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Quantum finite-depth memory channels: Case study

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We analyze the depth of the memory of quantum memory channels generated by a fixed unitary transformation describing the interaction between the principal system and internal degrees of freedom of the process device. We investigate the simplest case of a qubit memory channel with a two-level memory system. In particular, we explicitly characterize all interactions for which the memory depth is finite. We show that the memory effects are either infinite, or they disappear after at most two uses of the channel. Memory channels of finite depth can be to some extent controlled and manipulated by so-called reset sequences. We show that actions separated by the sequences of inputs of the length of the memory depth are independent and constitute memoryless channels.

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I. MEMORY EFFECTS

Schrödinger equation implies that an evolution of a closed quantum system is unitary. However, this ideal picture of closed and isolated quantum system is very difficult to achieve experimentally. Unavoidable interactions between the system and its environment result in a nonunitary evolution. Fortunately, under some specific though quite realistic conditions the dynamics of the system can be described without the necessity of explicit consideration of the environment's degrees of freedom. The crucial assumption of open system dynamics is that initially, the system is statistically completely independent of the environment degrees of freedom affecting its time dynamics. It means that a preparation procedure is completely uncorrelated from the evolution process.

For example, a photon source (e.g., laser) is independent of an optical cable used for the transmission. Only after inserted into the optical cable the photon is affected by its properties resulting in a state change. Although the interaction between the photon and the cable is driven by Schrödinger equation, the photon itself undergoes a nonunitary evolution. In particular, let us denote by ϱ_1 the initial state of the photon and by ξ the initial state of the environment represented by the optical cable. The input-output transformation then reads

$$\varrho_1 \to \varrho_1' = \mathrm{Tr}_{\mathrm{env}}[U\varrho_1 \otimes \xi_{\mathrm{env}} U^\dagger] = \mathcal{E}_1[\varrho_1]. \tag{1.1}$$

By definition, the mapping $\mathcal{E}$ describing the quantum process (channel) is linear, completely positive, and trace-preserving.

But not only the photon state has changed. Also, the environment degrees of freedom evolved into

$$\xi_{\mathrm{env}}' = \mathrm{Tr}_1[U\varrho_1 \otimes \xi_{\mathrm{env}} U^\dagger] = \mathcal{F}[\xi_{\mathrm{env}}]. \tag{1.2}$$

This *concurrent* [1] mapping $\mathcal{F}$ acting on the environment system is a valid channel, because it is linear, completely positive, and trace-preserving. Let us note that such concurrent channel depends only on the input system state, hence for any channel $\mathcal{E}$ acting on a system there exist many concurrent channels $\mathcal{F}$ acting on the memory, and vice versa.

If the same optical cable is used once more, then

$$\varrho_2' = \mathcal{E}_2[\varrho_2] = \mathrm{Tr}[U\varrho_2 \otimes \xi_{\mathrm{env}}' U^\dagger], \tag{1.3}$$

and $\mathcal{E}_1 \neq \mathcal{E}_2$ in general. Moreover,

$$\omega_{12} = \mathrm{Tr}_{\mathrm{env}}[U_2 U_1(\varrho_1 \otimes \varrho_2 \otimes \xi_{\mathrm{env}})U_1^\dagger U_2^\dagger] \neq \mathcal{E}_1[\varrho_1] \otimes \mathcal{E}_2[\varrho_2], \tag{1.4}$$

where U_1 (U_2) acts on the environment and the first (second) system. We see that subsequent usages of the same process device (e.g., optical cable) are not necessarily independent. Usually, a time intervals in between the usages are sufficiently large so that the environment relaxes into its original initial state, hence $\omega_{12} = \mathcal{E}_1 \otimes \mathcal{E}_1[\varrho_1 \otimes \varrho_2]$. If this holds for any number of uses, we say that the device is memoryless and its action can be fully described by means of quantum channels, i.e., completely positive trace-preserving linear maps. However, our goal is to investigate the cases when the relaxation processes are not sufficiently fast (or are not happening at all) to guarantee the same conditions for each run of the experiment (e.g., photon transmission). Such devices are described by quantum memory channels. In particular, we will focus on characterization and properties of those memory channels, for which the memory effects are finite.

The research subject of quantum memory channels is relatively new. Once the nature of the memory mechanism is known it can be exploited to increase the information transmission rates. Moreover, in this case the entangled encoding strategies can significantly overcome the factorized ones. Thus, the capacities (either classical or quantum) of quantum memory channels are not necessarily additive. Naturally, the research is mostly focused on investigation of transmission rates for particular classes of memory channels [2–14]. Recently, attention has been paid to an interesting class of so-called bosonic memory channels [15–23] and also to memory effects in the transmission of quantum states over the spin chains [24–26]. Our aim is to investigate the structural properties of quantum memory channels rather than to analyze their communication capabilities. A general framework and structural theorem for quantum memory channels was given in the seminal work of Kretschmann and Werner [27]. In [28], the discrimination of general quantum memory channels was investigated, and in [29] the concept of repeatable quantum memory channels was introduced and analyzed. In [27], the authors introduced the concept of forgetful quantum memory channels and showed that these memory channels form a dense subset of all quantum memory channels. For such memory channels the state of the memory is "forgotten" after a certain number of uses. In other words, after n uses of the memory channel the $(n+1)$th output state is approximately the same whatever was the original state of the memory. Our task is to identify those channels, for which the output state is *exactly* the same and to analyze the memory depth once the size of the memory system is fixed.

