Probability-based comparison of quantum states

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We address the following state comparison problem: is it possible to design an experiment enabling us to unambiguously decide (based on the observed outcome statistics) on the sameness or difference of two unknown state preparations without revealing complete information about the states? We find that the claim "the same" can never be concluded without any doubts unless the information is complete. Moreover, we prove that a universal comparison (that perfectly distinguishes all states) also requires complete information about the states. Nevertheless, for some measurements, the probability distribution of outcomes still allows one to make an unambiguous conclusion regarding the difference between the states even in the case of incomplete information. We analyze an efficiency of such a comparison of qudit states when it is based on the SWAP measurement. For qubit states, we consider in detail the performance of special families of two-valued measurements, enabling us to successfully compare at most half of the pairs of states. Finally, we introduce almost-universal comparison measurements which can distinguish almost all nonidentical states (up to a set of measure zero). The explicit form of such measurements with two and more outcomes is found in any dimension.

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I. INTRODUCTION

The exponential scaling of the number of parameters describing multipartite quantum systems stands behind the potential power of quantum information processing. However, the same feature makes a complete characterization (tomography) of unknown quantum devices intractable. Therefore, it is of practical interest to understand which properties of physical systems require the full tomography for their determination and for which of them is such a complete knowledge redundant. In this paper we analyze the resources needed for a comparison of quantum states. Suppose a given pair of quantum systems in unknown states. The question is what experiments (if any) are capable either of revealing with certainty the difference between the states or confirming their sameness as long as the the probability distribution of measurement outcomes is identified.

By the very nature of quantum theory, the events we observe in quantum experiments are random. That is, both quantum predictions and quantum conclusions are naturally formulated in terms of probabilities and uncertainty. Therefore, it is surprising that there are (very specific) situations (including special instances of the comparison problem) in which individual clicks enable us to make a nontrivial unambiguous prediction or conclusion. For example, if we are given a promise that the states are pure, then (with a nonzero probability) the difference of states can be confirmed unambiguously from a single experimental click [1,2]. This result can be also generalized to the comparison of many pure states [3–5], the comparison of ensembles of pure states [6], and the comparison of some pure continuous-variable states [7,8] (see also the review [9]). Unfortunately, such single-shot (nonstatistical) comparison strategy fails for general mixed states [5,10]. The reason is simple. The probability of any outcome is strictly nonvanishing provided that a bipartite system is in the completely mixed state, for which the subsystems are in the same state. That is, for any outcome there is a situation in which the systems are the same, hence the difference cannot be concluded unambiguously. In such a case any error-free conclusions need to be based on the observed probabilities of outcomes. Probability-based strategies were not considered in previous studies of quantum state comparison. Our aim in this paper is to introduce this concept and provide basic results in this area.

Trivially, if the experimentally measured probabilities provide complete information on quantum states of both systems individually, then they also contain all the information needed for the comparison. The question of our interest is whether the complete tomography is necessary. Our main goal is to design a comparison experiment providing as little redundant information as possible.

In Sec. II, we introduce the necessary mathematical notation and formulate the problem. In Sec. III, we address the existence of a universal comparison measurement. Section IV investigates the comparison performance of two-outcome measurements. Almost universal two-valued and many-valued comparison measurements are presented in Sec. V and conclusions are the content of Sec. VI.

II. PROBLEM FORMULATION

Any quantum state is associated with the density operator $\varrho \in \mathcal{S}(\mathcal{H})$ such that $\varrho \geqslant O$ and $\mathrm{tr}[\varrho] = 1$. Hereafter, $\mathcal{S}(\mathcal{H})$ stands for the set of all states of a system associated with the Hilbert space $\mathcal{H}$. The statistical features of quantum measurements are fully captured by means of a positive operator-valued measure (POVM) that is a collection E of positive operators (acting on $\mathcal{H}$ and called effects) $E_1, \ldots, E_n$ summing up to the identity (i.e., $\sum_{j=1}^{n} E_j = I$). For each state $\varrho \in \mathcal{S}(\mathcal{H})$ the measurement E assigns a probability distribution $\{p_j\}_{j=1}^{n} \equiv \vec{p}_{\mathsf{E}}$, where $p_j = \mathrm{tr}[E_j \varrho] \geqslant 0$ and $\sum_{j=1}^{n} p_j = 1$.

Let us now move on to the set of bipartite factorized states $\mathcal{S}_{\mathrm{fac}} = \{\varrho \otimes \xi : \varrho, \xi \in \mathcal{S}(\mathcal{H})\} \subset \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$, where the parties ϱ and ξ are the states to be compared. For a fixed measurement

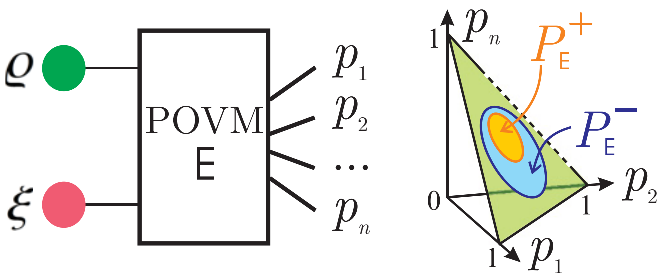

FIG. 1. (Color online) Illustration of probability-based comparison. If the observed probability distribution belongs to $P_{\mathsf{E}}^- \setminus P_{\mathsf{E}}^+$, then states ϱ and ξ are for sure different.

E we can ask how much information it reveals concerning the comparison of the subsystems.

Denote by $\mathcal{S}^+$ the subset of *twin-identical states*; that is, $\mathcal{S}^+ = \{\eta\otimes\eta : \eta \in \mathcal{S}(\mathcal{H})\} \subset \mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. Similarly, let us denote by $\mathcal{S}^-$ the subset of *nonidentical states*; that is, $\mathcal{S}^- = \{\varrho\otimes\xi : \varrho,\xi(\neq\varrho) \in \mathcal{S}(\mathcal{H})\} \subset \mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. Obviously, $\mathcal{S}_{\rm fac} = \mathcal{S}^+ \cup \mathcal{S}^-$. The goal of comparison is then to distinguish between sets of states $\mathcal{S}^+$ and $\mathcal{S}^-$. This goal can be achieved in our approach by considering two sets of probability distributions $P_{\mathsf{E}}^\pm = \{\vec{p} : p_j = {\rm tr}[E_j\omega],\ \omega \in \mathcal{S}^\pm\}$. In other words, since the measurement E performs the mapping $\mathcal{S}^\pm \mapsto P_{\mathsf{E}}^\pm$, one can unambiguously conclude that a bipartite state ω belongs to the set $\mathcal{S}^\pm$ if the observed probability distribution $\vec{p}_{\mathsf{E}} \in P_{\mathsf{E}}^\pm \setminus P_{\mathsf{E}}^\mp$ (see Fig. 1).

For a fixed POVM E on $\mathcal{H}\otimes\mathcal{H}$ we may introduce the following quantities:

$$D_{\mathsf{E}}(\varrho\otimes\xi,\mathcal{S}^+) = \inf_{\eta\otimes\eta\in\mathcal{S}^+} \sum_{j=1}^{n} |{\rm tr}[E_j(\varrho\otimes\xi - \eta\otimes\eta)]|, \quad (1)$$

$$D_{\mathsf{E}}(\eta\otimes\eta,\mathcal{S}^-) = \inf_{\varrho\otimes\xi\in\mathcal{S}^-} \sum_{j=1}^{n} |{\rm tr}[E_j(\varrho\otimes\xi - \eta\otimes\eta)]|. \quad (2)$$

While $D_{\mathsf{E}}(\varrho\otimes\xi,\mathcal{S}^+)$ quantifies how different the states ϱ and ξ are (with respect to measurement E), the value of $D_{\mathsf{E}}(\eta\otimes\eta,\mathcal{S}^-)$ tells us to which extent the equivalence of twin-identical states can be confirmed.

Before we proceed further let us make one important observation: for all $\epsilon > 0$ and any state $\eta\otimes\eta$ there exists a state $\varrho\otimes\xi$ such that $|{\rm tr}[E(\eta\otimes\eta - \varrho\otimes\xi)]| \leqslant \epsilon$ for any POVM effect E. In other words, in order to conclude that the states are the same no uncertainty in the specification of the probabilities $p_E(\omega) = {\rm tr}[E\omega]$ is allowed. Such an infinite precision is practically not achievable; however, for our purposes we will assume the probabilities are specified exactly. The proof of the statement above is relatively straightforward. Let us set $\varrho = \eta$ and $\xi = (1-\frac{\epsilon}{2})\eta + \frac{\epsilon}{2d}I$, thus, $\eta\otimes\eta - \varrho\otimes\xi = \frac{\epsilon}{2}\eta\otimes(\eta - \frac{1}{d}I)$. Since $|{\rm tr}[EX]| \leqslant \max_{E\in\mathsf{E}} {\rm tr}[|EX|] \leqslant {\rm tr}[|X|]$, it follows that

$$|{\rm tr}[E(\eta\otimes\eta - \varrho\otimes\xi)]| \leqslant \epsilon\, \frac{1}{2}{\rm tr}\left[\left|\eta\otimes\left(\eta - \frac{1}{d}I\right)\right|\right] \leqslant \epsilon. \quad (3)$$

In the last inequality we used the fact that the trace distance of states is bounded from above by one. Formula (3) is valid for any POVM effect E_j; therefore $\sum_{j=1}^{n} |{\rm tr}[E_j(\varrho\otimes\xi - \eta\otimes\eta)]| \leqslant n\epsilon \to 0$ when $\epsilon \to 0$. By definition of the greatest lower bound (infimum), the distance (2) vanishes for an arbitrary $\eta \in \mathcal{S}(\mathcal{H})$.

Our observation implies that, for any measurement E, we have $D_{\mathsf{E}}(\eta\otimes\eta,\mathcal{S}^-) = 0$. This seems to be in contradiction with measurements which provide us with complete information on the states of individual systems. We will refer to such measurements as *locally informationally complete* (LIC) measurements. Clearly, in case of an ideal LIC measurement the sameness can be verified. Where is the problem? Topologically, in the set of factorized states $\mathcal{S}_{\rm fac}$ with the trace-distance metrics, the subset $\mathcal{S}^+$ is closed and does not contain any interior point. Therefore, the distance (2) vanishes. However, it does not mean that the subset $\mathcal{S}^+$ is empty.

