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Simulation of indivisible qubit channels in collision models

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Abstract
A sequence of controlled collisions between a quantum system and its environment (composed of a set of quantum objects) naturally simulates (with arbitrary precision) any Markovian quantum dynamics of the system under consideration. In this paper we propose and study the problem of simulation of an *arbitrary* quantum channel via collision models. We show that a correlated environment is capable to simulate *non-Markovian* evolutions leading to any indivisible qubit channel. In particular, we derive the corresponding master equation generating a continuous time non-Markovian dynamics implementing the universal NOT gate being an example of the most non-Markovian quantum channels.

(Some figures may appear in colour only in the online journal)

1. Introduction

It is one of the basic postulates of quantum physics that the time evolution of *closed* quantum systems is governed by the Schrödinger equation [1]. As a result for any time interval of length $\tau > 0$, a state of a quantum system described by a density operator ϱ is transformed according to a unitary conjugation

$$\varrho \mapsto \varrho'_\tau = U_\tau \varrho U_\tau^\dagger, \tag{1}$$

where $U_\tau = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\tau H}$, H is the (for simplicity, time-independent) Hamiltonian of the system and $\hbar$ is the Planck constant. Moreover, it is natural to assume that $U_\tau = U_t U_s$, where $\tau = t + s$ and $t, s > 0$ are arbitrary durations of short time intervals that sum up to the whole time interval τ. However, if the system is open (interacting with its environment), such simple division of time evolution into incremental time steps does not necessarily possess a clear meaning. It is a relatively recent discovery [2] that certain non-unitary quantum evolutions are indivisible into shorter ones. In order to make this statement precise, let us introduce a mathematical framework modelling open system dynamics over finite time intervals.

Let us assume that initially the system and its environment are not correlated, so that the combined system–environment initial state can be described as $\varrho \otimes \xi$ (such an assumption is quintessential in order to preserve linearity of the dynamics—see e.g. [3] and references therein). Then the evolution of the joint system–environment state is governed by the Schrödinger equation and the state transformation over the time interval $[0, \tau]$ is described by a map

$$\varrho \mapsto \mathcal{E}_\tau[\varrho] = \mathrm{tr}_{\mathrm{env}}[\tilde{U}_\tau \varrho \otimes \xi \tilde{U}_\tau^\dagger], \tag{2}$$

where ξ is the initial state of the environment and $\mathrm{tr}_{\mathrm{env}}$ denotes the partial trace over the environmental degrees of freedom and $\tilde{U}_\tau$ is a unitary transformation describing the time evolution of both the system and the environment. Abstracted mathematical properties of $\mathcal{E}_\tau$ guaranteeing the existence of such a model for $\mathcal{E}_\tau$ are linearity ($\mathcal{E}(X+\lambda Y) = \mathcal{E}(X)+\lambda\mathcal{E}(Y)$), complete positivity (positivity of $\mathcal{E}_\tau \otimes \mathcal{I}_d$ for all d) and trace preservation ($\mathrm{tr}[\mathcal{E}_\tau[X]] = \mathrm{tr}[X]$). This result is known as the *Stinespring representation theorem* [4] and the maps $\mathcal{E}_\tau$ with such properties we call *quantum channels* (see for example [5]).

We say that a quantum channel $\mathcal{E}$ is divisible if $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are both non-unitary quantum channels. If this is not the case, then we say that the channel is indivisible. According to this definition, unitary channels are indivisible. This may sound quite counterintuitive, especially, if we recall how unitary operations are decomposed into gates in quantum computation [6]. Indeed, the gate decomposition is the crucial

tool in the analysis of quantum complexity. But let us stress a difference between the decomposition and the divisibility of channels. The primary task of the decomposition is to simplify the implementation of a multipartite quantum channel by means of elementary gates; thus, the decomposability captures the complexity of the process expressed in terms of the number of uses of the elementary gates. On the other hand, the divisibility addresses a more fundamental question of whether a given process can be understood as a sequential concatenation of processes. If not, then it is natural to ask how to simulate such quantum channels, in particular by means of some 'continuous' time evolution.