Let us note that the concept of finiteness of the memory we are going to use is different as the one introduced in Ref. [30], where the finiteness means the size of the memory system. In our case, the finiteness is related rather to the depth of memory effects. Our ultimate goal is to clearly formulate this concept and investigate the simplest case of qubit memory channels. We want to characterize those memory channels for which the memory depth is finite. Such memory channels can potentially mimic memoryless channels, paying the cost of larger inputs.

In the following, Sec. II, we will formalize the language of quantum memory channels. In Sec. III, we will formulate the problem in general settings. The qubit case will be investigated in details in Sec. IV. The results are summarized in Sec. V.

II. PRELIMINARIES

Let us denote by $\mathcal{H}$ a Hilbert space of the studied quantum system and by $\mathcal{L}(\mathcal{H})$ a set of bounded linear operators on $\mathcal{H}$. A state ϱ is any positive linear operator on $\mathcal{H}$ of unit trace, i.e., $\varrho \geq 0$ and $\mathrm{tr}[\varrho]=1$. A linear map $\mathcal{E}$ on the set of trace class operators is called a channel if it is completely positive $[\mathcal{L}(\mathcal{H}\otimes\mathcal{H}_{\mathrm{anc}}) \ni X \geq 0$ implies $(\mathcal{E}\otimes\mathcal{I}_{\mathrm{anc}})[X]\geq 0]$ and trace-preserving $(\mathrm{tr}[\mathcal{E}[X]]=\mathrm{tr}[X])$. The famous Stinespring dilation theorem says that any channel can be realized as a unitary channel on some extended Hilbert space, i.e.,

$$\mathcal{E}[X] = \mathrm{tr}_{\mathrm{anc}}[U(X \otimes \xi_{\mathrm{anc}})U^{\dagger}] \tag{2.1}$$

for some unitary operator $U \in \mathcal{L}(\mathcal{H}\otimes\mathcal{H}_{\mathrm{anc}})$ and some state ξ_{anc}.

By a process device, we will understand any fixed piece of hardware transforming quantum system from their initial state to some final state. In each individual use, it is described by some quantum channel, i.e., $\varrho \mapsto \varrho' = \mathcal{E}[\varrho]$. It is memoryless if its joint action on n subsequent inputs is factorized and in each run it is the same, i.e., $\mathcal{E}_{1\ldots n}=\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_1$ for all $n=1,2,\ldots$. If such property does not hold, then no single channel can be used to describe the quantum process device. The process device is in general described by an infinite sequence $\mathcal{E}_1, \mathcal{E}_{12}, \ldots$ of channel acting on $\mathcal{H}, \mathcal{H}\otimes\mathcal{H}, \ldots$, respectively. The causality requirement that the actual action does not depend on future inputs implies that

$$\mathrm{tr}_n\, \mathcal{E}_{12\ldots n}[X_{1,2,\ldots,n-1} \otimes Y_n] = \mathcal{E}_{1,2,\ldots,n-1}[X_{1,2,\ldots,n-1}],$$

for all X, Y. In the seminal work [27] it was shown that such causal quantum memory channel can be always expressed as a concatenation of unitary channels describing a sequence of interactions between the individual inputs and some fixed memory system, i.e.,

$$\mathcal{E}_{1,2,\ldots n}[\omega_{12\ldots n}] = \mathrm{tr}_{\mathrm{mem}}[U_n \ldots U_1(\omega_{12\ldots n} \otimes \xi_{\mathrm{mem}})U_1^{\dagger} \ldots U_n^{\dagger}],$$

where ξ_{mem} is a state of an ancillary system called memory and the bipartite unitary operator U_j acts nontrivially only on the jth input and the memory system. This representation is not unique and by definition we assume that we do not have direct access to the memory system.

In what follows, we shall restrict to a specific type of quantum memory models, in which the interactions are described by the same unitary operator, i.e., $U_1 = U_2 = \cdots = U_n = U$. Let us note that for general considerations this case covers the most general situation. In particular, let $U_1, U_2, \ldots$ be the sequence of unitaries defining a quantum memory channel (potentially $U_j \neq U_k$). We can define a unitary operator $W = \sum_{j=0}^{\infty} U_j \otimes |j+1\rangle\langle j|$ on $\mathcal{H}\otimes\mathcal{H}_{\mathrm{mem}}\otimes\mathcal{H}_{\infty}$, where $\mathcal{H}_{\infty}$ is the Hilbert space of the linear harmonic oscillator (being part of the memory system) and U_j are the unitaries associated with the quantum memory channel. In this sense, any quantum memory channel is generated by a fixed unitary operator $U=W$ and some initial memory state ξ_{mem}. However, such reduction requires infinite memory system.