Remark. The subset $\mathcal{S}^+$ is closed because the set $\mathcal{S}(\mathcal{H})$ is closed. To prove that $\mathcal{S}^+$ does not contain any interior point, assume the converse. Let $\eta_0\otimes\eta_0$ be an interior point of $\mathcal{S}^+$, then there exists a neighborhood $\mathcal{O}_\varepsilon(\eta_0\otimes\eta_0)$ such that $\mathcal{O}_\varepsilon(\eta_0\otimes\eta_0) \subset \mathcal{S}^+$. Choose an arbitrary point $\eta\otimes\eta \in \mathcal{O}_\varepsilon(\eta_0\otimes\eta_0)$ with $\eta \neq \eta_0$, then a nontrivial convex combination $[\lambda\eta_0\otimes\eta_0 + (1-\lambda)\eta\otimes\eta] \in \mathcal{O}_\varepsilon(\eta_0\otimes\eta_0) \subset \mathcal{S}^+$; that is, $\lambda\eta_0\otimes\eta_0 + (1-\lambda)\eta\otimes\eta = \zeta\otimes\zeta$ for some $\zeta \in \mathcal{S}(\mathcal{H})$. Taking the partial trace over the first subsystem, we obtain $\lambda\eta_0 + (1-\lambda)\eta = \zeta$. In view of this, $\zeta\otimes\zeta = \lambda^2\eta_0\otimes\eta_0 + \lambda(1-\lambda)(\eta_0\otimes\eta + \eta\otimes\eta_0) + (1-\lambda)^2\eta\otimes\eta$. Subtracting the two expressions obtained for $\zeta\otimes\zeta$ yields $(\eta-\eta_0)\otimes(\eta-\eta_0) = 0$ (i.e., $\eta = \eta_0$), which contradicts the choice $\eta \neq \eta_0$. Thus, $\mathcal{S}^+$ does not contain any interior point.

The vanishing value of the distance considered (2) is not completely relevant if one thinks about the ideal error-free experiments. In practice, experimental noise is unavoidable; hence, from the practical point of view a conclusion on the sameness of states can never be error free.

III. UNIVERSAL COMPARISON MEASUREMENT

We say the measurement E implements the comparison whenever $D_{\mathsf{E}}(\varrho\otimes\xi,S^+) > 0$ for some pairs ϱ,ξ. The state comparison measurement E is *universal* if $D_{\mathsf{E}}(\varrho\otimes\xi,S^+) > 0$ for all $\varrho,\xi(\neq\varrho)$. This is, for instance, achieved in case of the ideal LIC measurements: even though the value of $D_{\mathsf{E}}(\varrho\otimes\xi,S^+)$ can be arbitrarily small, it always remains strictly positive. As before, this situation is not very realistic in practice, because any error in the identification of outcome probabilities makes the conclusions (in some cases of ϱ and ξ) ambiguous. However, assuming the infinite precision in the specification of probabilities, the universality can be achieved and in what follows we will assume that probabilities are identified perfectly. The potential errors can be viewed as modifications of the sets $\mathcal{S}^+$ and $\mathcal{S}^-$ that we are aiming to distinguish. Nevertheless, our goal is to analyze the ideal case.

Let us now demonstrate that a universal comparison can be implemented if and only if the measurement is LIC.

To start with, we are reminded that any POVM E with effects E_j linearly maps a state $\omega \in \mathcal{S}(\mathcal{H}\otimes\mathcal{H})$ into the probability vector $\vec{p} = (p_1,p_2,\ldots)$, where $p_j = {\rm tr}[E_j\omega]$. For LIC measurements the induced mapping $\varrho\otimes\xi \mapsto \vec{\pi}$ is bijective. That proves the sufficiency. To prove the necessity let us assume the converse; that is, suppose the measurement E is not an LIC measurement but implements a universal comparison. Since E is not LIC, the probability assignment $\varrho\otimes\xi \mapsto \vec{p}_{\mathsf{E}}$ is injective. Let us denote by $\Pi^{\pm}$ and Π the images of $\mathcal{S}^{\pm}$ and $\mathcal{S}_{\rm fac}$ under some LIC measurement, respectively, and by $P_{\sf E}$ denote the image of $\mathcal{S}_{\rm fac}$ under the measurement $\sf E$. Clearly, the relation between $\vec{\pi}(\varrho\otimes\xi)$ and $\vec{p}_{\sf E}(\varrho\otimes\xi)$ is linear and injective; that is, there exist probability vectors $\vec{\pi}_1\in\Pi$ and $\vec{\pi}_2\in\Pi$ transformed into the same probability distribution $\vec{p}_{\sf E}(\varrho\otimes\xi)\in P_{\sf E}$. The distributions $\vec{\pi}_j$ transformed into the same probability vector $\vec{p}_{\sf E}(\varrho\otimes\xi)$ span a linear subspace (hyperplane) $H_{\varrho\otimes\xi}$ in the linear span of Π.

Consider an internal point $\eta\in\mathcal{S}(\mathcal{H})$, then the image $\vec{\pi}(\eta\otimes\eta)$ is an interior point of Π on the probability simplex. There exists $\epsilon_0>0$ such that, for all $0<\epsilon\leqslant\epsilon_0$, the neighborhood $O_\epsilon(\vec{\pi}(\eta\otimes\eta))$ belongs to Π (on the simplex). Moreover, the intersection $O_\epsilon(\vec{\pi}(\eta\otimes\eta))\cap H_{\eta\otimes\eta}$ cannot be a subset of Π^+ only, because Π^+ does not contain any interior point on the simplex [if it did, the distance $D_{\sf E}(\eta\otimes\eta,\mathcal{S}^-)$ would not vanish for all states η]. Thus, $O_\epsilon(\vec{\pi}(\eta\otimes\eta))\cap H_{\eta\otimes\eta}\cap\Pi^-$ is not empty and contains points of the form $\vec{\pi}(\tilde{\varrho}\otimes\tilde{\xi})$ such that $\tilde{\varrho}\neq\tilde{\xi}$. As both $\vec{\pi}(\eta\otimes\eta)$ and $\vec{\pi}(\tilde{\varrho}\otimes\tilde{\xi})$ belong to $H_{\eta\otimes\eta}$, we have $\vec{p}_{\sf E}(\tilde{\varrho}\otimes\tilde{\xi})=\vec{p}_{\sf E}(\eta\otimes\eta)$ and formula (1) yields $D(\tilde{\varrho}\otimes\tilde{\xi},\mathcal{S}^+)=0$; that is, $\sf E$ is not a universal comparison measurement (by definition). This contradiction concludes the proof of the necessity.

Let us summarize two main conclusions:

(1) In any locally informationally incomplete measurement the sameness of states cannot be confirmed.

(2) Universal comparison (concluding universally and unambiguously the difference of states) requires a locally informationally complete measurement.

A question that remains open is how to evaluate the overall performance of (universal or nonuniversal) comparison experiments. There are several options. We can use the volume of the subset $\mathcal{S}^-_{\rm comp}$ of states in $\mathcal{S}^-$ that can be successfully compared, or the average value of $D_{\sf E}(\varrho\otimes\xi,\mathcal{S}^+)$ with respect to some measure on the state space. In particular, these quantities read

$$|\mathcal{S}^-_{\rm comp}|_{\sf E}=\int\int_{\mathcal{S}^-}\mu(d\varrho)\mu(d\xi)h(D_{\sf E}(\varrho\otimes\xi,\mathcal{S}^+)),\quad(4)$$

$$\langle D_{\sf E}\rangle=\int\int_{\mathcal{S}^-}\mu(d\varrho)\mu(d\xi)D_{\sf E}(\varrho\otimes\xi,\mathcal{S}^+),\quad(5)$$

where $h(x)$ is the Heaviside function and $\mu(d\varrho)=\mu(d\xi)$ is a measure on the state space of individual subsystems. Quite common choices for the measure μ on density operators are the ones induced by metrics; namely, by the Bures distance and Hilbert-Schmidt distance (see, e.g., [12,13] and references therein). Let us stress that $|\mathcal{S}^-_{\rm comp}|_{\sf E}=1$ does not imply the comparison is universal, because there can be a set of measure zero for which $D_{\sf E}(\varrho\otimes\xi,S^+)=0$. In such a case we say that the comparison measurement is *almost universal*. It is of great interest to investigate whether there exist some almost universal comparison experiments and, in particular, how many outcomes such measurements require.

IV. TWO-VALUED COMPARISON EXPERIMENTS

Let us start our investigation with the simplest case of two-valued POVMs described by the effects E and $I-E$. In such a case,

$$D_{\sf E}(\varrho\otimes\xi,\mathcal{S}^+)=2D_E(\varrho\otimes\xi,\mathcal{S}^+),\quad(6)$$

where $D_E(\varrho\otimes\xi,\mathcal{S}^+)=\min_{\eta\otimes\eta\in\mathcal{S}^+}|{\rm tr}[E(\varrho\otimes\xi-\eta\otimes\eta)]|$. Two-valued measurements cannot be LIC, because they provide the only informative real number (the probability p_E, $p_{I-E}=1-p_E$) whereas the state $\varrho\otimes\xi$ is defined by $2(d^2-1)\geqslant 6$ real numbers. Thus, two-valued measurements are necessarily nonuniversal comparators. Nevertheless, it is of practical interest to understand how good their comparison performance is.

Let us consider a geometry of comparable states. Suppose $\varrho=\frac{1}{d}(I+\mathbf{r}\cdot\mathbf{\Lambda})$ and $\xi=\frac{1}{d}(I+\mathbf{k}\cdot\mathbf{\Lambda})$, where $\mathbf{\Lambda}=(\Lambda_1,\ldots,\Lambda_{d^2-1})$ is a vector formed of traceless Hermitian operators Λ_j such that ${\rm tr}[\Lambda_j\Lambda_k]=d\delta_{jk}$, and $\mathbf{r},\mathbf{k}\in\mathbb{R}^{d^2-1}$ are Bloch-like vectors which necessarily satisfy $|\mathbf{r}|,|\mathbf{k}|\leqslant\sqrt{d-1}$ (see, e.g., [14]). Using this notation, let us find such vectors $\mathbf{r}$ that the states ϱ are comparable with a fixed state ξ (the POVM effect E is fixed as well). The trace ${\rm tr}[E\varrho\otimes\xi]=\frac{1}{d}({\rm tr}[EI\otimes\xi]+\mathbf{r}\cdot\mathbf{K})$, where $\mathbf{K}={\rm tr}[E\mathbf{\Lambda}\otimes\xi]$. Therefore, the inequality $D_E(\varrho\otimes\xi,\mathcal{S}^+)>0$ boils down to either

$$\frac{1}{d}\mathbf{r}\cdot\mathbf{K}>\max_{\eta\otimes\eta\in\mathcal{S}^+}{\rm tr}[E\eta\otimes\eta]-\frac{1}{d}{\rm tr}[EI\otimes\xi],\quad(7)$$

or

$$\frac{1}{d}\mathbf{r}\cdot\mathbf{K}<\min_{\eta\otimes\eta\in\mathcal{S}^+}{\rm tr}[E\eta\otimes\eta]-\frac{1}{d}{\rm tr}[EI\otimes\xi].\quad(8)$$

These inequalities define two nonintersecting half-spaces in $\mathbb{R}^{d^2-1}$ separated by the distance

$$L=\frac{d}{|\mathbf{K}|}\Big(\max_{\eta\otimes\eta\in\mathcal{S}^+}{\rm tr}[E\eta\otimes\eta]-\min_{\eta\otimes\eta\in\mathcal{S}^+}{\rm tr}[E\eta\otimes\eta]\Big).\quad(9)$$

Thus, for any fixed ξ the set of successfully comparable Bloch-like vectors $\mathbf{r}$ is given by an intersection of two half-spaces with the state space.

