from the context, we omit the explicit mentioning of the dimensions and say $\widetilde{\mathsf{Q}}$ is EA dual to Q.

Using this concept, we may rephrase the goal of this paper as an identification of the set of channels $\mathsf{T}_{\text{ent}}\subset\mathsf{T}_2$ such that EA duals of all its elements coincide with the set of EB channels, i.e., $\widetilde{\{\mathcal{E}\}}=\mathsf{T}_{\text{EB}}$ for all $\mathcal{E}\in\mathsf{T}_{\text{ent}}$. Let us note that, for any Q, its EA dual $\widetilde{\mathsf{Q}}$ contains the EB channels, i.e., $\mathsf{T}_{\text{EB}}\subset\widetilde{\mathsf{Q}}$. It is also clear that $\widetilde{\mathsf{Q}}\subset\widetilde{\{\mathcal{E}\}}$, providing that $\mathcal{E}\in\mathsf{Q}$.

Example 3. Let us clarify the introduced concept with some examples:

(i) For unitary channels $\mathcal{U}[\rho]=U\rho U^\dagger$, the EA dual coincides with EB qubit channels, i.e., $\widetilde{\{\mathcal{U}\}}=\mathsf{T}_{\text{EB}}\subset\mathsf{T}_n$, which means that the system under consideration is maximally robust in sharing the entanglement unless an EB channel is applied to the second system.

(ii) For set T_{EB} of the EB channels, the EA-dual set equals the set of all channels, i.e., $\widetilde{\mathsf{T}}_{\text{EB}}=\mathsf{T}_n$. This only illustrates the fact that EB channels destroy any entanglement between the system under consideration and whatever second system. For each EB channel $\mathcal{E}$, its EA dual contains all channels, i.e., $\widetilde{\{\mathcal{E}\}}=\mathsf{T}_n$.

(iii) The EA dual of all channels is the set of EB channels, i.e., $\widetilde{\mathsf{T}}_d=\mathsf{T}_{\text{EB}}\subset\mathsf{T}_n$. Providing that $d=n$, we can write $\widetilde{\widetilde{\mathsf{T}}}_d=\mathsf{T}_d$.

Fixing d and n, the dual sets obviously satisfy the following properties:

$$\mathsf{Q}_1\subset\mathsf{Q}_2\Rightarrow\widetilde{\mathsf{Q}}_1\supset\widetilde{\mathsf{Q}}_2, \tag{23}$$

$$\widetilde{(\mathsf{Q}_1\cup\mathsf{Q}_2)} = \widetilde{\mathsf{Q}}_1\cap\widetilde{\mathsf{Q}}_2, \tag{24}$$

$$\widetilde{(\mathsf{Q}_1\cap\mathsf{Q}_2)} \supset \widetilde{\mathsf{Q}}_1\cup\widetilde{\mathsf{Q}}_2. \tag{25}$$

The results presented in the previous sections contain partial answers to EA-duality sets for 2-LEA qubit channels, depolarizing qubit channels, or amplitude-damping qubit channels, etc. However, further analysis is beyond the scope of this paper, and the full characterization of EA duals of these and other interesting subsets of qubit channels remains an open problem.

VIII. SUMMARY

In this paper, we have investigated the robustness of entanglement in two-qubit (spatially separated) systems under the influence of independent reservoirs. In particular, we paid attention to the characterization of the so-called EA channels. These are the channels that completely annihilate any entanglement initially present between the subsystems. In contrast, the so-called EB channels destroy any entanglement between the system they act on (two-qubit system in our case) and any other system. The dramatic differences, but also subtle relations, between these two concepts were discovered for particular classes of channels.

We succeeded to characterize all unital two-qubit EA channels of the factorized form $\mathcal{E}_1\otimes\mathcal{E}_2$, and the results are illustrated nicely in Fig. 3. We derived a sufficient condition (Proposition 4) for a local unital channel not to be EA, which guarantees the entanglement retention and enables performing entanglement-enabled experiments in the presence of such noise. We gave (Proposition 2) the complete characterization of unital EA channels $\mathcal{E}\otimes\mathcal{E}$ and showed (Proposition 3) that such (2-LEA) channels $\mathcal{E}$ form a convex subset of the set of all unital single-qubit channels.

For example, in the case of depolarizing channels $\mathcal{E}_1$ and $\mathcal{E}_2$ with rates q_1 and q_2, their product should be kept above $\frac{1}{3}$. Above this critical value, there still exist initial states of the two-qubit system for which the entanglement survives the effects of depolarizing noise.

Particular results have also been obtained for the case of extremal nonunital channels, for which the entanglement turns out to be more robust (in comparison with unital channels). In particular, they are EA only if one of the constituent channels is EB, meaning that (in this case) the set of 2-LEA channels coincides with EB channels.