Let us stress that only if the input states are uncorrelated, $\omega_{12\ldots n} = \varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_n$, then the transformation of each input state is described by a quantum channel. Otherwise, the channel model is not applicable. On one side, this is indeed a restrictive condition, however, on the other side, it is experimentally very relevant. The channel $\mathcal{E}_n$, defined as

$$\mathcal{E}_n[\rho_n] = \mathrm{tr}_{1,\ldots,n-1}[\mathcal{E}_{1,2,\ldots n}[\varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_n]], \tag{2.2}$$

transforming the nth input, in general, depends on all previous inputs $\varrho_1, \ldots, \varrho_{n-1}$. If this is the case for all n, then we say that the memory is infinite. The other extreme is the memoryless case, when $U = V \otimes V_{\mathrm{mem}}$ and the channel $\mathcal{E}_n$ is completely independent of any input. For example, if $\mathcal{H}_{\mathrm{mem}} \equiv \mathcal{H}$ and $U = V_{\mathrm{swap}}$ is the swap operation $(V_{\mathrm{swap}}\varrho \otimes \xi V_{\mathrm{swap}}^{\dagger} = \xi \otimes \varrho)$, then $\mathcal{E}_n[\varrho_n] = \varrho_{n-1}$, thus, $\mathcal{E}_n$ is a complete contraction of the state space into the state ϱ_{j-1}, which describes the $(n-1)$th input. In such case, the memory is of finite depth, because $\mathcal{E}_n$ depends solely on the input state ϱ_{n-1}.

In general, we say that a memory of the quantum memory channel generated by a unitary operator U is of depth Δ_U, if for each n the channel $\mathcal{E}_n$ does not depend on the initial memory state ξ_{mem}, neither on the particular choice of input states ϱ_j for all $j < n - \Delta_U$. Or, alternatively, the depth is Δ_U if for each n the channel $\mathcal{E}_n$ is independent of the inputs preceding $(n-\Delta_U)$th run of the process device including the original memory state ξ_{mem}. For example, the SWAP operator is of depth 1, i.e., $\Delta_{V_{\mathrm{swap}}} = 1$.

Our goal is to analyze which interactions U generate memory channels with finite memory irrespective of the initial state of the memory system.

III. FINITE-DEPTH MEMORY

The channel $\mathcal{E}_j$ transforming a given input ϱ_j is generated by the interaction U and the state of the ancilla ξ_j in the jth run of the process device. All the parameters the channel $\mathcal{E}_j$ depends on are only mediated through the memory state ξ_j. Choosing an orthogonal operator basis $\tau_0,\ldots,\tau_{d^2-1}$ of the memory system the memory state ξ takes the form

$$\xi=\sum_k m_k\tau_k, \tag{3.1}$$

and the resulting channel reads

$$\mathcal{E}_\xi[\varrho]=\sum_k m_k\,\mathrm{tr}_{\mathrm{mem}}[U\varrho\otimes\tau_k U^\dagger]. \tag{3.2}$$

Let us note that orthogonality is defined with respect to Hilbert-Schmidt scalar product $\langle A,B\rangle_{\mathrm{hs}}=\mathrm{tr}[A^\dagger B]$.

If for a fixed unitary operator U and arbitrary input state ϱ we have $\mathrm{tr}_{\mathrm{mem}}[U\varrho\otimes AU^\dagger]=O$ for some operator A, then the induced channels $\mathcal{E}$ are independent of parameter $\mathrm{tr}[\xi A]$. It follows from the fact that the operator $A/\mathrm{tr}[A^\dagger A]$ can be taken to be an element of the orthonormal operator basis $\{\tau_k\}$ and $\xi=\Sigma_k\mathrm{tr}[\xi\tau_k]\tau_k$. The set of all such operators A form a linear subspace of $\mathcal{L}(\mathcal{H})$ and we call the corresponding state parameters $\mathrm{tr}[\xi A]$ irrelevant, because $\mathcal{E}$ does not depend on them. Let us note that the identity operator I is never irrelevant, i.e., $\mathrm{tr}_{\mathrm{mem}}[U(\varrho\otimes I)U^\dagger]\neq O$. Therefore, without loss of generality we can set $\tau_0=I/\sqrt{d}$ and, consequently, due to orthogonality the other elements of the operator basis are traceless, i.e., $\mathrm{tr}[\tau_k]=0$ for all $k\neq 0$. Thus, the irrelevant operators are necessarily traceless. In such basis, the states ξ take the form $\xi=\frac{1}{d}I+\vec{m}\cdot\vec{\tau}$, hence they are uniquely represented by (d^2-1)-dimensional vectors $\vec{m}$ (so-called Bloch vectors). The entries of each vector $\vec{m}$ can be split into relevant and irrelevant ones. We will focus on the behavior of the relevant parameters mediating the memory effects.

Using the process device n times the memory undergoes an evolution

$$\xi_{n+1}=\mathcal{F}_n[\xi_n]=\cdots=\mathcal{F}_n\cdots\mathcal{F}_1[\xi_{\mathrm{mem}}], \tag{3.3}$$

where $\mathcal{F}_j$ is defined via $\mathcal{F}_j[\xi_j]=\mathrm{tr}_{\mathrm{sys}}[U\varrho_j\otimes\xi_jU^\dagger]$ and $\xi_1=\xi_{\mathrm{mem}}$ is the initial state of the memory system. Let us define a channel $\mathcal{G}=\mathcal{F}_n\cdots\mathcal{F}_1$. This channel potentially depends on all input states $\varrho_1,\ldots,\varrho_n$, hence, consequently, the memory state ξ_{n+1} and also the channel $\mathcal{E}_{n+1}$ depend on ξ_1 and all inputs $\varrho_1,\ldots,\varrho_n$. If the memory is finite and of the depth n, then $\mathcal{E}_{n+1}$ does not depend on ξ_1 whatever collection of input states $\varrho_1,\ldots,\varrho_n$ was used. This happens if the relevant parameters of ξ_{n+1} do not depend on the memory state ξ_1. Let us note that ξ_n still may depend on input states $\varrho_1,\ldots,\varrho_n$, however, it is independent on any input preceding ϱ_1. As it is required this feature is invariant in time. That is, $\mathcal{E}_{s+n+1}$ is independent of memory state ξ_s and also on all input states ϱ_j with $j\leq s$.