A. SWAP-based comparison

As we have already mentioned in Sec. I, if we restrict ourselves only to pure states, then there exists a strategy to perform an unambiguous comparison (via the SWAP measurement). In such an approach, the sameness of the states cannot be concluded and this is related to the absence of the universal NOT operation [11]. However, the strategy (if successful) can reveal the difference between the states in a single shot; hence, no collection of statistics is needed.

The key observation for such a conventional strategy is that the support of twin-identical pure states spans only the symmetric subspace of $\mathcal{H}\otimes\mathcal{H}$. Suppose projections $E_{\rm sym},E_{\rm asym}$ onto symmetric and antisymmetric subspaces of $\mathcal{H}\otimes\mathcal{H}$. Since $E_{\rm sym}+E_{\rm asym}=I$ they form a two-valued POVM $\mathsf{E}_{\rm SWAP}$. Let us note that $E_{\rm sym}=\frac{1}{2}(I+S)$, $E_{\rm asym}=\frac{1}{2}(I-S)$, where S is the SWAP operator acting as $S(|\psi\otimes\varphi\rangle)=|\varphi\otimes\psi\rangle$ for all $|\psi\rangle,|\varphi\rangle\in\mathcal{H}$. It is straightforward to see that, for any twin-identical pure state $|\varphi\otimes\varphi\rangle$, one has ${\rm tr}[E_{\rm asym}|\varphi\otimes\varphi\rangle\langle\varphi\otimes\varphi|]=0$; however, ${\rm tr}[E_{\rm asym}|\varphi\otimes\psi\rangle\langle\varphi\otimes\psi|]=\frac{1}{2}(1-|\langle\varphi|\psi\rangle|^2)>0$ if $|\psi\rangle\neq|\varphi\rangle$. Therefore, recording an outcome $E_{\rm asym}$ allows us to unambiguously conclude that the states are different. No statistics is needed.

Let us see how this strategy works in the case of general mixed states. A direct calculation yields

$$p_{\text{sym}} = \text{tr}[E_{\text{sym}}\varrho \otimes \xi] = \tfrac{1}{2}(1 + \text{tr}[\varrho\xi]), \tag{10}$$

$$p_{\text{asym}} = \text{tr}[E_{\text{asym}}\varrho \otimes \xi] = \tfrac{1}{2}(1 - \text{tr}[\varrho\xi]), \tag{11}$$

where we used the identity $\text{tr}[S\varrho \otimes \xi] = \text{tr}[\varrho\xi]$. The purity of a state, $\text{tr}[\eta^2]$, is bounded from below by $1/d$, where $d = \dim \mathcal{H}$. Thus,

$$p_{\text{sym}}(\varrho \otimes \xi) \in \left[\frac{1}{2},1\right) \equiv P_{\text{sym}}^{-}, \tag{12}$$

$$p_{\text{sym}}(\eta \otimes \eta) \in \left[\frac{d+1}{2d},1\right] \equiv P_{\text{sym}}^{+}, \tag{13}$$

where $P_{\text{sym}}^{\pm}$ is the image of $\mathcal{S}^{\pm}$ under the POVM effect E_{sym}. It follows that, by measuring the probability $p_{\text{sym}} < (d+1)/2d$, we can with certainty conclude that the states are different. In particular, $D_{\text{SWAP}}(\varrho \otimes \xi, \mathcal{S}^{+}) = \max\{0, \frac{1}{d} - \text{tr}[\varrho\xi]\}$.

Associating ϱ and ξ with the Bloch-like vectors $\mathbf{r},\mathbf{k} \in \mathbb{R}^{d^2-1}$ as above ($|\mathbf{r}|,|\mathbf{k}| \leqslant \sqrt{d-1}$), for the SWAP-based measurement we obtain $\mathbf{K}_{\pm} = \text{tr}[(I \pm S)\mathbf{\Lambda} \otimes \xi] = \pm\mathbf{k}$. Also, we find explicitly $\text{tr}[(I \pm S)I \otimes \xi] = d \pm 1$, $\max_{\eta\otimes\eta\in\mathcal{S}^+} \text{tr}[(I + S)\eta \otimes \eta] = 2$, $\max_{\eta\otimes\eta\in\mathcal{S}^+} \text{tr}[(I - S)\eta \otimes \eta] = 1 - \frac{1}{d}$, $\min_{\eta\otimes\eta\in\mathcal{S}^+} \text{tr}[(I + S)\eta \otimes \eta] = 1 + \frac{1}{d}$, and $\min_{\eta\otimes\eta\in\mathcal{S}^+} \text{tr}[(I - S)\eta \otimes \eta] = 0$. Then it is straightforward to see that, for the SWAP-based measurement, one of inequalities (7) and (8) is never fulfilled and the other one reduces to $\mathbf{r} \cdot \mathbf{k} < 0$. That is, for each fixed $\mathbf{k}(\neq\mathbf{0})$ the set of successfully comparable Bloch-like vectors $\mathbf{r}$ is given by an intersection of a single half-space with the state space. Let us stress that for qubits ($d = 2$) the state space is exactly the Bloch ball $|\mathbf{r}| \leqslant 1$, so the set of comparable vectors $\mathbf{r}$ is the hemisphere. Thus, for qubits $|\mathcal{S}_{\text{comp}}^{-}|_{\text{SWAP}} = \frac{1}{2}$ (see the next paragraph). In other words, the difference of states from the same hemisphere is not detected in the SWAP measurement. This implies that the approximate universality is lost.

Due to unitary invariance of the measure μ we can always treat one of the states in $D_E(\varrho \otimes \xi, \mathcal{S}^{+})$ as diagonal, say ϱ. Then $\text{tr}[\varrho\xi] = \sum_{j=1}^{d} \varrho_{jj}\xi_{jj}$ and the integration area of $|\mathcal{S}_{\text{comp}}^{-}|_{\text{SWAP}}$ is split into $d!$ subsets labeled by the permutation of the labels $j_1,\ldots,j_d$ identifying the ordering $\varrho_{j_1j_1} \geqslant \cdots \geqslant \varrho_{j_dj_d}$ of eigenvalues of ϱ. The (normalized) volume of each of these subsets is $1/(d!)$. If ϱ and ξ are from mutually opposite subsets (labelled as $j_1,\ldots,j_d$ and $j_d,\ldots,j_1$, respectively), then $\text{tr}[\varrho\xi] = \sum_{j=1}^{d} \varrho_{jj}\xi_{jj} \leqslant \frac{1}{d}$ (see the remark below), meaning that such pairs ϱ and ξ can be successfully compared. Therefore, we find the lower bound $|\mathcal{S}_{\text{comp}}^{-}|_{\text{SWAP}} \geqslant 1/(d!)$. If ϱ,ξ are from the same subset, then $\text{tr}[\varrho\xi] = \sum_{j=1}^{d} \varrho_{jj}\xi_{jj} \geqslant \frac{1}{d}$ (see the remark below), hence the contribution to $|\mathcal{S}_{\text{comp}}^{-}|_{\text{SWAP}}$ is vanishing and we can bound the fraction of comparable states also from above, $|\mathcal{S}_{\text{comp}}^{-}|_{\text{SWAP}} \leqslant 1 - 1/(d!)$. For qubits both bounds coincide and give the value $|\mathcal{S}_{\text{comp}}^{-}|_{\text{SWAP}} = \frac{1}{2}$.

Remark. The d-dimensional probability vectors $\boldsymbol{\varrho} = (\varrho_{j_1j_1},\ldots,\varrho_{j_dj_d})$ obeying the ordering $\varrho_{j_1j_1} \geqslant \cdots \geqslant \varrho_{j_dj_d}$ form a convex subset on a $(d-1)$ simplex, with the extremal points being $(1,0,0,\ldots,0)$, $(\frac{1}{2},\frac{1}{2},0,\ldots,0),\ldots,(\frac{1}{d},\frac{1}{d},\frac{1}{d},\ldots,\frac{1}{d})$. Analogously, the d-dimensional probability vectors $\boldsymbol{\xi} = (\xi_{j_1j_1},\ldots,\xi_{j_dj_d})$ obeying the ordering $\xi_{j_1j_1} \leqslant \cdots \leqslant \xi_{j_dj_d}$ form a convex subset on $(d-1)$ simplex, with the extremal points being $(0,\ldots,0,0,1)$, $(0,\ldots,0,\frac{1}{2},\frac{1}{2}),\ldots,(\frac{1}{d},\ldots,\frac{1}{d},\frac{1}{d},\frac{1}{d})$. Since the function $f(\boldsymbol{\varrho},\boldsymbol{\xi}) = \boldsymbol{\varrho}\cdot\boldsymbol{\xi} = \sum_{j=1}^{d} \varrho_{jj}\xi_{jj} = \text{tr}[\varrho\xi]$ is linear with respect to ϱ_{jj} and ξ_{jj}, then its extremal (maximum) value is achieved for some pair $(\boldsymbol{\varrho},\boldsymbol{\xi})$ of extremal probability vectors $\boldsymbol{\varrho}$ and $\boldsymbol{\xi}$. It can be easily checked that, for all pairs of extremal probability vectors, $\boldsymbol{\varrho}\cdot\boldsymbol{\xi} \leqslant \frac{1}{d}$ (i.e., $\text{tr}[\varrho\xi] \leqslant \frac{1}{d}$). Similarly, for the ordering $\xi_{j_1j_1} \geqslant \cdots \geqslant \xi_{j_dj_d}$ one has $\text{tr}[\varrho\xi] \geqslant \frac{1}{d}$.

B. Qubit "diagonal" comparison experiments

In the previous subsection we have shown that the SWAP-based comparison enables us (in the case of qubits) to successfully detect the difference for half (up to a set of zero measure) of the pairs $\varrho \otimes \xi$. Can one do better with some other two-valued measurement?

Arguing as in the beginning of Sec. IV, we can conclude that, for a given effect E, the states ϱ comparable with a fixed state ξ form a cut of the Bloch ball or two cuts of this ball by parallel planes. Note that, for the SWAP-based comparison, the complete mixture cannot be conclusively compared with any other state. So we will be interested in finding such a two-valued measurement for which the complete mixture can be unambiguously distinguished from some other states.

A general two-qubit effect takes the form

$$E = \sum_{l,m=0}^{3} \varepsilon_{lm}\sigma_l \otimes \sigma_m, \tag{14}$$

where we use the notation $\sigma_0 = I$ and $\sigma_1,\sigma_2,\sigma_3$ denote the Pauli operators $\sigma_x,\sigma_y,\sigma_z$, respectively. Real coefficients ε_{lm} read $\varepsilon_{lm} = \frac{1}{4}\text{tr}[E\sigma_l \otimes \sigma_m]$ and satisfy the the constraints $O \leqslant E \leqslant I$.

