Unambiguous comparison of quantum measurements

Mario Ziman[1,2], Teiko Heinosaari[3] and Michal Sedlák[1]
[1]*Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia*
[2]*Faculty of Informatics, Masaryk University, Botanická 68a, Brno, Czech Republic*
[3]*Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen, Denmark*
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The goal of comparison is to reveal the difference of compared objects as fast and reliably as possible. In this paper we formulate and investigate the unambiguous comparison of unknown quantum measurements represented by non-degenerate sharp POVMs. We distinguish between measurement devices with apriori labeled and unlabeled outcomes. In both cases we can unambiguously conclude only that the measurements are different. For the labeled case it is sufficient to use each unknown measurement only once and the average conditional success probability decreases with the Hilbert space dimension as $1/d$. If the outcomes of the apparatuses are not labeled, then the problem is more complicated. We analyze the case of two-dimensional Hilbert space. In this case single shot comparison is impossible and each measurement device must be used (at least) twice. The optimal test state in the two-shots scenario gives the average conditional success probability 3/4. Interestingly, the optimal experiment detects unambiguously the difference with nonvanishing probability for any pair of observables.

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I. INTRODUCTION

Unavoidable randomness is one of the most important paradigms of quantum theory. As a consequence, its typical predictions and conclusions have a statistical and probabilistic essence. However, there are exceptions [2, 3, 4, 5]. For example, a photon passing through a vertical polarizer will pass the second vertical polarizer with probability 1. In such "certain" cases, the goal is not to acquire a complete description of quantum systems, but rather to identify some features of interest. In this paper we investigate a variant of an unambiguous quantum comparison problem [6, 7, 8, 9], i.e., a task in which the aim is to compare a pair of quantum devices.

It is an interesting question how the quantum systems can be compared and which of the quantum properties are comparable. For example, the velocity of quantum particles is a well defined property only under very specific conditions. In general, the probability distribution of velocities is the correct description of the dynamical properties of quantum particles. Therefore, in quantum case it is typical that the comparison problem is naturally a statistical problem. This is in contradiction with the usual approach to comparison tasks, which are based on individual events rather than on statistics. It could seem counter-intuitive, but individual experimental clicks can provide us with a definite and unambiguous answer even if the description is statistical. In general, the goal is to design an experiment accepting quantum devices as free parameters and producing events we can associate with three conclusions: i) same, ii) different, and iii) no conclusion.

So far, the unambiguous comparison problem has been studied in the cases of pure states [6, 8] and unitary channels [7, 9]. In this paper we analyze the unambiguous comparison of quantum measurements. Suppose that we are given a pair of experimental setups implementing qubit measurements, each of them designed by a different experimentalist. Is there a way to unambiguously compare their performance? Especially, are they same or different? As independent experimentalists we can think of these experimental setups as black boxes, producing outcomes after a qubit is inserted. Our conclusions then have to be based on the acquired measurement outcomes.

For quantum measurements, there are two natural variations of the comparison problem. First of all, we can ask whether the given black boxes are identical. This means that they produce the same measurement outcome statistics in any state. In particular, also the labeling of the outcomes is similar. For instance, two Stern-Gerlach apparatuses oriented in opposite directions are considered to be different in this strict sense. However, they can be made identical by simply re-labeling the outcomes in one of them. Thus, the other way to compare two black boxes is to ask whether they are *equivalent*, i.e., identical after suitable re-labeling of the outcomes.

As an example, suppose we are comparing whether two Stern-Gerlach apparatuses are identical. A singlet state of two qubits inserted into the measurements cannot lead to the same outcomes unless the measurement devices (including the labeling) are different. If labeling of the outcomes is not given or it is part of the comparison problem, then we can perform this singlet-based test for all possible labelings independently. Finding the same unambiguous conclusions in all of them leads to a conclusion also for measurements without apriori labels. Since for each of the Stern-Gerlach apparatuses we have two different choices of labels, we need to perform the singlet-based comparison four times, i.e. each of the apparatuses is used 4 times. We will show that there are also better strategies in which each of the unlabeled apparatuses is used only twice.