The goal is to investigate for which n the concurrent channel $\mathcal{G}$ is deleting all relevant parameters of the memory system whatever sequence $\varrho_1,\ldots,\varrho_n$ is used. The action of the channel $\mathcal{G}$ on Bloch vectors $\vec{m}$ takes the form of an affine mapping, i.e., $\vec{m}\mapsto\vec{g}+G\vec{m}$, where $g_k=\frac{1}{d}\mathrm{tr}[\tau_k^\dagger\mathcal{G}[I]]$ and $G_{kl}=\mathrm{tr}[\tau_k^\dagger\mathcal{G}[\tau_l]]$ for $k,l=1\ldots,d^2-1$. Since $\mathcal{G}$ is a composition of channels $\mathcal{F}_1,\ldots,\mathcal{F}_n$, using the corresponding vectors $\vec{f}_j$ and matrices F_j, the action can be expressed as

$$\vec{m}_1\to\vec{m}_{n+1}=(F_n\cdots F_1)\vec{m}+(F_n\cdots F_2)\vec{f}_1+\cdots+\vec{f}_n,$$

thus $G=F_n\cdots F_1$ and $\vec{g}=(F_n\cdots F_2)\vec{f}_1+\cdots+\vec{f}_n$. The requirement of finite depth of the memory implies that relevant parameters of $\vec{m}_{n+1}$ are independent of $\vec{m}_1$ for all input states $\varrho_1,\ldots,\varrho_n$, hence, G is singular and maps any vector $\vec{m}_1$ into the subspace spanned by "irrelevant" operators τ_k. Let us note that product of nonsingular matrices is not singular. Since we do require that G is singular for all sequences of inputs it follows that each F_j must be singular. If for some input state ϱ the matrix F is not singular, then sequence $\varrho^{\otimes n}$ induces a nonsingular matrix $G=F^n$ for arbitrary n. In such case, the memory depth is infinite. Therefore, the singularity of the matrices F for all input states ϱ is a necessary (but not sufficient) condition for U to generate a finite quantum memory channel.

Let us note that a finite-depth memory channel does not create any correlations between outputs separated by n uses if all inputs are factorized (see Appendix). Consequently, its actions (separated by n uses) are independent. In this way the memory process device can be used to implement a memoryless channel, using first n inputs as a reset sequence which will set the memory system to some particular (although not arbitrary) state ignoring the outputs and then performing the channel on next input. The proof of this statement is given in Appendix.

IV. CASE STUDY: TWO-DIMENSIONAL MEMORY

In this section, we will investigate qubit memory channels with a two-dimensional memory system. The question is, what are the possible values of Δ in such very specific settings? Let us use the basis of Pauli operators $\sigma_x,\sigma_y,\sigma_z$ to express the qubit states. Then, the memory state takes the form $\xi_1=\frac{1}{2}(I+\vec{m}_1\cdot\vec{\sigma})$ and can be represented by a three-dimensional Bloch vector $\vec{m}_1$. Similarly, let us assume that the system is initially prepared in a state $\varrho_1=\frac{1}{2}(I+\vec{r}_1\cdot\vec{\sigma})$. The action of the concurrent channel $\mathcal{F}_1[\xi_1]=\mathrm{tr}_{\mathrm{sys}}[U\varrho_1\otimes\xi_1U^\dagger]$ can be expressed by means of vector $\vec{f}_1=\frac{1}{2}\mathrm{tr}[\vec{\sigma}\mathcal{F}[I]]$ and matrix $F_{1,jk}=\frac{1}{2}\mathrm{tr}[\sigma_j\mathcal{F}_1[\sigma_k]]$. In particular, in the language of Bloch vectors the channel takes an affine form $\vec{m}_1\to\vec{t}+T\vec{m}_1$, hence, in the nth run the memory system is transformed as $\vec{m}_n\to\vec{f}_n+F_n\vec{m}_n$, where by $\vec{m}_n$ we denoted the state of the memory before the nth use of the process device. As before, the initial memory $\vec{m}_1$ is transformed as follows:

$$\vec{m}_1\to\vec{m}_{n+1}=(F_n\cdots F_1)\vec{m}_1+(F_n\cdots F_2)\vec{f}_1+\cdots+\vec{f}_n.$$

A general two-qubit unitary transformation can be expressed as follows (see, for example, [31]):

$$U=(V_1\otimes W_1)e^{i\Sigma_j\alpha_j\sigma_j\otimes\sigma_j}(V_2\otimes W_2), \tag{4.1}$$

where V_j,W_j are single-qubit unitary operators and α_j are real numbers. We learned that in order to generate a quantum memory channel with finite depth of the memory for all input sequences, it is necessary for U that the induced concurrent channels $\mathcal{F}_j$ are singular. Since local unitary rotations $V_j\otimes W_j$ do not affect the singularity, it is sufficient for now to analyze only the unitary operators of the form $U=e^{i\Sigma_j\alpha_j\sigma_j\otimes\sigma_j}$.