For the sake of simplicity let us consider only the diagonal case; that is, we will assume $\varepsilon_{lm} = \delta_{lm}\varkappa_m$, with $\varkappa_m \in \mathbb{R}$ for all $m = 0,\ldots,3$. Then

$$E_{\text{diag}} = \begin{pmatrix} \varkappa_0 + \varkappa_3 & 0 & 0 & \varkappa_1 - \varkappa_2 \\ 0 & \varkappa_0 - \varkappa_3 & \varkappa_1 + \varkappa_2 & 0 \\ 0 & \varkappa_1 + \varkappa_2 & \varkappa_0 - \varkappa_3 & 0 \\ \varkappa_1 - \varkappa_2 & 0 & 0 & \varkappa_0 + \varkappa_3 \end{pmatrix}. \tag{15}$$

Applying the positivity constraints on E of this form we obtain the following conditions:

$$0 \leqslant \varkappa_0 - \varkappa_1 - \varkappa_2 - \varkappa_3 \leqslant 1, \tag{16}$$

$$0 \leqslant \varkappa_0 - \varkappa_1 + \varkappa_2 + \varkappa_3 \leqslant 1, \tag{17}$$

$$0 \leqslant \varkappa_0 + \varkappa_1 - \varkappa_2 + \varkappa_3 \leqslant 1, \tag{18}$$

$$0 \leqslant \varkappa_0 + \varkappa_1 + \varkappa_2 - \varkappa_3 \leqslant 1. \tag{19}$$

The system of inequalities (16)–(19) has a nontrivial solution whenever $0 < \varkappa_0 < 1$. Indeed, if $\varkappa_0$ is fixed, then each two-sided inequality of the system determines a geometric fiber between two planes in the reference frame $(\varkappa_1,\varkappa_2,\varkappa_3)$, with the distance between the planes being equal to $\frac{1}{\sqrt{3}}$. If $0 < \varkappa_0 \leqslant \frac{1}{4}$, then four fibers intersect to yield a tetrahedron. The intersection becomes a truncated tetrahedron if $\frac{1}{4} < \varkappa_0 < \frac{1}{2}$ and finally transforms into an octahedron for the case $\varkappa_0 = \frac{1}{2}$. If $\varkappa_0 \geqslant \frac{1}{2}$ then the solution is a body obtained by the

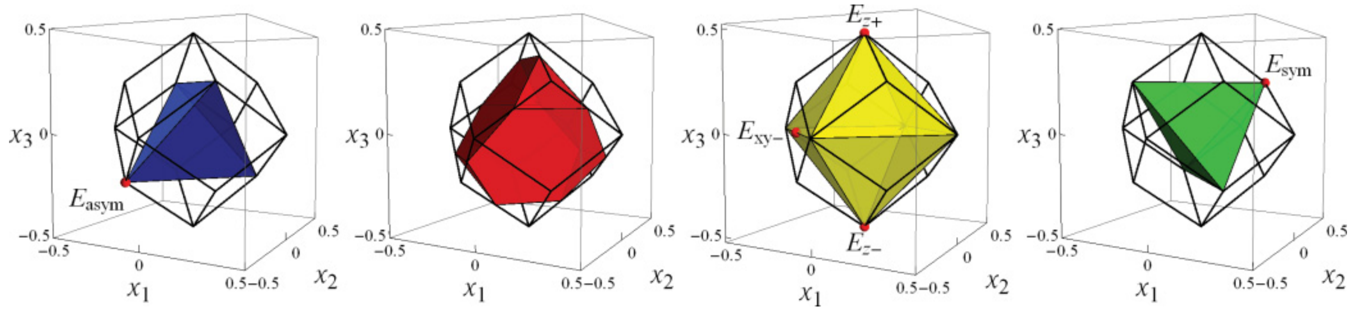

FIG. 2. (Color online) Body $B(\varkappa_0)$ [i.e., the region of parameters $(\varkappa_1,\varkappa_2,\varkappa_3)$] when (15) is a true POVM effect. Parameter $\varkappa_0$ takes values $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, and $\frac{3}{4}$ for figures from left to right. The union $\cup_{\varkappa_0\in[0,1]}B(\varkappa_0)$ is a rhombododecahedron and is depicted by solid lines. POVM effects E_{asym}, E_{sym}, $E_{z\pm}$ are vertices and POVM effects $E_{xy\pm}$ are face centers of this convex polytope.

inversion $(\varkappa_1,\varkappa_2,\varkappa_3)\to(-\varkappa_1,-\varkappa_2,-\varkappa_3)$ of the body labeled by parameter $(1-\varkappa_0)$. For instance, if $\frac{3}{4}\leqslant\varkappa_0<1$, then the intersection is a tetrahedron inverted with respect to that in the case $0<\varkappa_0\leqslant\frac{1}{4}$. Given $\varkappa_0$ we will refer to the intersection as body $B(\varkappa_0)$ (see Fig. 2).

The probabilities of the measurement outcome, corresponding to the POVM effect E_{diag}, can be readily calculated for the nonidentical (different) and twin-identical (same) states and read

$$p_{diff}=p_{diag}(\varrho\otimes\xi)=\varkappa_0+\sum_{m=1}^{3}\varkappa_m r_m k_m, \qquad (20)$$

$$p_{same}=p_{diag}(\eta\otimes\eta)=\varkappa_0+\sum_{m=1}^{3}\varkappa_m h_m^2, \qquad (21)$$

where we used $\varrho=\frac{1}{2}(I+\mathbf{r}\cdot\boldsymbol{\sigma})$, $\xi=\frac{1}{2}(I+\mathbf{k}\cdot\boldsymbol{\sigma})$, and $\eta=\frac{1}{2}(I+\mathbf{h}\cdot\boldsymbol{\sigma})$. Using the normalization constraints for Bloch vectors $\mathbf{r}$, $\mathbf{k}$, and $\mathbf{h}$, we obtain from Eqs. (20) and (21) that probabilities p_{diff} and p_{same} satisfy the relations

$$p_{diff}\in[\varkappa_0-\varkappa_{max},\varkappa_0+\varkappa_{max}]\equiv \mathrm{cl}(P^-_{diag}), \qquad (22)$$

$$p_{same}\in[\varkappa_0-|\varkappa_-|,\varkappa_0+\varkappa_+]\equiv P^+_{diag}, \qquad (23)$$

respectively, where we introduced the notations $\varkappa_{max}=\max\{|\varkappa_1|,|\varkappa_2|,|\varkappa_3|\}$, $\varkappa_-=\min\{0,\varkappa_1,\varkappa_2,\varkappa_3\}$, and $\varkappa_+=\max\{0,\varkappa_1,\varkappa_2,\varkappa_3\}$. Clearly, $P^+_{diag}\subset\mathrm{cl}(P^-_{diag})$; hence, there is a two-valued POVM that allows us to make a nontrivial conclusion about the difference of some states. Notice that the case $\varkappa_0=\frac{3}{4}$, $\varkappa_1=\varkappa_2=\varkappa_3=\frac{1}{4}$ gives the SWAP-based comparison measurement for which, whenever the measured probability p_{diag} satisfies $p_{diag}<\frac{3}{4}$, the states ϱ and ξ are unambiguously different.

1. Fixed-pair comparison

Surprisingly, there are pairs of states ϱ and ξ such that no measurement E of the form (15) can reveal their difference. A direct calculation gives that the difference for a pair of states ϱ and ξ can be concluded in the diagonal comparison experiment if

$$D_{diag}(\varrho\otimes\xi,\mathcal{S}^+)=\min_{|\mathbf{h}|\leqslant 1}\left|\sum_{m=1}^{3}\varkappa_m\left(r_m k_m-h_m^2\right)\right|>0. \qquad (24)$$

Suppose that $r_m k_m$ is nonnegative for all m. Setting $h_m=\sqrt{r_m k_m}$, the distance (24) is vanishing for arbitrary measurement of the considered diagonal form. Let us stress that the requirement of positivity of $r_m k_m$ for all m means that signs of the Bloch vector components coincide, hence, $\mathbf{k}$ and $\mathbf{r}$ belong to the same octant of the Bloch ball. Let us stress, however, that the octants depend on the choice of the axes (Pauli operators) and, for a given pair of states, we can always fix the coordinate system in such a way that they belong to two different octants. The only exceptions are collinear vectors $\mathbf{k}$ and $\mathbf{r}=c\mathbf{k}$ for $c\geqslant 0$. In fact, a pair of parallel Bloch vectors (pointing in the same direction) is indistinguishable by any diagonal measurement irrelevant of the choice of coordinate system. In particular, it follows that none of these measurements is capable of distinguishing (in the comparison sense) the complete mixture $\varrho=\frac{1}{2}I$ from any other state, because $p_{diff}(\frac{1}{2}I\otimes\xi)=p_{same}(\frac{1}{2}I\otimes\frac{1}{2}I)=\varkappa_0$.

It is natural to ask for which pairs ϱ,ξ their difference can be identified by a suitably selected E of the considered diagonal form and whether there are some "nondiagonal" measurements enabling us to compare a pair of states containing the complete mixture.

In order to get an insight into the power of diagonal measurements, let us assume that $\varkappa_m\geqslant 0$, $m=1,2,3$ and fix $\mathbf{k}$. Define a new vector $\mathbf{K}=(\varkappa_1k_1,\varkappa_2k_2,\varkappa_3k_3)$. If $\mathbf{K}\cdot\mathbf{r}>0$, then we can find $\mathbf{h}$ such that $\mathbf{K}\cdot\mathbf{r}=\sum_{m=1}^{3}\varkappa_m h_m^2$; hence $D_{diag}(\varrho\otimes\xi,\mathcal{S}^+)=0$. If $\mathbf{K}\cdot\mathbf{r}<0$, then $D_{diag}(\varrho\otimes\xi,\mathcal{S}^+)=|\mathbf{K}\cdot\mathbf{r}|+\sum_{m=1}^{3}\varkappa_m h_m^2>0$ for any $\mathbf{h}\neq\mathbf{0}$. Therefore, the minimum is achieved for $\eta\otimes\eta=\frac{1}{2}I\otimes\frac{1}{2}I$ and the distance reads

$$D_{diag}(\varrho\otimes\xi,\mathcal{S}^+)=\begin{cases}0 & \text{if}\quad \mathbf{K}\cdot\mathbf{r}\geqslant 0\\ |\mathbf{K}\cdot\mathbf{r}| & \text{otherwise.}\end{cases} \qquad (25)$$

In other words, the considered diagonal measurement E enables us to verify the difference between ϱ and ξ for all ϱ satisfying the inequality $\mathbf{K}\cdot\mathbf{r}<0$. The condition $\mathbf{K}\cdot\mathbf{r}=0$ determines a plane containing the complete mixture (center of the Bloch ball); hence, for any measurement of the considered type and any state ξ the set of successfully comparable states ϱ is exactly a hemisphere of the Bloch ball. Figure 3 illustrates this situation for the following choices of the diagonal measurements (POVMs):

$$\mathsf{E}_{SWAP}=\left\{E_{sym}=\frac{1}{4}\left(3\cdot I\otimes I+\sum_{m=1}^{3}\sigma_m\otimes\sigma_m\right),\right.$$
$$\left.E_{asym}=\frac{1}{4}\left(I\otimes I-\sum_{m=1}^{3}\sigma_m\otimes\sigma_m\right)\right\}; \qquad (26)$$