The paper is organized as follows. In Section II we shortly recall the mathematical description of quantum observables. Sections III and IV explain the concepts of unknown quantum measurement apparatuses and apriori information. The unambiguous comparison of measurements with labeled outcomes is presented in Section V and for apparatuses with unlabeled outcomes in Section VI. In the last Section VII we summarize the obtained results. Some of the techical details are given in the Appendix.

II. OBSERVABLES

The statistics of quantum measurements is described by positive operator valued measures (POVM). In what follows we consider only measurements with finite number of outcomes. For simplicity, we assume that these outcomes form an index set $J_n = \{1, \ldots, n\}$. The associated POVM is a mapping $\mathcal{A}$ from J_n into the set of effects $\mathcal{E}(\mathcal{H})$, i.e. a set of positive operators E on Hilbert space $\mathcal{H}$ such that $O \leq E \leq I$, where O is the zero operator and I is the identity operator. Moreover, the POVM is normalized to identity i.e. $\mathcal{A}_1 + \cdots + \mathcal{A}_n = I$, where $\mathcal{A}_i \equiv \mathcal{A}(i)$. The effects serve as a proper mathematical representation of observed quantum events (experimental clicks). A probability to observe an effect E is given by the trace formula

$$p_E = \mathrm{tr}\left[\varrho E\right] , \tag{2.1}$$

where ϱ is a state of the measured quantum system.

For an operator X, we denote by Π_X the projection onto the support of X. For effects and states we then have $E \leq \Pi_E$ and $\varrho \leq \Pi_\varrho$. Moreover, the condition $\mathrm{tr}\left[\varrho E\right] > 0$ is equivalent to $\Pi_E \Pi_\varrho \neq O$.

We say that observable is *sharp* if each effect composing the POVM is a projection, i.e., $E_j = E_j^2$ for all j. If, moreover, $E_j\mathcal{H}$ is a one-dimensional subspace of $\mathcal{H}$ for each j, then the observable is *non-degenerate*. In such case we can write $E_j = |\psi_j\rangle\langle\psi_j| \equiv \psi_j$ and $\langle\, \psi_j \,|\, \psi_k \,\rangle = \delta_{jk}$. In fact, each orthonormal basis of the Hilbert space defines a sharp non-degenerate POVM. We denote by $\mathcal{M}$ the set of all non-degenerate sharp observables. It is closed under the action of the unitary group $U(d)$ transforming $\mathcal{A} = \{\mathcal{A}_j\} \in \mathcal{M}$ into an observable $\mathcal{A}^U$ consisting of effects $\mathcal{A}_j^U = U\mathcal{A}_j U^\dagger$.

III. UNKNOWN BLACK BOX

We shall think of an unknown measurement apparatus as of a black box accepting physical systems and producing one of n distinguished outcomes. For sharp non-degenerate observables each of the outcomes is associated with a one-dimensional projection. We distinguish two types of black boxes leading to two different concepts of equivalence of observables and affecting the formulation of the comparison problem, too. In principle, we can meet with measurement outcomes that are either labeled, or not. If the outcomes are not labeled, we assign a number $j \in J_n$ to each of them. However, in such case the ambiguity of relabeling must be taken into account and equivalence of observables should be compatible with this freedom. Let us spell these definitions explicitly.

Definition 1. Observables $\mathcal{A} : J_n \to \mathcal{E}(\mathcal{H})$ and $\mathcal{B} : J_n \to \mathcal{E}(\mathcal{H})$ are *identical* if $\mathcal{A}_j = \mathcal{B}_j$ for all j.

Definition 2. Observables $\mathcal{A} : J_n \to \mathcal{E}(\mathcal{H})$ and $\mathcal{B} : J_n \to \mathcal{E}(\mathcal{H})$ are *equivalent* (in the unlabeled sense) if there exist a permutation $\pi : J_n \to J_n$ such that $\mathcal{A}_j = \mathcal{B}_{\pi(j)}$ for all j.

It follows from the definition that equivalence class of an unlabeled observable consists of POVMs with the same range, i.e. the elements of the set of unlabeled measurements can be understood as unordered collections of effects summing up to identity. The comparison of unlabeled measurements can hence be seen as a comparison of ranges of POVMs.