For the considered unitary operator $U=e^{i\Sigma_j\alpha_j\sigma_j\otimes\sigma_j}$, the matrix F takes the form

$$F(\vec{r})=\begin{pmatrix} c_yc_z & r_zc_ys_z & -r_ys_yc_z \\ -r_zc_xs_z & c_xc_z & r_xs_xc_z \\ r_yc_xs_y & -r_xs_xc_y & c_xc_y \end{pmatrix}, \tag{4.2}$$

where $c_j=\cos 2\alpha_j$ and $s_j=\sin 2\alpha_j$. Let us note that due to symmetry of U with respect to exchange of the system and the memory, the same matrix describes the channel acting on the system, only the role of $\vec{r}$ is replaced by the initial state of the memory $\vec{m}_1$.

Evaluating the determinant we get

$$\det F(\vec{r})=r_x^2s_x^2c_y^2c_z^2+r_y^2c_x^2s_y^2c_z^2+r_z^2c_x^2c_y^2s_z^2+c_x^2c_y^2c_z^2.$$

It vanishes if and only if at least one of the following conditions hold:

$$\cos 2\alpha_x=\cos 2\alpha_y=0; \tag{4.3}$$

$$\cos 2\alpha_x=\cos 2\alpha_z=0; \tag{4.4}$$

$$\cos 2\alpha_y=\cos 2\alpha_z=0. \tag{4.5}$$

If exactly one of the above conditions holds, for instance $\cos 2\alpha_x=\cos 2\alpha_y=0$, then

$$F(\vec{r})=(\cos 2\alpha_z)\begin{pmatrix} 0 & 0 & \pm r_y \\ 0 & 0 & \pm r_x \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.6}$$

is a matrix of rank one and $F(\vec{r}_2)F(\vec{r}_1)=O$. Setting $F_j=F(\vec{r}_j)$ we get for all j

$$\vec{m}_{j+1}=F_j\vec{m}_j+\vec{f}_j=F_j\vec{f}_{j-1}+\vec{f}_j. \tag{4.7}$$

Since $\vec{f}_j$ depends only on input state ϱ_j the state of the memory ξ_{j+1} depends only on input state ϱ_j and ϱ_{j-1}, i.e., on preceding two input states. Therefore, the memory depth equals $\Delta=2$. That is, the jth input state is transformed by a channel $\mathcal{E}_j$

$$\vec{r}_j'=E_j\vec{r}_j+\vec{e}_j, \tag{4.8}$$

where $E_j,\vec{e}_j$ depends via the memory state $\vec{m}_j$ on input states ϱ_{j-1} and ϱ_{j-2}.

Due to the already mentioned symmetry of U, it follows that the channel $\mathcal{E}_\xi$ acting on the system qubit does not depend on the value of m_z, because

$$E=(\cos 2\alpha_z)\begin{pmatrix} 0 & 0 & \pm m_y \\ 0 & 0 & \pm m_x \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

The unitary operators $U=\exp(i\Sigma_j\alpha_j\sigma_j\otimes\sigma_j)$ generating the considered finite memory channels $(\alpha_x,\alpha_y\in\{\pm\pi/4\})$ are of the form

$$U_{\alpha_z}=\frac{1}{2}[I+\sigma_{zz}+ie^{-2i\alpha_z}(\sigma_{xx}+\sigma_{yy})]\sigma_{xx}^{h_x}\sigma_{yy}^{h_y}, \tag{4.10}$$

where $\sigma_{jj}=\sigma_j\otimes\sigma_j$, $h_j=H(-\alpha_j)$ $(j=x,y,z)$ and $H(\cdot)$ is the Heavyside step function. The remaining options α_x, $\alpha_z\in\{\pm\pi/4\}$ and α_y, $\alpha_z\in\{\pm\pi/4\}$ correspond to unitary operators that are locally unitarily equivalent to U_{α_z}. In particular, it is sufficient to relabel the basis, i.e., instead of using the eigenbasis of σ_z we use eigenbasis of σ_x, or σ_y in which the unitary transformations $U_{\alpha_x},U_{\alpha_y}$ takes the same form.

The freedom as specified in Eq. (4.1) is a bit larger than that. Replacing the unitary operator U_{α_z} by a more general one $U=V_1\otimes W_1U_{\alpha_z}V_2\otimes W_2$, the concurrent channel $F(\vec{r})$ takes the form

$$F'(\vec{r})=S'F(\vec{r})R, \tag{4.11}$$

where S' and R are orthonormal matrices corresponding to unitary operators W_1 and W_2, respectively. Since orthogonal matrices do not affect the singularity, the matrices $F'(\vec{r})$ are singular. Moreover, it can be rewritten in a more convenient form as $R^{-1}SF(\vec{r})R$, where $S=RS'$ is a suitable orthogonal matrix. Using a sequence of input states $\varrho_1\otimes\cdots\otimes\varrho_n$ and defining $F_j'=F'(\vec{r}_j)$, we get

$$G'=F_n'\cdots F_1'=R^{-1}SF_n\cdots SF_1R. \tag{4.12}$$

The question is for which values of n and for which rotations S the matrices G' (generated by sequences $F_1,\ldots,F_n$) maps memory states into the irrelevant subspace.