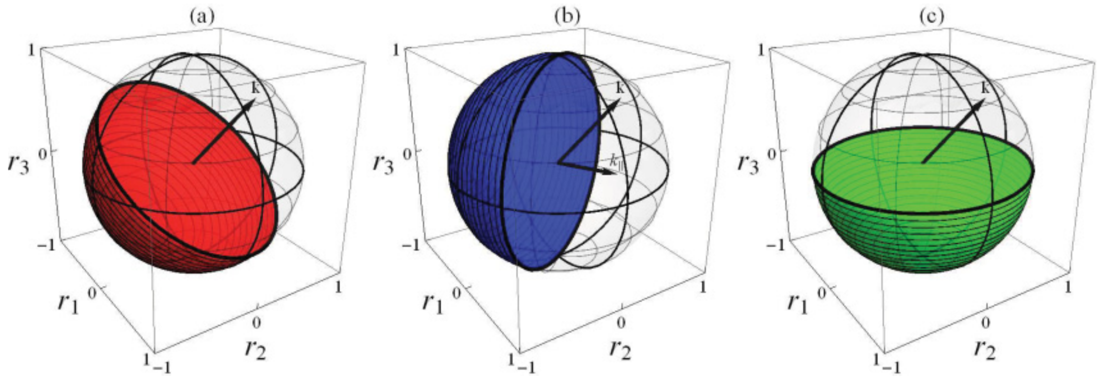

FIG. 3. (Color online) States ϱ in Bloch ball (determined by vectors $\mathbf{r}$) which can be distinguished from a fixed state ξ (given by vector $\mathbf{k}$) by using measurement $\mathsf{E}_{\rm SWAP}$ (a), E_{xy} (b), and E_z (c).

$$\mathsf{E}_{xy} = \Bigg\{ E_{xy+} = \frac{1}{4}\left(3\cdot I\otimes I + \sum_{m=1}^{2}\sigma_m\otimes\sigma_m\right), \quad E_{xy-} = \frac{1}{4}\left(I\otimes I - \sum_{m=1}^{2}\sigma_m\otimes\sigma_m\right)\Bigg\}; \tag{27}$$

$$\mathsf{E}_z = \left\{ E_{z\pm} = \frac{1}{2}(I\otimes I \pm \sigma_3\otimes\sigma_3)\right\}. \tag{28}$$

In particular, for $\mathsf{E}_{\rm SWAP}$ the "comparable hemisphere" is orthogonal to the vector $\mathbf{k}$. For E_{xy} the "comparable" hemisphere is orthogonal to the vector $\mathbf{k}_\parallel = (k_1,k_2,0)$ being a projection of $\mathbf{k}$ onto the xy plane. Finally, for E_z any state from the northern hemisphere is "comparable" with any state from the southern hemisphere.

It is worth noting that we have restricted ourselves to the specific form of POVM effects (15). However, even for such a simplified problem the solution looks rather sophisticated.

2. Average performance

The fact that, for any given diagonal two-valued measurement, the states within the same octant are not comparable means that none of them is universal neither in an approximative way. Nevertheless, it is of interest to understand which of them perform better than the others and which do not perform at all. In particular, we are interested in the answer to the following question: How many qubit states ϱ,ξ can be distinguished? As is briefly outlined in Sec. III, to answer this question it is necessary to introduce some measure μ on the state space. Once it is introduced, we can evaluate the quantities $|S^-_{\rm comp}|_{\mathsf{E}}$ (relative volume of the successfully comparable states; i.e., the comparison universality factor) and $\langle D_{\mathsf{E}}\rangle$ (average distance; i.e., the comparison quality factor).

In contrast with pure states, for density operators there exist many equivalently well-motivated measures (see, e.g., Refs. [12,13] and references therein). We will employ two most commonly used ones; namely, the Hilbert-Schmidt measure $\mu_{\rm HS}$ and the Bures measure $\mu_{\rm B}$:

$$\mu_{\rm HS}(d\varrho) = \frac{3}{4\pi} r^2 \sin\theta dr d\theta d\varphi, \tag{29}$$

$$\mu_{\rm B}(d\varrho) = \frac{r^2\sin\theta}{\pi^2\sqrt{1-r^2}} dr d\theta d\varphi, \tag{30}$$

where we used the following parametrization of Bloch vectors: $\mathbf{r} = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$ with $r\in[0,1]$, $\theta\in[0,\pi]$, and $\varphi\in[0,2\pi]$. Both these measures are spherically symmetric and the former one corresponds to the uniform coverage of the entire Bloch ball [12]; that is, the Hilbert-Schmidt measure $\mu_{\rm HS}(T)$ of any compact set $T\subset\mathcal{S}(\mathcal{H}_2)$ equals the geometrical volume $\int_{\varrho(\mathbf{r})\in T} d^3\mathbf{r}$ of the corresponding body inside the Bloch ball divided by $4\pi/3$. The Bures measure (30) ascribes higher weights to the states with higher purity (that are closer to the surface of the Bloch ball).

For calculation purposes it is convenient to introduce the following (relative) density of states:

$$N^{\rm diff}_{E,\mu}(p) = \lim_{\Delta p\to 0}\frac{1}{\Delta p}\int\int_{{\rm tr}[E\varrho\otimes\xi]\in[p;p+\Delta p]}\mu(d\varrho)\mu(d\xi), \tag{31}$$

whose physical meaning is that $N^{\rm diff}_{E,\mu}(p)\Delta p$ equals the fraction of pairs $\varrho\otimes\xi$ resulting in the measurement outcome probability within the region $[p,p+\Delta p]$ for the effect E. Using the introduced function, we can readily write

$$|S^-_{\rm comp}|_{\mathsf{E}} = \int_{p\in P_E^-\setminus P_E^+} N^{\rm diff}_{E,\mu}(p)dp, \tag{32}$$

$$\langle D_{\mathsf{E}}\rangle = 2\int_{p\in P_E^-\setminus P_E^+} |p-p_0|\, N^{\rm diff}_{E,\mu}(p)dp, \tag{33}$$

where $P_E^\pm$ stands for the image of $\mathcal{S}^\pm$ under the action of POVM effect E, p_0 is simultaneously the frontier point of P_E^+ and the inner point of P_E^- [if there are two such points, then $\langle D_{\mathsf{E}}\rangle$ is a sum of two integrals (33) in the corresponding regions of variable p].

In what follows we will analyze the three examples from the previous section (comparison measurements $\mathsf{E}_{\rm SWAP}$, E_{xy}, and E_z) and compare their performance. The associated densities are depicted in Fig. 4 and explicitly written in the Appendix. We focus on these POVMs because they represent three different types of boundary (extremal in case of $\mathsf{E}_{\rm SWAP}$ and E_z) points of the set of diagonal measurements (see Fig. 2). Other diagonal measurements will exhibit intermediate behavior with respect to these three.

We have numerically tested that for diagonal measurements the relative volume of the set of comparable states $|\mathcal{S}^+_{\rm comp}|_{\mathsf{E}} \leqslant \frac{1}{2}$ irrelevant of the measure used. The considered

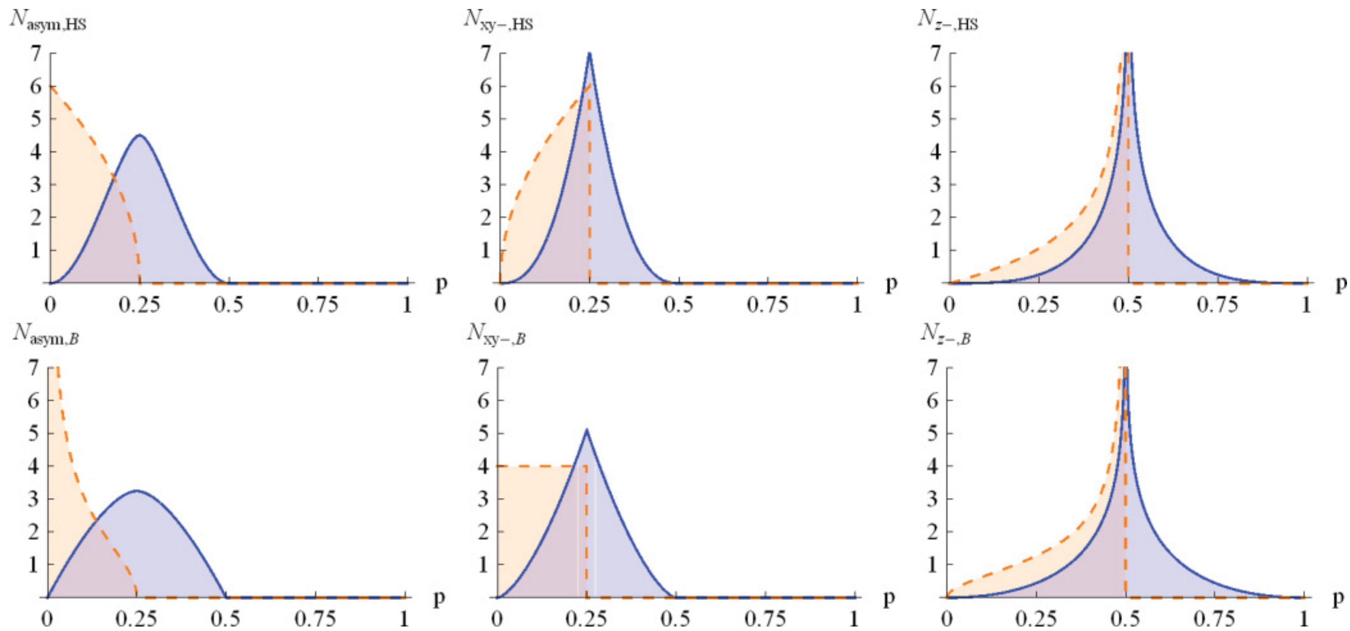

FIG. 4. (Color online) Densities of states $N^{\text{diff}}_{E,\mu}(p)$ (solid) and $N^{\text{same}}_{E,\mu}(p)$ (dashed) for different POVM effects: E_{asym} (left column), E_{xy-} (middle column), and E_{z-} (right column); and for different measures: the Hilbert-Schmidt measure (top row) and the Bures measure (bottom row).

three examples saturate this value, i.e., $|\mathcal{S}^{+}_{\text{comp}}|_{\mathsf{E}_{\text{SWAP},xy,z}} = \frac{1}{2}$. In fact, there exist many measurements (of the considered family) for which the comparable set is of this size and the question is whether there are some interesting differences in their performance. As a figure of merit for this purpose we employ the average distance $\langle D_{\mathsf{E}}\rangle$, which is closely related to the quality of the fixed-pair comparison and partially quantifies also the difference of states.