Dynamics of open quantum systems is modelled by the so-called master equations that are usually derived under the assumption of Markovianity [7, 8]. Mathematically, this means that a one-parametric set of channels $\mathcal{E}_t$, being the solution of the Markovian master equation (equivalently the Lindblad master equation), satisfies the semigroup property $\mathcal{E}_s \circ \mathcal{E}_t = \mathcal{E}_{s+t}$ for all $s, t \geqslant 0$ and the initialization condition $\lim_{t\to 0} \mathcal{E}_t = \mathcal{I}$. Formally, we can write $\mathcal{E}_t = \mathrm{e}^{\mathcal{L}t}$, where $\mathcal{L}$ is known as the Lindbladian. We say a channel $\mathcal{E}$ is Markovian if there exist the Lindbladian $\mathcal{L}$ and time τ such that $\mathcal{E} = \mathrm{e}^{\mathcal{L}\tau}$. Otherwise, the channel is called non-Markovian. Let us note that all indivisible channels are non-Markovian, but there are divisible non-Markovian channels [2, 9].

Simple collision models, in which the system's evolution is modelled via a sequence of (weak) interactions of the system with sequentially selected particles from the environment, provide a natural playground for simulations of quantum dynamics of open quantum systems. It is known (see, for instance, [10, 11]) that these models can approximate (with arbitrary precision) any evolution governed by a Lindblad master equation [12, 13], i.e. implement any Markovian quantum channel. Our aim in this paper is to investigate whether a general quantum channel (in particular an indivisible one) can be simulated in the framework of simple collision models. Let us note that a related collision model was studied in [14], where the authors derived master equations for a family of correlated memory evolutions with Markovian dependences between subsequent (memory) channel inputs. In this paper, we concentrate on the collision model (simulating non-divisible evolutions) generating the memory effects between the system (individual channel input) and its environment.

The rest of this paper is structured as follows. In section 2, we define the framework of collision models and introduce the concept of stroboscopic simulations. In section 3, we design a collision model for implementation of *all* indivisible qubit channels. Finally, we discuss our results and present conclusions of our investigation in section 4.

2. Simple collision model

By a collision model of a specific system dynamics, we understand a sequence of interactions (collisions) between the system and particles from the environment. Essential property of the model considered in this paper is that each particle of the environment interacts with the system at most once, while the environment particles do not interact between themselves.

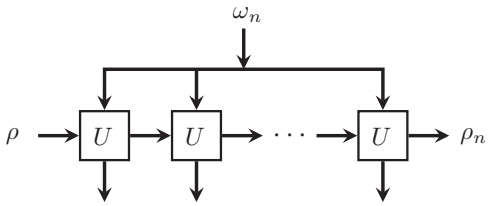

Figure 1. Schematic illustration of a simple collision model.

More formally, let U_j be a bipartite unitary operator describing an individual collision between the system particle and the jth particle of the environment. The collision model is a concatenation of unitary channels $U_1 \ldots U_n$ (see figure 1). Let us denote by ω_n the initial state of the environment composed of n particles and by ϱ the initial state of the system.

Suppose $\varrho \to \varrho_t = \mathcal{E}_t(\varrho)$ is a solution of some Lindblad master equation. The question we would like to answer is: 'How to simulate the process $\mathcal{E}_t$ in the framework of a collision model?' Let us assume that U_δ and ξ determine the Stinespring dilation of $\mathcal{E}_\delta$. Then the sequence of concatenations of U_δ applied in each step jointly on the system and a 'fresh' particle from the environment in the state ξ results in a discrete evolution

$$\varrho \mapsto \varrho_n = \mathcal{E}_\delta^n(\varrho) = \mathrm{tr}_{\mathrm{env}}[(U_1 \ldots U_n)(\varrho \otimes \xi^{\otimes n})(U_1 \ldots U_n)^\dagger], \tag{3}$$