The matrix R corresponds merely to changing the basis of memory system and as such does not affect the depth of memory of the memory channel and can be left arbitrary. We will not consider it in further calculations. The unitary matrix $W'=W_2W_1$ corresponding to S does not change the relevance of parameters, because for all operators τ and arbitrary U

$$\begin{aligned}\mathrm{tr}_{\mathrm{mem}}[(I\otimes W')U(\varrho\otimes\tau)U^\dagger(I\otimes W'^\dagger)] &=\sum_{abcd}\mathrm{tr}[W'|a\rangle\langle b|\tau|d\rangle\langle c|W'^\dagger]A_{ab}\varrho A_{cd}^\dagger \\ &=\sum_{abcd}\mathrm{tr}[|a\rangle\langle b|\tau|d\rangle\langle c|]A_{ab}\varrho A_{cd}^\dagger=\mathrm{tr}_{\mathrm{mem}}[U\varrho\otimes\tau U^\dagger],\end{aligned} \tag{4.13}$$

where we used the expression $U=\Sigma_{ab}A_{ab}\otimes|a\rangle\langle b|$ for some orthonormal basis $\{|a\rangle\}$ and operators A_{ab} such that U is unitary.

As we have seen in Eq. (4.9), there is only one irrelevant parameter m_z, because only m_z does not enter the expression in Eq. (4.9). Consequently, we require for all sequences $F_1,\ldots,F_n$ the following conditions:

$$SF_nS\ldots SF_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1 & x_2 & x_3 \end{pmatrix}, \tag{4.14}$$

where x_1,x_2,x_3 are arbitrary numbers, and n will be the depth of this channel.

Let us denote by S_{kl} the entries of S and define $a_{j,k}=\cos(2\alpha_z)(\pm r_{j,y}S_{k1}\pm r_{j,x}S_{k2})$ with $j=1,\ldots,n$ and $k,l=1,2,3$. Then

$$SF_j=\begin{pmatrix}0&0&a_{j,1}\\0&0&a_{j,2}\\0&0&a_{j,3}\end{pmatrix} \tag{4.15}$$

and the Eq. (4.14) reads

$$a_{1,3}\ldots a_{n-1,3}\begin{pmatrix}0&0&a_{n,1}\\0&0&a_{n,2}\\0&0&a_{n,3}\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\x_1&x_2&x_3\end{pmatrix}.$$

Since this relation must hold for all states ϱ, i.e., for all Bloch vectors $\vec{r}_1,\ldots,\vec{r}_n$, it is necessary that $a_{j,3}=\pm r_{j,y}S_{31}\pm r_{j,x}S_{32}=0$ for all vectors $\vec{r}_j$, thus, $S_{31},S_{32}=0$. Rotation matrices S satisfying such constraint are necessarily of the form

$$S=\begin{pmatrix}q\cos 2\beta & q\sin 2\beta & 0\\-\sin 2\beta & \cos 2\beta & 0\\0&0&q\end{pmatrix}, \tag{4.16}$$

where $q=\pm 1$ and $\beta\in[0,2\pi]$. Therefore,

$$SF_j=\begin{pmatrix}0&0&q(r_{j,y}\cos 2\beta+r_{j,x}\sin 2\beta)\\0&0&-r_{j,y}\sin 2\beta+r_{j,x}\cos 2\beta\\0&0&0\end{pmatrix} \tag{4.17}$$

are matrices of the same form as for F_j only. The same arguments imply that the depth is either 1, or 2, because $SF_2SF_1=O$ for all possible matrices F_1,F_2. The unitary matrix W' corresponding to S equals to

$$W'=\begin{pmatrix}0&1\\-1&0\end{pmatrix}^{(1-q)/2}\begin{pmatrix}e^{i\beta}&0\\0&e^{-i\beta}\end{pmatrix}. \tag{4.18}$$

In conclusion, the memory is finite only if the quantum memory channel is induced by unitary operator U of the form (in some factorized basis)

$$U=(V_1\otimes W_2^{\dagger}W')U_{\alpha}(V_2\otimes W_2). \tag{4.19}$$

Moreover, in such case necessarily $\Delta_U\leq 2$, hence, the memory depth (if not infinite) is surprisingly quite limited. If $\alpha_x=\alpha_y=\alpha_z=\pi/4$, then F_j is a zero matrix, $F_j\equiv O$, and $U_{\pi/4}=e^{i\pi/4\Sigma_j\sigma_j\otimes\sigma_j}=V_{\text{swap}}$ is the swap operator. In such case,

$$\vec{m}_j\mapsto\vec{m}_{j+1}=\vec{f}_j=\vec{r}_j; \tag{4.20}$$

$$\vec{r}_j\mapsto\vec{r}_j'=\vec{m}_j=\vec{r}_{j-1}, \tag{4.21}$$

where jth output state equals to $(j-1)$th input state, i.e., $\Delta_{V_{\text{swap}}}=1$. In summary, the depth of the memory Δ_U in the considered case of single-qubit memory systems can achieve only the values 0,1,2, or infinity.

Classical bits

Let us shortly discuss the case of classical memory channels. Quantum description covers the classical one in a sense that classical states are density operators orthogonal in some fixed (factorized) basis, i.e., they represent probability distributions expressed as diagonal matrices. Similarly, unitary operators are replaced by permutations, which form a very specific subgroup of all unitary operators. Having in mind these restrictions, all the discussed concepts are applicable for classical systems as well.