The performance of three comparison measurements E_{SWAP}, E_{xy}, and E_z is compared in Table I. It is clear that, whichever measure μ is used, the mean value $\langle D_{\mathsf{E}}\rangle_\mu$ is greater for the measurements E_{SWAP} and E_z than that for the measurement E_{xy}. Furthermore, although both POVMs E_{SWAP} and E_z lead to the same expectation values $\langle D_{\mathsf{E}}\rangle_\mu$, the former one gives rise to less dispersion $\mathsf{D}_\mu[D_{\mathsf{E}}]$ and relative standard deviation $\sqrt{\mathsf{D}_\mu[D_{\mathsf{E}}]}/\langle D_{\mathsf{E}}\rangle_\mu$. In addition, from Fig. 4 it follows that the effect E_{asym} results in the smallest density of states in the vicinity of the point p_0. Such a feature is very demanding because the values close to p_0 are the most affected by potential statistical errors, which are unavoidable in practice.

TABLE I. Effectiveness of probability-based comparison based on different POVM effects. Mean values $\langle\cdot\rangle_\mu$ and dispersions $\mathsf{D}_\mu[\cdot]$ of the distance $D_{\mathsf{E}}(\varrho\otimes\xi,\mathcal{S}^{+})$ for different kinds of measure μ.

	E_{SWAP}	E_{xy}	E_z
$\langle D_{\mathsf{E}}\rangle_{\text{HS}}$	0.070 32	0.055 22	0.070 32
$\langle D_{\mathsf{E}}\rangle_{\text{B}}$	0.090 06	0.070 74	0.090 06
$\mathsf{D}_{\text{HS}}[D_{\mathsf{E}}]$	0.010 06	0.006 95	0.015 06
$\mathsf{D}_{\text{B}}[D_{\mathsf{E}}]$	0.015 33	0.010 62	0.023 14
$\sqrt{\mathsf{D}_{\text{HS}}[D_{\mathsf{E}}]}/\langle D_{\mathsf{E}}\rangle_{\text{HS}}$	1.426	1.510	1.745
$\sqrt{\mathsf{D}_{\text{B}}[D_{\mathsf{E}}]}/\langle D_{\mathsf{E}}\rangle_{\text{B}}$	1.375	1.457	1.689

Using the mentioned figures of merit, we can draw a conclusion that the measurement E_{SWAP} performs (on average) the best among the considered measurements. There is yet another fact in favor of this. Figure 4 contains also the densities of the same states $N^{\text{same}}_{E,\mu}(p)$ defined by

$$N^{\text{same}}_{E,\mu}(p) = \lim_{\Delta p\to 0}\frac{1}{\Delta p}\int_{\text{tr}[E\eta\otimes\eta]\in[p;p+\Delta p]}\mu(d\eta). \qquad (34)$$

The value of the quantity $N^{\text{same}}_{E,\mu}(p)\Delta p$ tells us the number of states of the form $\eta\otimes\eta$ for which the probability $\text{tr}[E\eta\otimes\eta]$ belongs to the region $[p;p+\Delta p]$ (explicit formulas for the involved effects E are given in the Appendix). One can clearly see from Fig. 4 that, for the SWAP-based comparison (unlike the other two), the distribution is concentrated far from the border point p_0. It is evident that, if the measured experimentally probability p satisfies $p\in P^{+}_{\text{asym}}$, then one cannot judge whether states are the same or different. However, even in this case it is possible to extract additional information. In fact, the measured probability p sets a limit on the maximum trace distance between the states ϱ and ξ (provided they are different), because $\text{tr}|\varrho-\xi| \leqslant 2\sqrt{2p}$. Consequently, the smaller the measured probability p the closer the states ϱ and ξ are.

C. "Nondiagonal" qubit measurements lacking in almost universality

We have argued that "diagonal" qubit measurements are not able to decide on the difference of states defined by codirectional Bloch vectors; in particular, the states $\frac{1}{2}I\otimes\varrho$ are not in the comparable sets of any measurement from this family. We address the question whether this feature is general. In other words, whether there exists a two-valued

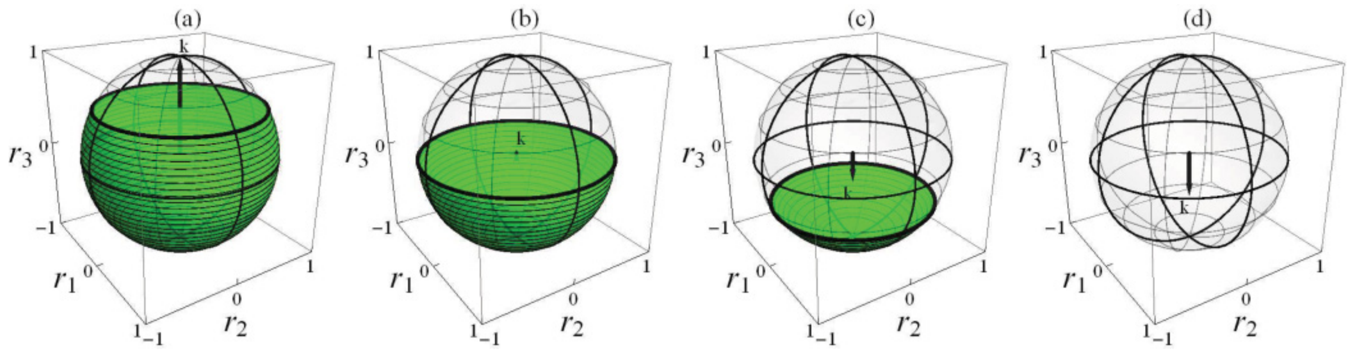

FIG. 5. (Color online) States ϱ inside Bloch ball (determined by vectors $\mathbf{r}$) which can be distinguished from a particular fixed state $\xi = \mathrm{diag}(\Xi_1,\Xi_2)$ [given by vector $\mathbf{k} = (0,0,\Xi_1 - \Xi_2)$] by using the nondiagonal POVM effect $E = |\Xi_2 \otimes \Xi_1\rangle\langle\Xi_2 \otimes \Xi_1| \equiv \mathrm{diag}(0,0,1,0)$ for the following cases: $\xi = \mathrm{diag}(1,0)$ (a), $\xi = \mathrm{diag}(\frac{1}{2},\frac{1}{2})$ (b), $\xi = \mathrm{diag}(\frac{1}{3},\frac{2}{3})$ (c), and $\xi = \mathrm{diag}(\frac{1}{4},\frac{3}{4})$ (d).

qubit measurement allowing us to decide on the difference between the complete mixture and some other state.

Consider a qubit state ξ and its spectral decomposition $\xi = \sum_{i=1,2} \Xi_i |\Xi_i\rangle\langle\Xi_i|$ with eigenvalues $\Xi_1 \geqslant \Xi_2$. Suppose a two-valued measurement E with the effects $E_1 = |\Xi_2 \otimes \Xi_1\rangle\langle\Xi_2 \otimes \Xi_1|$ and $E_2 = I - E_1$. Then we have

$$D_{\mathsf{E}}(\varrho \otimes \xi, \mathcal{S}^+) = 2 \inf_{\eta\otimes\eta\in\mathcal{S}^+} |\varrho_{22}\Xi_1 - \eta_{11}\eta_{22}| = \begin{cases} 2\varrho_{22}\Xi_1 - \frac{1}{2} & \text{if} \quad \varrho_{22} > (4\Xi_1)^{-1} \\ 0 & \text{otherwise,} \end{cases} \quad (35)$$

where $\varrho_{ii} = \langle\Xi_i|\varrho|\Xi_i\rangle$ and $\eta_{ii} = \langle\Xi_i|\eta|\Xi_i\rangle$, $i = 1,2$. If, for instance, $\Xi_1 = 1$ (i.e., ξ is the north pole of the Bloch ball), then states ϱ with $\varrho_{22} > \frac{1}{4}$ form the comparable set for ξ. They lie below latitude 60° north and include also the maximally mixed state [see Fig. 5(a)]. If ξ is the maximally mixed state ($\Xi_1 = \Xi_2 = \frac{1}{2}$), then it is unambiguously distinguished from any state from the southern hemisphere of the Bloch ball [Fig. 5(b)]. Applying unitary transformations of the form $U\varrho U^\dagger$ and $(U \otimes U)E_{1,2}(U^\dagger \otimes U^\dagger)$, we can draw a conclusion that maximally mixed state can be effectively compared with any other qubit state, which answers our question in a positive way.

We find that, on average, the fraction of comparable states is smaller than $\frac{1}{2}$. In particular, $|\mathcal{S}^-_{\mathrm{comp}}|_{\mathrm{HS}} = \frac{3}{8}(6 - 7\ln 2) = 0.43$ and $|\mathcal{S}^-_{\mathrm{comp}}|_{\mathrm{B}} = 0.42$, which means that the measurements $\mathsf{E}_{\mathrm{SWAP}}$, E_{xy}, and E_z outperform the considered nondiagonal measurement in this parameter. The average distance $\langle D_{\mathsf{E}}\rangle$ reads 0.1342 and 0.1524, respectively. We can see that the quality factor $\langle D_{\mathsf{E}}\rangle$ is increased by the expense of the lower universality factor $|\mathcal{S}^-_{\mathrm{comp}}|$.

Surprisingly, for a fixed pure state $\xi = |\Xi\rangle\langle\Xi|$ it is possible to design a two-outcome measurement E_Ξ such that $D_\Xi(\varrho \otimes \xi, \mathcal{S}^+) > 0$ for all states $\varrho \neq \xi$. In fact, suppose a POVM E_Ξ with effects $E_{\Xi 1} = \mathrm{diag}(\frac{1}{4},\frac{3}{8},\frac{1}{8},\frac{5}{8})$ and $E_{\Xi 2} = I - E_{\Xi 1}$ specified in the orthonormal basis $\{|\Xi \otimes \Xi\rangle, |\Xi \otimes \Xi_\perp\rangle, |\Xi_\perp \otimes \Xi\rangle, |\Xi_\perp \otimes \Xi_\perp\rangle\}$. Then, $p_{\Xi 1}(\varrho \otimes \xi) = \frac{1}{8}(2 - \varrho_{22})$ and $p_{\Xi 1}(\eta \otimes \eta) = \frac{1}{8}(2 + 3\eta_{22}^2)$, where $\varrho_{22} = \langle\Xi_\perp|\varrho|\Xi_\perp\rangle$ and $\eta_{22} = \langle\Xi_\perp|\eta|\Xi_\perp\rangle$. Hence $p_{\Xi 1}(\varrho \otimes \xi) \in [\frac{1}{8},\frac{1}{4})$ and $p_{\Xi 1}(\eta \otimes \eta) = [\frac{1}{4},\frac{5}{8}] \equiv P^+_{\Xi 1}$. Therefore, if $\varrho \neq \xi$, then we necessarily observe a probability $p_{\Xi 1}$ outside the interval $P^+_{\Xi 1}$, which unambiguously identifies the difference of ϱ and ξ. Such a two-outcome experiment can be used to check whether a copy ϱ of the etalon pure state ξ was really produced. Nonetheless, in spite of the seeming effectiveness of this measurement, its average performance is quite low. To be precise, the fraction of comparable states $\varrho \otimes \xi$ on average reads $|\mathcal{S}^-_{\mathrm{comp}}|_{\mathrm{HS}} = 0.097$, or $|\mathcal{S}^-_{\mathrm{comp}}|_{\mathrm{B}} = 0.131$, and the quality factor is $\langle D_\Xi\rangle_{\mathrm{HS}} = 0.0049$, or $\langle D_\Xi\rangle_{\mathrm{B}} = 0.0079$.