where $U_j = U_\delta \otimes I_{\bar{j}}$ and $I_{\bar{j}}$ denote the identity operator on all particles of the environment but the jth one. As a result the described collision model *stroboscopically simulates* the continuous Markovian evolution, i.e. for all $\delta > 0$ we have $\mathcal{E}_t \approx \mathcal{E}_\delta^n$ with $n = [t/\delta]$ ($[x]$ denotes the nearest integer to x). The parameter δ determines the quality of the (stroboscopic) approximation of the time-continuous (Markovian) dynamics $\mathcal{E}_t$ for $t \geqslant 0$.

It was shown in [15] that if U is a controlled unitary operator, then the sequence of collisions simulates the pure decoherence process, in which the diagonal elements of density operators (with respect to the so-called decoherence basis) are preserved, while the off-diagonal ones vanish exponentially. Setting U to be the partial SWAP interaction [16, 17], the system exponentially converges to the original state of the particles in the reservoir. The information about the original state of the system is diluted into the correlations among all particles. Because of these features the process is called quantum homogenization and represents a quantum information analogue of the thermalization process, in which the temperature is replaced by the concept of quantum state.

In both examples described above, it is assumed that the initial state of the reservoir is factorized, i.e. $\omega_n = \xi^{\otimes n}$. In this factorized setting, the channel $\mathcal{E}_1$ is called a *generating channel* of the collision process. Indeed, if this is the case, then the induced discrete dynamics $\mathcal{E}_n \equiv \mathcal{E}_1^n$ fulfils the conditions of a discrete semigroup, i.e. $\mathcal{E}_n \circ \mathcal{E}_m = \mathcal{E}_{n+m}$ for all positive integers n, m. Without loss of generality, we may set $\mathcal{E}_0 = \mathcal{I}$. It is important to stress that such a semigroup feature does not guarantee that the discrete dynamics stroboscopically approximates some Markovian continuous-time evolution. In

other words, the generated channels $\mathcal{E}_n$ are not necessarily Markovian. For instance, by definition indivisible channels $\mathcal{E}_{\text{indivisible}}$ can be elements of a discrete semigroup only as the generating channels, i.e. $\mathcal{E}_{\text{indivisible}} = \mathcal{E}_1$.

But what if a general initial state of the environment is allowed? Could we then find a way to stroboscopically simulate (using the collision model) a continuous-time evolution leading to an arbitrary quantum channel?

Let us formulate the problem: we say that $\mathcal{E}$ is a *stroboscopically simulated channel* if for every n there exists a bipartite interaction U (between the system and a specific particle from the environment) and an initial state ω_n of the environment is such that $\mathcal{E}_n \equiv \mathcal{E}$, where

$$\mathcal{E}_n(\varrho) = \text{tr}_{\text{env}}[(U_1 \dots U_n)(\varrho \otimes \omega_n)(U_1 \dots U_n)^\dagger],$$

and $U_j = U \otimes I_{\bar{j}}$. If the channel can be approximated in the introduced sense, then the collision models reveal the characteristic features of Markovian dynamics even for non-Markovian channels. In particular, the process can be realized in arbitrarily small (non-unitary) steps although the steps themselves are not described by valid channels. Our aim is to address the question of which channels can be stroboscopically simulated. In what follows we will focus on implementation of indivisible quantum channels, which are, intuitively, the 'most non-Markovian' examples of quantum evolutions.

3. Stroboscopic simulation of indivisible qubit channels

The existence of indivisible channels was observed for the first time in [2], where the authors analysed qubit channels in detail. In particular, they found that indivisible qubit channels are unitarily equivalent to the subclass of Pauli channels

$$\mathcal{E}(\varrho) = q_x\sigma_x\varrho\sigma_x + q_y\sigma_y\varrho\sigma_y + q_z\sigma_z\varrho\sigma_z, \tag{4}$$

where q_x, q_y, q_z are positive ($q_xq_yq_z \neq 0$) and $q_x+q_y+q_z = 1$. For instance, the choice $q_x = q_y = q_z = 1/3$ defines the best quantum approximation $\mathcal{E}_{\text{NOT}}$ of the universal quantum NOT gate for a qubit $\mathcal{F}_{\text{NOT}} : \varrho \mapsto (I - \varrho)$ [18, 19]. The question is: How to stroboscopically simulate some continuous time dynamics leading to these channels?