A single usage of a measurement device tells us that an effect E associated with the observed outcome has support overlapping with the support of ϱ, i.e. $\Pi_E\Pi_\varrho \neq O$. However, in the unlabeled case this information does not tell us too much about the particular effect associated with the observed outcome. Let us consider an unlabeled measurement described by effects $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ forming a particular POVM once the ordering is fixed. In fact, since we assume that the labeling is chosen in random way, for each artificially named outcome the predicted probability is the same, i.e.

$$p_j(\mathcal{A}) = \frac{1}{n!}\sum_\pi \mathrm{tr}\left[\varrho\mathcal{A}_{\pi(j)}\right] = \frac{(n-1)!}{n!}\sum_{j'} \mathrm{tr}\left[\varrho\mathcal{A}_{j'}\right] = \frac{1}{n} ,$$

where we used the fact that $n!$ is the total number of permutations on J_n, $(n-1)!$ is the number of them having a specific label j' on the fixed (jth) position.

Using the apparatus once more we can distinguish whether the observed outcomes coincide, or not. After fixing the labels $1, \ldots, n$ of the measurement device, the probability to observe a pair of outcomes j, k reads

$$\begin{aligned} p_{jk}(\mathcal{A}) &= \frac{1}{n!}\sum_\pi \mathrm{tr}\left[\varrho\mathcal{A}_{\pi(j)} \otimes \mathcal{A}_{\pi(k)}\right] \\ &= \frac{(n-1)!}{n!}\sum_{j'} \mathrm{tr}\left[\varrho\mathcal{A}_{j'} \otimes \mathcal{A}_{j'}\right] \qquad \text{if } j = k\,; \\ &= \frac{(n-2)!}{n!}\sum_{j' \neq k'} \mathrm{tr}\left[\varrho\mathcal{A}_{j'} \otimes \mathcal{A}_{k'}\right] \qquad \text{if } j \neq k\,, \end{aligned} \tag{3.1}$$

where $(n-2)!$ is the number of permutations resulting in fixed operators $\mathcal{A}_{j'}, \mathcal{A}_{k'}$ for outcomes j, k. Let us note that the values of p_{jk} do not depend on particular values of j, k, but only on their relative relation whether $j = k$, or $j \neq k$. Consequently, the probability to find the same/different outcomes in two shots reads

$$p_{\rm same} \;=\; np_{jj}=\sum_j {\rm tr}\left[\varrho\mathcal{A}_j\otimes\mathcal{A}_j\right]\,,$$
$$p_{\rm diff} \;=\; n(n-1)p_{jk}=\sum_{j\neq k}{\rm tr}\left[\varrho\mathcal{A}_j\otimes\mathcal{A}_k\right]\,.$$

We used the fact that for n-valued measurement used twice there are in total n pairs of same outcomes and $n(n-1)$ pairs of different outcomes. In this two-shot scenario the probabilities $p_{\rm same}, p_{\rm diff}$ depend on particular properties of effects $\mathcal{A}_1,\dots,\mathcal{A}_n$, hence they contain some information about $\mathcal{A}$.

IV. APRIORI INFORMATION

From now on, we assume that otherwise unknown measurement apparatuses are described by sharp non-degenerate observables. This assumption represents a very important part of our apriori information. As such, they are in direct correspondence with orthonormal bases and have the same number of outcomes as the dimension of the Hilbert space ($n=d$). Let us fix an orthonormal basis $|\psi_1\rangle,\dots,|\psi_d\rangle$ and denote by $\mathcal{A}_j^U$ the projections onto vectors $U|\psi_j\rangle$, where U is a unitary operator defined on $\mathcal{H}$. The projections $\mathcal{A}_1^U,\dots,\mathcal{A}_d^U$ form a non-degenerate sharp observable $\mathcal{A}^U$. Moreover, every non-degenerate sharp observable is of the form $\mathcal{A}^U$ for some unitary operator U. We assume that each $\mathcal{A}^U$ is equally likely, therefore we have to average our expectations over all observables $\mathcal{A}^U$ using the Haar measure on the unitary group $U(d)$.