V. ALMOST-UNIVERSAL COMPARISON MEASUREMENTS

We have shown in Sec. III that any universal comparison measurement is necessarily an LIC measurement. However, the question of an existence of an almost universal comparison measurement (which is not LIC) remains open. In Sec. IV, we have considered such examples of two-valued measurements that the fraction of unambiguously comparable qubit states ϱ, ξ does not exceed $\frac{1}{2}$. In what follows we find a two-valued POVM, for which this fraction equals 1. This makes such a two-outcome measurement almost universal.

Consider a general two-valued measurement $\mathsf{E}_{2\text{-out}}$ composed of the effects $E = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and $I - E$ that are diagonal in some Hilbert space basis $|j \otimes k\rangle$ of two qubits. For $\mathsf{E}_{2\text{-out}}$ to be almost universal, the set of indistinguishable states $\varrho \otimes \xi$ ($\varrho \neq \xi$) must have measure zero. Let us recall the geometrical picture presented in the beginning of Sec. IV: for any fixed qubit state ξ, the comparable states ϱ form a cut or two cuts of the Bloch ball; in latter case the cuts being separated by the distance L [see formula (9)]. The almost universality requires $L = 0$ for all states ξ from a set of measure 1 in $\mathcal{S}(\mathcal{H}_2)$. In other words, the almost universality requires $\max_{\eta\otimes\eta\in\mathcal{S}^+} \mathrm{tr}[E\eta \otimes \eta] = \min_{\eta\otimes\eta\in\mathcal{S}^+} \mathrm{tr}[E\eta \otimes \eta]$; that is, $\mathrm{tr}[E\eta \otimes \eta] = \mathrm{const}$ for all $\eta \in \mathcal{S}(\mathcal{H})$. On the other hand, the only invariant which is quadratic with respect to η is $(\mathrm{tr}[\eta])^2$, which means that $\mathrm{tr}[E\eta \otimes \eta] = \lambda_1\eta_{11}^2 + (\lambda_2 + \lambda_3)\eta_{11}\eta_{22} + \lambda_4\eta_{22}^2 \propto \eta_{11}^2 + 2\eta_{11}\eta_{22} + \eta_{22}^2$; that is, $\lambda_1 = \lambda_4 = \frac{1}{2}(\lambda_2 + \lambda_3) \equiv \lambda \in [0,1]$. Denoting $\frac{1}{2}(\lambda_2 - \lambda_3) \equiv \mu \in [-\min(\lambda,1-\lambda), \min(\lambda,1-\lambda)]$, the operator $E = \mathrm{diag}(\lambda,\lambda+\mu,\lambda-\mu,\lambda)$ is an effect indeed. The constructed effect determines an almost-universal comparison because $P^+_E = \{\lambda\}$ and $P^-_E = [\lambda - |\mu|, \lambda + |\mu|]$. The distance (6) is

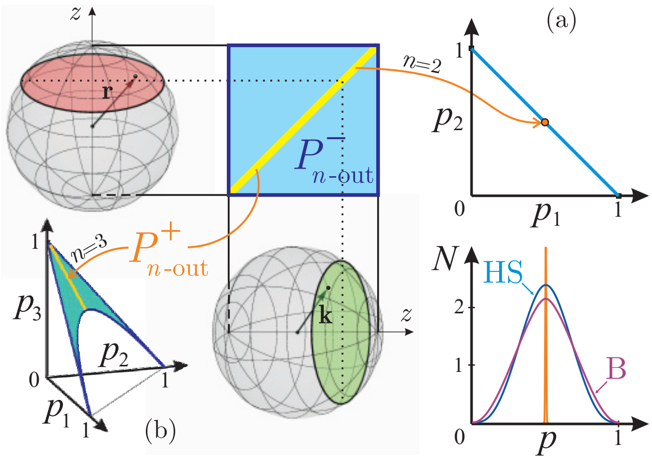

FIG. 6. (Color online) Almost-universal comparison measurement (for qubits) with n outcomes: (a) $n=2$, the images $P^{\pm}_{2\text{-out}}$ on 1 simplex and the densities $N^{\text{diff}}_{E,\mu}(p)$ for Hilbert-Schmidt (HS) and Bures (B) measures, with δ peak being ascribed to $N^{\text{same}}_{E,\mu}(p)$; (b) $n=3$, the images $P^{\pm}_{3\text{-out}}$ on 2 simplex. In both cases the states ϱ and ξ can be distinguished whenever their Bloch vectors satisfy $r_z \neq k_z$.

easily calculated and reads

$$D_{2\text{-out}}(\varrho\otimes\xi,\mathcal{S}^+) = 2|\mu(\varrho_{11}-\xi_{11})|. \tag{36}$$

Let the basis $|j\otimes k\rangle$ be composed of eigenvectors of $\sigma_z\otimes\sigma_z$. For a given state ξ the set of comparable states ϱ equals the whole Bloch ball except for a circle of states satisfying $r_z = k_z$, where r_z and k_z are z components of the corresponding Bloch vectors $\mathbf{r}$ and $\mathbf{k}$, respectively [see Fig. 6(a)].

The distance (36) takes maximal value for maximal μ (i.e., when $\mu=\lambda=\frac{1}{2}$). In this case $E=\text{diag}(\frac{1}{2},1,0,\frac{1}{2})$. The quality factor of such two-valued almost-universal comparison is $\langle D_{2\text{-out}}\rangle_{\text{HS}} = 18/35 \approx 0.51$ or $\langle D_{2\text{-out}}\rangle_{\text{B}} = 256/(45\pi^2)\approx 0.58$; namely, substantially greater than for "diagonal" two-valued measurements (cf. Sec. IV). The densities (31) and (34) for the effect E are given in the Appendix and depicted also in Fig. 6(a).

The qubit example of two-valued almost-universal comparison measurement can be straightforwardly generalized to any dimension d. In fact, the choice $E=\frac{1}{2}[A\otimes I + I\otimes(I-A)]$, where $O\leqslant A\leqslant I$, guarantees that $\text{tr}[E\eta\otimes\eta]=\frac{1}{2}$ for all states $\eta\in\mathcal{S}(\mathcal{H})$. Therefore, the distance (9) vanishes and the comparison is almost universal. The distance (6) equals $D_{2\text{-out}}(\varrho\otimes\xi,\mathcal{S}^+) = |\text{tr}[A(\varrho-\xi)]|$. For conventionally parametrized states $\varrho=\frac{1}{d}(I+\mathbf{r}\cdot\mathbf{\Lambda})$ and $\xi=\frac{1}{d}(I+\mathbf{k}\cdot\mathbf{\Lambda})$, one can choose $A=A_j\equiv\frac{1}{2}(I+\sqrt{2/d}\,\Lambda_j)$ that ensures $O\leqslant A_j\leqslant I$. Then $D_{2\text{-out}_j}(\varrho\otimes\xi,\mathcal{S}^+)=\frac{1}{\sqrt{2d}}|r_j-k_j|>0$ whenever $r_j\neq k_j$.

The considered two-valued almost-universal measurement $\mathsf{E}_{2\text{-out}_j}$ compares j components of (d^2-1)-dimensional Bloch-like vectors $\mathbf{r}$ and $\mathbf{k}$. Combining all these measurements $\mathsf{E}_{2\text{-out}_j}$, $j=1,\dots,d^2-1$, into a single measurement allows us to distinguish vectors $\mathbf{r}$ and; that is, to perform universal comparison measurements with $2(d^2-1)$ effects. Such a measurement will be LIC, in total agreement with the results of Sec. III.

Although a measurement with two outcomes can be almost universal, the practical realization of two-outcome measurements can be rather complicated. In some physical experiments, many-valued measurements are naturally performed instead of two-valued ones. In view of this fact, it is reasonable to consider almost universal many-valued measurements, which are closer to practical realization.

We begin with the qubit case and three-valued measurement $\mathsf{E}_{3\text{-out}}$ composed of effects $E_1=\text{diag}(0,1,0,0)$, $E_2=\text{diag}(0,0,1,0)$, and $E_3=\text{diag}(1,0,0,1)$ defined in the basis $|j\otimes k\rangle$. For twin-identical states $\eta\otimes\eta$, the probabilities of outcomes satisfy $p_1^{\text{same}}=p_2^{\text{same}}\leqslant\frac{1}{4}$, $p_3^{\text{same}}=1-p_1^{\text{same}}-p_2^{\text{same}}\geqslant\frac{1}{2}$. In other words, the set $\mathcal{S}^+$ is mapped onto a line inside the probability simplex [see Fig. 6(b)]. On the other hand, the elements of $\mathcal{S}^-$, $\varrho\otimes\xi$, give rise to probability vectors $\vec{p}^{\,\text{diff}}$ intersecting the line of twin-identical states, $P^+_{3\text{-out}}$, if and only if density matrices ϱ and ξ have the same diagonal elements. However, the subset of the states $\varrho\otimes\xi$ satisfying this peculiar requirement has zero measure in $\mathcal{S}^-$. Hence, almost all pairs of states ϱ and ξ can be compared by the described three-outcome measurement $\mathsf{E}_{3\text{-out}}$ (Fig. 6). A direct calculation of the distance (1) yields

$$D_{3\text{-out}}(\varrho\otimes\xi,\mathcal{S}^+)=\frac{1}{2}\begin{cases}|r_z-k_z| & \text{if}\quad r_zk_z\geqslant 0\\ |r_z-k_z|+|r_zk_z| & \text{otherwise.}\end{cases} \tag{37}$$

It is not hard to see that the calculation of $|\mathcal{S}^-_{\text{comp}}|$ by formula (4) results in 1; that is, the comparison measurement $\mathsf{E}_{3\text{-out}}$ is indeed almost universal. The average distance (5) is $\langle D_{3\text{-out}}\rangle_{\text{HS}}=0.29$, or $\langle D_{3\text{-out}}\rangle_{\text{B}}=0.33$. The quality factor is smaller than that for two-valued almost-universal measurements because the average probabilities of outcomes are smaller ($\sim\frac{1}{3}$ vs $\sim\frac{1}{2}$ for two-valued measurements).