Suppose that the environment is composed of three-dimensional quantum systems (qutrits). Then one of the possible dilations for indivisible qubit channels is the following: $\mathcal{E}(\varrho) = \text{tr}_{\text{env}}[C\varrho \otimes \omega C^\dagger]$, where $C = \sigma_x \otimes |x\rangle\langle x| + \sigma_y \otimes |y\rangle\langle y| + \sigma_z \otimes |z\rangle\langle z|$ is a controlled unitary interaction and ω satisfies the conditions $\langle k|\omega|k\rangle = q_k$ for $k = x, y, z$. We assume that $|x\rangle, |y\rangle, |z\rangle$ form an orthonormal basis of the qutrit's Hilbert space. Let us stress that C is not only unitary, but also Hermitian. Therefore $U_\eta = \text{e}^{\text{i}\eta C} = \cos\eta I + \text{i}\sin\eta C$ defines a family of interactions. Moreover, each member of this family is a controlled-U operator, $U_\eta = \sum_k \text{e}^{\text{i}\eta\sigma_k} \otimes |k\rangle\langle k|$.

Consider a model generated by collisions U_η. The concatenation of n such collisions implements a global (system plus environment) unitary transformation

$$U_{\eta,1} \dots U_{\eta,n} = \sum_k \exp[\text{i}n\eta\sigma_k] \otimes |k^{\otimes n}\rangle\langle k^{\otimes n}|. \tag{5}$$

Suppose ω_n is the initial state of the environment composed of n qutrits such that $\langle k^{\otimes n}|\omega_n|k^{\otimes n}\rangle = q_k$. After the nth collision, the system evolution reads

$$\begin{aligned}\mathcal{E}_n(\varrho) &= \text{tr}_{\text{env}}[(U_{\eta,1} \dots U_{\eta,n})(\varrho \otimes \omega_n)(U_{\eta,1} \dots U_{\eta,n})^\dagger] \\ &= \sum_k q_k \exp[\text{i}n\eta\sigma_k]\varrho \exp[-\text{i}n\eta\sigma_k],\end{aligned}$$

hence setting the strength of the interaction $\eta = \pi/(2n)$; the resulting collision model simulates the implementation of any indivisible qubit channel in n steps.

3.1. Evaluation of $||\mathcal{E}_{j+1} - \mathcal{E}_j||$

In this section we will show that for the considered collision model, the individual collisions induce arbitrarily small disturbances of the system. Assume n is fixed, i.e. $\eta = \pi/(2n)$. After the jth interaction, the state of the system undergoes the transformation

$$\begin{aligned}\mathcal{E}_j(\varrho) &= \sum_k q_k \exp\left[\text{i}\pi\frac{j}{2n}\sigma_k\right]\varrho\exp\left[-\text{i}\pi\frac{j}{2n}\sigma_k\right] \\ &= \cos^2\left(\frac{j\pi}{2n}\right)\varrho + \sin^2\left(\frac{j\pi}{2n}\right)\sum_k q_k\sigma_k\varrho\sigma_k \\ &\quad + \frac{\text{i}}{2}\sin\left(\frac{j\pi}{n}\right)\sum_k q_k[\sigma_k, \varrho],\end{aligned}$$

where $[A, B] = AB - BA$ is the commutator of operators A, B. Let us denote by $\mathcal{E}$ the target quantum channel, i.e. $\mathcal{E} = \mathcal{E}_n$ (we fix $\eta = \pi/(2n)$) and define $\mathcal{F}(\cdot) = [F, \cdot]$ with $F = \text{i}\sum_k q_k\sigma_k$. Then the distance between two subsequent steps equals