A classical bit is the simplest classical system having the quantum bit as its quantum counterpart. The states are expressed as density operators $p|0\rangle\langle 0|+(1-p)|1\rangle\langle 1|$ and there are only two permutations corresponding to I and σ_x, which flips the bit values. Assuming the memory system is also of the size of a single classical bit, there are only $4!=24$ permutations U describing the classical memory channels of a single bit. Analyzing all of them we find that the memory depth can be 0,1, or infinity, because U_α describes a permutation only if $\alpha=\pi/4$, i.e., when it is the SWAP operator.

V. CONCLUSION

For each quantum memory channel describing any quantum process device we can assign a parameter Δ_U meaning that its nth run depends at most on the previous Δ_U uses. Equivalently, the input-output action is irrelevant of the state of the memory after the $(n-\Delta_U)$th use. We call this number the depth of the memory. We investigated in details the simplest case of qubit memory channels with the memory system composed of a single qubit, as well. We showed that values of the memory depth are restricted and $\Delta_U\in\{0,1,2,\infty\}$. Let us note that in the analogous situation for classical systems $\Delta_U\in\{0,1,\infty\}$. In particular, $\Delta_U=0$ if U is factorized, $\Delta_U=1$ if U is the SWAP operator (up to local unitaries) and $\Delta_U=2$ if

$$U=|\varphi\rangle\langle\varphi'|\otimes|\psi\rangle\langle\psi'|+|\varphi_\perp\rangle\langle\varphi_\perp'|\otimes|\psi_\perp\rangle\langle\psi_\perp'|$$
$$+ie^{-2i\alpha}(|\varphi\rangle\langle\varphi_\perp'|\otimes|\psi_\perp\rangle\langle\psi'|+|\varphi_\perp\rangle\langle\varphi'|\otimes|\psi\rangle\langle\psi_\perp'|),$$

where $|\psi\rangle=W_2^{\dagger}W'|0\rangle$ [see Eq. (4.18)], $|\psi'\rangle=W_2^{\dagger}\sigma_{xx}^{h_x}\sigma_{yy}^{h_y}|0\rangle$, $|\varphi\rangle=V_1|0\rangle$, $|\varphi'\rangle=V_2^{\dagger}\sigma_{xx}^{h_x}\sigma_{yy}^{h_y}|0\rangle$. In all other cases the memory is infinite.

If the memory depth is finite, then a sequence of input states can be used to reset the memory system into a fixed state irrelevant of the initial state of the memory and inputs preceding the reset input sequence. Applying the same reset sequence guarantees that in each (Δ_U+1)th use locally the same channel is implemented. In the Appendix it is shown that actions of the process device separated by reset sequences are indeed uncorrelated.

That is, in each $(n+1)$th run of the process device, the same quantum channel is independently implemented providing that the same reset sequence is used. In this way, memory channels can be used as memoryless ones. However, that it is an open problem whether any channel can be implemented on some finite-depth memory channel in this way and also whether there is some bound on the size of the reset sequence and the memory system. So far, we know that if we restrict ourselves to single-qubit memory, then such channels are represented by rank-1 matrices and the reset sequence is of length at most 2.

In summary, for most of the qubit memory channels, the memory effects have infinite depth. Based on our investigation of the simplest physical model, we can make a rather surprising conjecture that the dimension of the memory puts constraints on the memory depth Δ_U. Unfortunately, we have not succeeded to find any simple analytic bound expressing this relation. Similarly, the characterization of general unitary operators generating fine-depth memory channels remains open.

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APPENDIX: CORRELATIONS

Theorem 1. *Consider a unitary memory channel U of the depth n, i.e.,* $\Delta_U = n$. *Then, the actions of the process device separated by n uses (reset sequence) are not correlated providing that the reset sequences are not correlated, i.e.,*

$$\mathcal{E}[\omega_{n+1,2n+2}] = (\mathcal{E}_{n+1} \otimes \mathcal{E}_{2n+2})[\omega_{n+1,2n+2}], \tag{A1}$$

where $\omega_{n+1,2n+2}$ *is the joint state of* $(n+1)$th *and* $2(n+1)$th *inputs and* $\mathcal{E}_j$ *denotes the action of the memory channel on its* jth *input.*

Proof. Let us denote by $\Xi = R_1 \otimes \cdots \otimes R_n$ the sequence of input states forming the so-called reset sequence. This sequence, together with the memory system ξ, is inducing a channel $\mathcal{E}_{\Xi}^{\xi}$ on the $(n+1)$th process device input state

$$\mathcal{E}_{\Xi}^{\xi}[\omega] = \mathrm{tr}_{\mathrm{res,mem}}[U^{(n+1)}(\Xi \otimes \omega \otimes \xi)U^{(n+1)\dagger}], \tag{A2}$$

where $U^{(n+1)}$ is the $n+1$-fold concatenation of the channel U and ω is the state of input system. Let us express the interaction U as follows:

$$U = \sum_{a,b} A_{ab} \otimes |a\rangle\langle b|, \tag{A3}$$

where A_{ab} are operators acting on the principal system and vectors $\{|a\rangle\}$ form an orthonormal basis of the Hilbert space of the memory system. The unitarity of U imposes the following normalization conditions on operators A_{ab}:

$$\sum_a A_{ab}^{\dagger} A_{ac} = \delta_{bc} I, \quad \sum_b A_{ab} A_{cb}^{\dagger} = \delta_{ac} I. \tag{A4}$$