Let us note that the considered example of a 3-outcome POVM is nothing else but a coarse graining of a local projective measurement applied on each of the system independently. In particular, $E_1=E_{01}$, $E_2=E_{10}$, and $E_3=E_{00}+E_{11}$, where $E_{jk}=|j\otimes k\rangle\langle j\otimes k|$ $(j,k=0,1)$ are the effects forming the local (factorized) projective measurement $\mathsf{E}_{4\text{-out}}$. In other words, if one performs the same (along the same direction) Stern-Gerlach experiment on both spin-$\frac{1}{2}$ systems, then the resulting four-outcome POVM $\mathsf{E}_{4\text{-out}}$ performs an almost-universal comparison. So does the POVM $\mathsf{E}_{3\text{-out}}$, where the outcomes "00" and "11" are unified into a single one.

This qubit example of three-outcome measurement can also be generalized for an almost-universal comparison of d-dimensional systems performed by a measurement with $d(d-1)+1$ outcomes. Let Q be a projective measurement associated with effects $Q_j=|\psi_j\rangle\langle\psi_j|$, where $\{|\psi_j\rangle\}_{j=1}^d$ form an orthonormal basis in $\mathcal{H}$. Suppose the same measurement is performed on both d-dimensional systems and define a coarse-grained POVM E_{cg} with effects $E_0=\sum_{j=1}^d Q_j\otimes Q_j$ and $E_{jk}=Q_j\otimes Q_k$ if $j\neq k$. If the states are the same, then $p_{jk}^{\text{same}}=p_{kj}^{\text{same}}$ and $p_0^{\text{same}}=1-2\sum_{j<k}p_{jk}^{\text{same}}$. Clearly, $\vec{p}^{\,\text{diff}}\in P^+_{\text{cg}}$ if $\langle\psi_j|\varrho|\psi_j\rangle=\langle\psi_j|\xi|\psi_j\rangle$ for all $j=1,\dots,d$. Thus, a collection of all nonidentical states $\varrho\otimes\xi\in\mathcal{S}^+$, whose

images $\vec{p}^{\rm diff} \in P_{\rm cg}^{+}$, is a subset of $\mathcal{S}^{-}$ defined by $d-1$ real parameters (in view of normalization) and, hence, this subset is less parametric than the set $\mathcal{S}^{-}$ defined by $2(d^2-1)$ real parameters (i.e., the measure of this subset is zero). In conclusion, the measurement $\mathsf{E}_{\rm cg}$ consisting of $d(d-1)+1$ effects $\{E_0, E_{jk}\}$ is an example of the almost-universal comparison measurement for d-dimensional quantum systems.

VI. SUMMARY

In its essence the comparison is a binary decision problem, which we believe plays a very important role in our everyday lives. In this paper we addressed its quantum version, namely, a comparison of a pair of unknown sources of generally mixed states. We designed a new comparison strategy based on the observed statistics (not individual outcomes) of a particular comparison measurement device. It turns out that, basing upon the observed probabilities of measurement outcomes, one can sometimes draw an unambiguous conclusion on the difference between the states. In fact, it seems that a vast majority of measurements are capable of comparing some pairs of states. However, as we have shown in this paper, the universal comparison of two arbitrary states requires locally informationally complete measurements; hence, the complete tomography of both sources is necessary and sufficient in order to perfectly distinguish between twin-identical states $\eta \otimes \eta$ and nonidentical states $\varrho \otimes \xi$ ($\varrho \neq \xi$).

Furthermore, we analyzed the comparison performance of two-valued qubit measurements. We defined the family of "diagonal" measurements including the so-called SWAP-based comparison measurement, which is known to be useful for the single-shot unambiguous comparison of pure states. We have shown that none of these diagonal measurements is able to decide on the difference of a completely mixed state from any other mixed state. Consequently, the fraction of comparable states, $|\mathcal{S}^{-}_{\rm comp}|_{\mathsf{E}}$, is at most $\frac{1}{2}$ for diagonal measurements. We compared in detail the average performance of three diagonal measurements $\mathsf{E}_{\rm SWAP}, \mathsf{E}_{xy}$, and E_z that are boundary (extremal in case of $\mathsf{E}_{\rm SWAP}$ and E_z) points of diagonal measurements. Although for all of these measurements $|\mathcal{S}^{-}_{\rm comp}|_{\mathsf{E}} = \frac{1}{2}$, we found differences in the distribution of distances $D_{\mathsf{E}}(\varrho \otimes \xi, \mathcal{S}^{+})$. Using these considerations, we concluded that the SWAP-based comparison performs (on average) better than the other two examples.

We also provided nondiagonal comparison measurements enabling us to decide on the difference between an arbitrary state ξ and the complete mixture $\frac{1}{2}I$ or between a pure state ξ and any other state (in both cases the measurement depends on ξ). In this sense, for a given ξ the measurements of this kind overcome the performance of any diagonal measurement. However, their average performance over the set of all states results in $|\mathcal{S}^{-}_{\rm comp}|_{\mathsf{E}} < \frac{1}{2}$. Despite this shortcoming, any pair of qubit states can be compared in a suitable nondiagonal two-valued measurement.

In the remaining part we presented almost-universal comparison measurements in any dimension; that is, such measurements that the size of the comparable set is maximal, $|\mathcal{S}^{-}_{\rm comp}|_{\mathsf{E}} = 1$, but there are still some pairs of states (forming a subset of measure zero) for which their difference cannot be certified. The almost-universal comparison can even be realized by two-outcome measurements. For qubits, the constructed two-valued almost-universal measurement is shown to exhibit the best average performance. We succeeded in finding the explicit form of two-valued almost universal comparators in any dimension. Nonetheless, many-outcome almost universal comparison measurements may turn out to be more feasible for practical implementation than two-valued ones. For d-dimensional systems we theoretically constructed such measurements with $d(d-1)+1$ outcomes (3 in case of qubits). Each measurement is just a coarse-graining of a local measurement, where both systems are measured by the same (d-valued) projective measurement (e.g., the Stern-Gerlach apparatus oriented along z direction in the case of spin particles).

In summary, we have shown that the universal comparison of states (in the considered settings) is not possible, but that there still exist simple almost-universal comparators. In particular, two-outcome measurements are already sufficient (in any dimension) for almost universality. A nice feature of the proposed many-outcome almost-universal comparison measurements is their experimental simplicity. We left many open questions, especially concerning an optimality of the almost-universal comparison measurements. As concerns the universal comparison, an optimality is directly related to the optimal complete tomography.

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APPENDIX: DENSITIES OF STATES

The densities of states $N^{\rm diff}_{E,\mu}(p)$ and $N^{\rm same}_{E,\mu}(p)$ are introduced in Eqs. (31) and (34), respectively. It is worth noting that the domain of functions $N^{\rm diff}_{E,\mu}(p)$ and $N^{\rm same}_{E,\mu}(p)$ is P^{-}_{E} and P^{+}_{E}, respectively. Below we present the explicit formulas of these densities for the effects $E_{\rm asym}$, E_{xy-}, and E_{z-} specified in Eqs. (26)–(28). We calculate the densities by using either the Hilbert-Schmidt measure (29) or the Bures measure (30) and the obtained densities are depicted in Fig. 4.

As far as POVM effect $E_{\rm asym}$ is concerned, $P^{-}_{\rm asym} = (0,\frac{1}{2}]$ and $P^{+}_{\rm asym} = [0,\frac{1}{4}]$, consequently $p_0 = \frac{1}{4}$ and the density of states $N^{\rm diff}_{{\rm asym},\mu}(p)$ is symmetrical with respect to the point p_0. The density of states can be calculated explicitly in the corresponding domains for the Hilbert-Schmidt measure and expressed in quadratures for the Bures measure; namely,

$$N^{\text{diff}}_{\text{asym,HS}}(p) = \frac{9}{2}[1+(4p-1)^2(2\ln|4p-1|-1)],$$

$$N^{\text{diff}}_{\text{asym,B}}(p) = \frac{32}{\pi^2}\int_{|4p-1|}^{1}\sqrt{\frac{r^2-(4p-1)^2}{1-r^2}}dr,$$

$$N^{\text{same}}_{\text{asym,HS}}(p) = 6\sqrt{1-4p},$$

$$N^{\text{same}}_{\text{asym,B}}(p) = 4\sqrt{1-4p}/(\pi\sqrt{p}).$$

Similarly, for the effect E_{xy-} we have $P^-_{xy-} = (0,\frac{1}{2}]$, $P^+_{xy-} = [0,\frac{1}{4}]$, and $p_0 = \frac{1}{2}$, but the densities of states differ from those obtained above, and in the corresponding domains they read

$$N^{\text{diff}}_{xy-,\text{HS}}(p) = \frac{9}{2}[(1+2(4p-1)^2)\arccos|4p-1| \\ -3|4p-1|\sqrt{1-(4p-1)^2}],$$

$$N^{\text{diff}}_{xy-,\text{B}}(p) = \frac{16}{\pi^2}\int_{\arcsin|4p-1|}^{\pi-\arcsin|4p-1|}d\theta \\ \times\int_{\frac{|4p-1|}{\sin\theta}}^{1}\sqrt{\frac{r^2\sin^2\theta-(4p-1)^2}{(1-r^2)\sin^2\theta}}dr,$$

$$N^{\text{same}}_{xy-,\text{HS}}(p) = 12\sqrt{p},\quad N^{\text{same}}_{xy-,\text{B}}(p) = 4.$$

The effect E_{z-} is characterized by regions $P^-_{z-} = (0,1]$, $P^+_{z-} = [0,\frac{1}{2}]$, and $p_0 = \frac{1}{2}$ and gives rise to the following densities of states:

$$N^{\text{diff}}_{z-,\text{HS}}(p) = \frac{9}{4}\{[1+(2p-1)^2](1-\ln|2p-1|)-2\},$$

$$N^{\text{diff}}_{z-,\text{B}}(p) = \frac{16}{\pi^2}\int_{|2p-1|}^{1}\frac{\sqrt{[r_z^2-(2p-1)^2](1-r_z^2)}}{r_z^2}dr_z,$$

$$N^{\text{same}}_{z-,\text{HS}}(p) = 3p/\sqrt{1-2p},$$

$$N^{\text{same}}_{z-,\text{B}}(p) = 4\sqrt{2p}/(\pi\sqrt{1-2p}).$$

Also, we present the calculated densities (31) and (34) for the effect $E = \text{diag}(\frac{1}{2},1,0,\frac{1}{2})$ of the almost universal two-valued comparison measurement from Sec. V. The result is

$$N^{\text{diff}}_{E,\text{HS}}(p) = \frac{12}{5}(1-|2p-1|)^3(|2p-1|^2+3|2p-1|+1)$$

$$N^{\text{diff}}_{E,\text{B}}(p) = \frac{16}{\pi^2} \\ \times\int_{-1}^{1-2|2p-1|}\sqrt{(1-r_z^2)[1-(r_z+2|2p-1|)^2]}dr_z,$$

$$N^{\text{same}}_{E,\text{HS}}(p) = N^{\text{same}}_{E,\text{B}}(p) = \delta\left(p-\frac{1}{2}\right).$$

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