$$\Delta = ||\mathcal{E}_{j+1} - \mathcal{E}_j|| = ||C_{j+1}(\mathcal{I} - \mathcal{E}) + D_{j+1}\mathcal{F}||, \tag{6}$$

where

$$\begin{aligned}C_{j+1} &= \cos^2\frac{(j+1)\pi}{2n} - \cos^2\frac{j\pi}{2n} \\ &= \frac{1}{2}\left(\cos\frac{(j+1)\pi}{n} - \cos\frac{j\pi}{n}\right),\end{aligned} \tag{7}$$

$$D_{j+1} = \frac{1}{2}\left(\sin\frac{(j+1)\pi}{n} - \sin\frac{j\pi}{n}\right). \tag{8}$$

Let us note that the norm we have in mind here is the completely bounded norm (see for instance [20]), namely

$$||\mathcal{A}|| = \sup_{n, X:||X||_1=1} ||(\mathcal{A} \otimes \mathcal{I}_n)(X)||_1,$$

where $||\cdot||_1 = \text{tr}|\cdot|$.

Using the identities $\cos^2\alpha = (1 + \cos 2\alpha)/2$, $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$, and $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$, we obtain

$$C_{j+1} = -\sin\frac{(2j+1)\pi}{n}\sin\frac{\pi}{n}, \tag{9}$$

$$D_{j+1} = \cos\frac{(2j+1)\pi}{n}\sin\frac{\pi}{n}. \tag{10}$$

This allows us to conclude that the distance is bounded as follows:

$$\begin{aligned}\Delta &\leqslant |C_{j+1}| \cdot ||\mathcal{I} - \mathcal{E}|| + |D_{j+1}| \cdot ||\mathcal{F}|| \\ &\leqslant \sin\frac{\pi}{n}\left(\left|\sin\frac{(2j+1)\pi}{n}\right| \cdot ||\mathcal{I} - \mathcal{E}|| \right. \\ &\quad \left. + \left|\cos\frac{(2j+1)\pi}{n}\right| \cdot ||\mathcal{F}||\right) \\ &\leqslant K \sin\frac{\pi}{n} \quad (\longrightarrow 0 \quad \text{for } n \to \infty), \end{aligned} \tag{11}$$

where $K \leqslant ||\mathcal{I} - \mathcal{E}|| + ||\mathcal{F}|| \leqslant 2 + ||\mathcal{F}|| < \infty$, because for channels $||\mathcal{E}|| = 1$ and $||\mathcal{F}|| < \infty$. In fact, as shown in [20] for any linear qubit map $||\mathcal{A}|| \leqslant 2\sqrt{2}\sup_{||X||_1=1} ||\mathcal{A}(X)||_1$; hence, $||\mathcal{F}|| \leqslant 2\sqrt{2}\sup_{||X||_1=1} ||FX - XF||_1 \leqslant 4\sqrt{2}\frac{||FX||_1}{||X||_1} \leqslant 4\sqrt{2}||F||$, where the relations $||FX||_1 \leqslant ||F|| \cdot ||X||_1$ and $||F|| = \sup_\psi ||F\psi||/||\psi|| \leqslant 2$ were used. That is, $K \leqslant 2+8\sqrt{2}$, which allows us to conclude that the system's changes in individual steps can be made arbitrarily small as n goes to infinity. Therefore, in this limit the evolution is continuous.