Much attention has also been focused on such a fundamental entity as the convexity of the set of all two-qubit channels T_{chan} and the set of EA channels T_{EA}. We have revealed that sets T_{chan} and T_{EA} have common extremal points corresponding to local channels. We have constructed a class of purely EA-extremal channels (phase-damping channels), i.e., channels that are extremal for T_{EA} but are internal for T_{chan}.

Finally, we have introduced an important concept of EA duality between sets of channels defined on individual subsystems. This concept gives another perspective on the classification of channels with respect to their EA potentials.

To conclude, the presented analysis contains a partial characterization of local EA two-qubit channels. This class of channels is of particular importance and interest in the domain of quantum-information processing. Although the complete understanding of the phenomenon of EA is still missing, the presented results represent important and practical steps toward this direction. Our analysis shows that the phenomenon of being EA is not a rare one and, hence, deserves further attention. From a practical point of view, we have analyzed, in detail, classes of physically relevant qubit channels (depolarizing, phase damping, amplitude damping) and among them, have identified the good and the bad ones.

ACKNOWLEDGMENTS

This research was carried out while S.N.F. was visiting at the Research Center for Quantum Information, Institute of Physics, Slovak Academy of Sciences. S.N.F. is grateful for their very kind hospitality. This work was supported by EU integrated Project Nos. 2010-248095 (Q-ESSENCE), APVV DO7RP-0002-10, and VEGA 2/0092/09 (QWAEN). S.N.F. thanks the Russian Foundation for Basic Research (Projects No. 09-02-00142, 10-02-00312, and 11-02-00456), the Russian Science Support Foundation, the Dynasty Foundation, and the Ministry of Education and Science of the Russian Federation (Projects No. 2.1.1/5909, П558, and 14.740.11.1257). T.R. acknowledges APVV Grant No. LPP-0264-07 (QWOSSI), and M.Z. acknowledges the support of SCIEX Grant No. 10.271.

[1] E. Schrödinger, Math. Proc. Cambridge Philos. Soc. **31**, 555 (1935).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[3] K. Życzkowski, P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. A **65**, 012101 (2001).

[4] I. Sinayskiy, E. Ferraro, A. Napoli, A. Messina, and F. Petruccione, J. Phys. A: Math. Theor. **42**, 485301 (2009).

[5] C.-J. Shan, J.-B. Liu, W.-W. Cheng, T.-K. Liu, Y.-X. Huang, and H. Li, Commun. Theor. Phys. **51**, 1013 (2009).

[6] M. Scala, R. Migliore, A. Messina, and L. L. Sánchez-Soto, Eur. Phys. J. D **61**, 199 (2011).

[7] W. Cui, Z. Xi, and Y. Pan, J. Phys. A: Math. Theor. **42**, 155303 (2009).

[8] F. Altintas and R. Eryigit, J. Phys. A: Math. Theor. **43**, 415306 (2010).

[9] E. Ferraro, M. Scala, R. Migliore, and A. Napoli, Phys. Scr., T **140**, 014042 (2010).

[10] J.-G. Li, J. Zou, and B. Shao, Phys. Rev. A **82**, 042318 (2010).

[11] J.-G. Li, J. Zou, and B. Shao, Phys. Lett. A **375**, 2300 (2011).

[12] M. Merkli, G. Berman, F. Borgonovi, and K. Gebresellasie, e-print arXiv:1001.1144.

[13] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. S. Ribeiro, and L. Davidovich, Science **316**, 579 (2007).

[14] T. Yu and J. H. Eberly, Science **323**, 598 (2009).

[15] W. A. de Brito and R. V. Ramos, Phys. Lett. A **360**, 251 (2006).

[16] M. Tiersch, F. de Melo, and A. Buchleitner, Phys. Rev. Lett. **101**, 170502 (2008).

[17] T. Konrad, F. de Melo, M. Tiersch, C. Kasztelan, A. Aragão, and A. Buchleitner, Nat. Phys. **4**, 99 (2008).

[18] H. Zhang, J. Luo, T.-T. Ren, and X.-P. Sun, Chin. Phys. Lett. **27**, 090303 (2010).

[19] L. Moravčíková and M. Ziman, J. Phys. A: Math. Theor. **43**, 275306 (2010).

[20] A. S. Holevo, Russ. Math. Surveys **53**, 1295 (1998).

[21] C. King, J. Math. Phys. **43**, 1247 (2002).

[22] P. W. Shor, J. Math. Phys. **43**, 4334 (2002).

[23] M. B. Ruskai, Rev. Math. Phys. **15**, 643 (2003).

[24] M. Horodecki, P. W. Shor, and M. B. Ruskai, Rev. Math. Phys. **15**, 629 (2003).

[25] A. S. Holevo, Probl. Inf. Transm. **44**, 3 (2008).

[26] M. Tiersch, Ph.D. thesis, University of Freiburg, 2009.

[27] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[28] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[29] M. B. Ruskai, S. Szarek, and E. Werner, Linear Algebra Appl. **347**, 159 (2002).

[30] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).

[31] M.-D. Choi, Linear Algebra Appl. **10**, 285 (1975).

[32] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).

[33] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002).