Defining the operators

$$M_{a_n a_0} = \sum_{a_1,\ldots,a_{n-1}} A_{a_1 a_0} \otimes \cdots \otimes A_{a_n a_{n-1}} \tag{A5}$$

acting on the Hilbert space of the reset sequence $\mathcal{H}_{\mathrm{res}}$ we get

$$\mathcal{E}_{\Xi}^{\xi}[\omega] = \sum_{\substack{a_0,a_n,a_{n+1}\\ c_0,c_n}} \xi_{a_0 c_0} \mathrm{tr}[M_{a_n a_0} \Xi M_{c_n c_0}^{\dagger}] A_{a_{n+1} a_n} \omega A_{a_{n+1} c_n}^{\dagger} = \sum_{a,c} \xi_{ac} \Omega_{ac}(\Xi,\omega), \tag{A6}$$

where $\xi_{ac} = \langle a|\xi|c\rangle$, $A_{a_j a_{j-1}}$ acts on jth input of the reset sequence and $\Omega_{ac}(\Xi,\omega)$ are operators defined on the $(n+1)$th principal system. These operators depend on Ξ, ω, but not on the state ξ.

Then, the finite-memory depth condition implies that for all memory states ξ, ξ' following relation holds:

$$\mathcal{E}_{\Xi}^{\xi}[\omega] = \mathcal{E}_{\Xi}^{\xi'}[\omega] \equiv \mathcal{E}_{\Xi}[\omega], \tag{A7}$$

for all states ω. Especially, for memory states $\xi = |a\rangle\langle a|$ we get $\mathcal{E}_{\Xi}^{|a\rangle\langle a|}[\omega] = \Omega_{aa}(\Xi,\omega) = \Omega_0(\Xi,\omega)$ for all values of a. Using a general state Ξ we obtain

$$\mathcal{E}_{\Xi}^{\xi}[\omega] = \Omega_0(\Xi,\omega) + \sum_{a \neq c} \xi_{ac} \Omega_{ac}(\Xi,\omega), \tag{A8}$$

and, consequently, the condition (A7) implies that $\Omega_{ac}(\Xi,\omega) = O$ for all $a \neq c$. In summary,

$$\Omega_{ac}(\Xi,\omega) = \sum_{a_n,a_{n+1},c_n} \mathrm{tr}[M_{a_n a} \Xi M_{c_n c}^{\dagger}] A_{a_{n+1} a_n} \omega A_{a_{n+1} c_n}^{\dagger} = \delta_{ac} \Omega_0(\Xi,\omega), \tag{A9}$$

and

$$\mathcal{E}_{\Xi}[\omega] = \Omega_0(\Xi,\omega). \tag{A10}$$

Next, we add another reset sequence Ξ_2 followed by next input ω_2 and analyze the joint action of the finite-depth memory process device on the inputs ω_1 and ω_2. In such case

$$\mathcal{E}_{\Xi_1 \otimes \Xi_2}^{\xi}[\omega_1 \otimes \omega_2] = \mathrm{tr}_{\mathrm{res,mem}}[U^{(2n+2)}(\Xi_1 \otimes \Xi_2 \otimes \omega_{12} \otimes \xi)U^{(2n+2)\dagger}] = \sum \xi_{a_0 c_0} \mathrm{tr}[M_{a_n a_0} \Xi_1 M_{c_n c_0}^{\dagger}] A_{a_{n+1} a_n} \omega_1 A_{c_{n+1},c_n}^{\dagger} \otimes \mathrm{tr}[M_{a_{2n+1} a_{n+1}} \Xi_2 M_{c_{2n+1} c_{n+1}}^{\dagger}] A_{a_{2n+2} a_{2n+1}} \omega_2 A_{a_{2n+2},c_{2n+1}}^{\dagger} = \sum \xi_{a_0 c_0} \mathrm{tr}[M_{a_n a_0} \Xi_1 M_{c_n c_0}^{\dagger}] A_{a_{n+1} a_n} \omega_1 A_{c_{n+1},c_n}^{\dagger} \otimes \delta_{a_{n+1},c_{n+1}} \Omega_0(\Xi_2,\omega_2) = \Omega_0(\Xi_1,\omega_1) \otimes \Omega_0(\Xi_2,\omega_2) = (\mathcal{E}_{\Xi_1} \otimes \mathcal{E}_{\Xi_2})[\omega_1 \otimes \omega_2],$$

where we have used twice the identity in Eq. (A9). Let us note that due to linearity the inputs (separated by the reset sequence Ξ_2) does not have to be factorized and altogether are described by a density operator ω_{12}. In conclusion, the actions separated by reset sequences take the "memoryless" form

$$\mathcal{E}_{\Xi_1\otimes\Xi_2}=\mathcal{E}_{\Xi_1}\otimes\mathcal{E}_{\Xi_2}. \tag{A11}$$

This completes the proof. □

[1] Not to be confused with *conjugate* or *complementary* channel. Let us denote $\mathcal{L}(\mathcal{H})$ as the set of bounded operators on a Hilbert space $\mathcal{H}$. Concurrent channel is then defined as a mapping from environment operators to environment operators: $\mathcal{L}(\mathcal{H}_{\text{env}})\mapsto\mathcal{L}(\mathcal{H}_{\text{env}})$, whereas conjugate or complementary channel is a mapping from input operators to environment operators: $\mathcal{L}(\mathcal{H}_{\text{input}})\mapsto\mathcal{L}(\mathcal{H}_{\text{env}})$. For further references on conjugate or complementary channels, see Refs. [32,33].

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