3.2. Master equation

Replacing the integer parameter j in the expression for $\mathcal{E}_j$ by a continuous parameter t, we formally define a one-parametric set of channels

$$\mathcal{E}_t = \mathcal{E} + \cos^2(\alpha t)(\mathcal{I} - \mathcal{E}) + \sin(\alpha t)\cos(\alpha t)\mathcal{F},$$

where $\alpha = \pi/(2n)$ and linear maps $\mathcal{E}, \mathcal{F}$ are defined in the previous paragraph. Let us note that $\mathcal{E}_t$ form the same set of channels irrelevant of the value of α. Moreover, this one-parametric set of channels is continuous in t and $\mathcal{E}_0 = \mathcal{I}$. A collision model (determined by the value of α) stroboscopically simulates continuous-time quantum evolution given by $\mathcal{E}_t$, where different values of α define the quality of the simulation. In the reminder of this section, we will derive the master equation generating $\mathcal{E}_t$.

Formally, the evolution of density operators is generated by the first-order differential equation

$$\frac{\mathrm{d}\varrho_t}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{E}_t}{\mathrm{d}t}\mathcal{E}_t^{-1}(\varrho_t) \equiv \mathcal{L}_t(\varrho_t), \tag{12}$$

where

$$\mathcal{L}_t(X) = \frac{\mathrm{i}}{\hbar}\sum_j h_j[X, \sigma_j] + \frac{1}{2}\sum_{j,k} c_{jk}([\sigma_j, X\sigma_k] + [\sigma_j X, \sigma_k]) \tag{13}$$

is the generator of the dynamics [12, 13]. Any such generator defines dynamics which is trace preserving and if (time-dependent coefficients) h_j, c_{jk} are real, then also the hermiticity of operators is preserved. If these parameters are time independent ($\mathcal{L}_t = \mathcal{L}$) and the matrix composed of entries c_{jk} is positive, then the generated dynamics is also completely positive and Markovian. In such a case we can write $\mathcal{E}_t = \mathrm{e}^{\mathcal{L}t}$ and $\mathcal{L}$ is the Lindbladian.

For the sake of simplicity, let us illustrate the derivation of the driving master equation ($\mathcal{L}_t$) for the case of the target channel $\mathcal{E} = \mathcal{E}_{\mathrm{NOT}}$. This channel transforms the Bloch vector $\vec{r}$ into $\vec{r}' = -\vec{r}/3$, hence, implements a Bloch ball (shrinking) inversion. In this case the map $\mathcal{F}$ induces the Bloch vector transformation $\vec{r} \to \vec{u} \times \vec{r}$ with $\vec{u} = (1/3, 1/3, 1/3)$. Let us stress that $\mathcal{F}(I) = O$; hence, the map is not trace preserving and maps any operator into a traceless one.

In the Bloch sphere parametrization, the channels take the form of an affine 4×4 matrix. Define 3×3 matrices

$$I = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad A = \begin{pmatrix}0&-1&1\\1&0&-1\\-1&1&0\end{pmatrix},$$

$$S = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}. \tag{14}$$

Then the one-parametric dynamics is given by the following 4×4 matrix:

$$\mathcal{E}_t = \begin{pmatrix}1 & 0 & 0 & 0\\ \vec{0} & x(t)I + a(t)A\end{pmatrix},$$

where $x(t) = \frac{1}{3}(4\cos^2\alpha t - 1)$ and $a(t) = \frac{1}{6}\sin(2\alpha t)$. A direct calculation gives

$$\frac{\mathrm{d}\mathcal{E}_t}{\mathrm{d}t} = \frac{1}{3}\alpha\begin{pmatrix}0 & 0 & 0 & 0\\ \vec{0} & 4\sin(2\alpha t)I + \cos(2\alpha t)A\end{pmatrix}, \tag{15}$$

and

$$\mathcal{E}_t^{-1} = \begin{pmatrix}1 & 0 & 0 & 0\\ \vec{0} & \frac{1}{3x(t)}I - \frac{a(t)}{\det\mathcal{E}_t}[x(t)A - a(t)S]\end{pmatrix}, \tag{16}$$

where $\det\mathcal{E}_t = 3x(t)[a(t)^2 + x(t)^2]$. Thus, for the generator we obtain

$$\mathcal{L}_t = \frac{\mathrm{d}\mathcal{E}_t}{\mathrm{d}t}\mathcal{E}_t^{-1} = \begin{pmatrix}0 & 0 & 0 & 0\\ \vec{0} & b(t)I + c(t)A + d(t)S\end{pmatrix}, \tag{17}$$

with

$$b(t) = \frac{2}{9}a(t)\left[12\alpha + \frac{1}{a(t)^2 + x(t)^2}\right], \tag{18}$$

$$c(t) = \frac{1}{9x(t)}\left[\frac{\alpha(3x(t) - 1)}{2} - \frac{a(t)[3x(t) + a(t)]}{x(t)[a(t)^2 + x(t)^2]}\right], \tag{19}$$

$$d(t) = \frac{a(t)[3a(t) - x(t)]}{9x(t)[a(t)^2 + x(t)^2]}. \tag{20}$$

Using the methods and formulae described in [11], we obtain the non-Markovian master equation in the operator form

$$\begin{aligned}\frac{\mathrm{d}\varrho_t}{\mathrm{d}t} &= -\frac{\mathrm{i}c(t)}{2\hbar}[\varrho_t, H] - \frac{b(t)}{2}\left(\sum_j \sigma_j\varrho_t\sigma_j - 3\varrho_t\right) \\ &\quad + d(t)\sum_{j\neq k}\sigma_j\varrho_t\sigma_k, \end{aligned} \tag{21}$$

where $H = \sigma_x + \sigma_y + \sigma_z$. The time dynamics induced by the collision model is illustrated in figure 2.

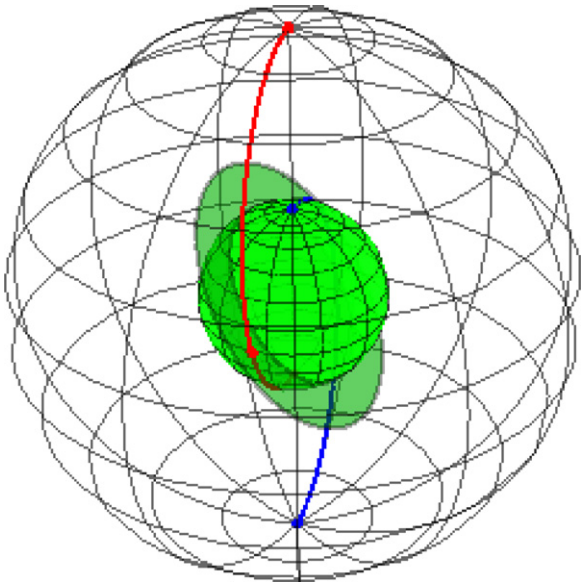

Figure 2. The collision model simulating the continuous time evolution towards the universal NOT gate (shrinked Bloch sphere inversion). In particular, the transformation of the Bloch sphere (red and blue lines capture the time evolution of the eigenstates of σ_z operator, $|0\rangle$ and $|1\rangle$, respectively) is depicted for the time interval $t \in [0, n]$. For $t = \frac{2}{3}n$ the channel $\mathcal{E}_t$ is not invertible (det $\mathcal{E}_{t=2n/3} = 0$) and at this time the Bloch sphere is mapped onto a two-dimensional disk. Let us note that images of eigenstates of σ_z operator are internal points of the disk. In fact, the whole disk is the image of pure states only[4].

4. Conclusions

In this paper we opened the question of whether a continuous time (stroboscopic) simulation of quantum channels within simple collision models, i.e. via a sequence of interactions of the system with particles forming the environment, is possible. Using the environment composed of three-dimensional quantum particles, we design a collision model simulating arbitrary indivisible qubit channel. Indivisible channels could be coined as the most non-Markovian ones if one quantifies the *Markovianity of a channel* $\mathcal{E}$ as the maximal number (n) of non-unitary channels $\mathcal{E}_1, \ldots, \mathcal{E}_n$ such that $\mathcal{E} = \mathcal{E}_1 \ldots \mathcal{E}_n$. The smaller the number, the more non-Markovian the channel. Let us stress that there are several recent proposals for measures of Markovianity of continuous time evolutions [21–23]. These measures can be applied not only to the derived master equation for implementation of the $\mathcal{E}_{\mathrm{NOT}}$, but can also be modified to the settings of discrete time evolutions, which are naturally generated by simple collision models. However, a more detailed analysis along these lines goes beyond the scope of this paper.

Another important point we want to stress is the necessity of initial correlations between the particles of the reservoir, which are introducing the memory mechanism present in any non-Markovian evolution. It is of interest to understand whether there is some deeper relation between the correlation structure of ω_n and memory features of the system's dynamics (Markovianity). In order to answer this question, one needs to understand the ambiguity of the stroboscopic simulation. For example, each convex decomposition into unitary channels induces a different collision model (as described in the previous section) for the same channel $\mathcal{E}$. However, for this class of collision models, there is no qualitative difference neither in the initial states nor in the derived evolutions. In the considered collision model, the correlations are relatively strong and all particles involved in the dynamics are mutually pairwise correlated, but it could happen that there are qualitatively different collision models for which the structure of correlations is completely different, especially, it could be that particles entering the jth and the kth collisions are initially uncorrelated if $|j - k|$ is sufficiently large.

In summary, we introduced the problem of stroboscopic simulations of general quantum channels. In this paper we reported the case study of qubit indivisible channels. In fact, we designed a collision model for stroboscopic implementation of any channel (of d-dimensional quantum system) from the family of random unitary channels (see the appendix). Since indivisible channels are the most non-Markovian, it is natural to conjecture that any (qubit) channel can be simulated in the stroboscopic manner. In fact, it was shown in [2] that non-unital qubit channels are infinitesimally divisible and can be approximated by a concatenation of Markovian channels that can be stroboscopically simulated by factorized initial states of the environment. Moreover, since unital qubit channels are necessarily random unitary and for them we have an explicit collision model (irrelevant of their Markovianity), we may conclude that collision models can stroboscopically approximate any qubit channel. We believe that the considered collision model deserves further investigation that would finally result in a better understanding of non-Markovian features of general continuous time quantum evolutions and implementations of non-Markovian quantum channels.

[4] An animation of this evolution can be found at http://youtu.be/C55F4FKysh0.

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Appendix. Simulation of all random unitary channels

In this appendix we will extend the presented stroboscopic simulation of qubit indivisible channels to any random unitary channel acting on the system of arbitrary dimension. A channel is called random unitary if $\mathcal{E} = \sum_j q_j V_j \varrho V_j^\dagger$, $q_j \geqslant 0$,

$\sum_j q_j = 1$ and V_j are unitary operators. Suppose $1 \leqslant j \leqslant d$ and define a collision model generated by the interaction

$$U = \sum_{j=1}^{d} |j\rangle\langle j| \otimes V_j^{1/n},$$

where $V_j^{1/n}$ are unitary operators such that $\left(V_j^{1/n}\right)^n = V_j$. Assuming that initially $\omega_j = \sum_j q_j |j^{\otimes n}\rangle\langle j^{\otimes n}|$, we find (analogously as for the qubit case) that after the kth collision

$$\mathcal{E}_k(\varrho) = \sum_{j=1}^{d} q_j \left(V_j^{k/n}\right) \varrho \left(V_j^{k/n}\right)^\dagger; \tag{A.1}$$

thus, $\mathcal{E}_n = \mathcal{E}$. Such construction works for any value of n and therefore we can conclude that an arbitrary random unitary channel can be stroboscopically simulated. Let us note that this collision model defines a non-Markovian evolution also for Markovian channels. The results of section 3 can be easily generalized for any random unitary channel in finite-dimensional Hilbert space. It is sufficient to change $\sigma_k \to H_k$, where H_k is a Hamiltonian generating the unitary V_k such that $\exp\left[\mathrm{i}\frac{\pi}{2} H_k\right] = V_k$.

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