If the outcomes are labeled, then due to our apriori information a particular sequence of outcomes $\vec{j}=(j_1,\dots,j_r)\in J_d\times\cdots\times J_d$ can be observed in r usages of the apparatus with the average probability

$$\overline{q}_{\vec{j}}=\int dU {\rm tr}\left[\varrho\mathcal{A}_{j_1}^U\otimes\cdots\otimes\mathcal{A}_{j_r}^U\right]\,. \tag{4.1}$$

Further, let us discuss how the considered apriori information affects the formulas for probabilities in the unlabeled case. For the purposes of later analysis it is sufficient to investigate only the experiments in which the apparatus is used at most twice. Thus, if the observable is unlabeled and $r=2$, then the average probability to observe the outcomes j,k reads

$$\overline{p}_{jk}=\int dU p_{jk}(\mathcal{A}^U)\,, \tag{4.2}$$

where $p_{jk}(\mathcal{A}^U)$ is specified in Eq.(3.1). Since

$$\int dU\mathcal{A}_j^U\otimes\mathcal{A}_k^U=\int dU\mathcal{A}_{j'}^U\otimes\mathcal{A}_{k'}^U \tag{4.3}$$

for all $j\neq k, j'\neq k'$ and $j=k, j'=k'$, respectively, it follows that

$$\sum_j\int dU\mathcal{A}_j^U\otimes\mathcal{A}_j^U \;=\; d\int d\psi\;\psi\otimes\psi\,,$$
$$\sum_{j\neq k}\int dU\mathcal{A}_j^U\otimes\mathcal{A}_k^U \;=\; d(d-1)\int d\psi d\psi_\perp\;\psi\otimes\psi_\perp\,,$$

where $d\psi_\perp$ denotes the integration over all vectors orthogonal to ψ. To simplify the expressions we replaced the integration over unitary group by integration over pure states ψ. In summary, we get

$$\overline{p}_{jj} \;=\; \int d\psi\;{\rm tr}\left[\varrho\psi\otimes\psi\right]\,, \tag{4.4}$$
$$\overline{p}_{jk} \;=\; \int d\psi d\psi_\perp\;{\rm tr}\left[\varrho\psi\otimes\psi_\perp\right]\,. \tag{4.5}$$

We see that $\overline{p}_{jk}$ and $\overline{p}_{jj}$ do not depend on particular values of indexes j,k, which are anyway chosen by us and cannot be distinguished. As before, we can discriminate only whether the outcomes are the same, or different, with probabilities given by formulas

$$p_{\rm same} \;=\; d\overline{p}_{jj}=d\int d\psi\;{\rm tr}\left[\varrho\psi\otimes\psi\right]\,,$$
$$p_{\rm diff} \;=\; d(d-1)\overline{p}_{jk}=d(d-1)\int d\psi d\psi_\perp\;{\rm tr}\left[\varrho\psi\otimes\psi_\perp\right]\,.$$

In comparison, for labeled observables in two shots we distinguish d^2 different outcomes with probabilities

$$\overline{q}_{jj}=\int d\psi\;{\rm tr}\left[\varrho\psi\otimes\psi\right]\,,\quad \overline{q}_{jk}=\int d\psi d\psi_\perp\;{\rm tr}\left[\varrho\psi\otimes\psi_\perp\right]\,.$$

V. COMPARISON OF LABELED OBSERVABLES

In the considered measurement comparison problem we are given a pair of measurement devices measuring some non-degenerate sharp observable $\mathcal{A}$ and $\mathcal{B}$. In this section we assume that the outcomes of these devices are labeled by numbers $1,\dots,d$. We start with the simplest experimental scenario in which each of the apparatuses is used only once. Our goal is to find a test state ϱ and divide the potential outcomes (j,k) into three families associated with three conclusions: i) observables are identical, ii) observables are different (not identical), iii) no conclusion (inconclusive result).

Using a pair of labeled measurements (each of them once) we distinguish d^2 different outcomes (j,k) appearing with probabilities q_{jk} that depend on the equivalence of $\mathcal{A}$ and $\mathcal{B}$

$$\overline{q}_{jk}(\mathcal{A}\neq\mathcal{B}) \;=\; \int dUdV{\rm tr}\left[\varrho\mathcal{A}_j^U\otimes\mathcal{B}_k^V\right]\,, \tag{5.1}$$
$$\overline{q}_{jk}(\mathcal{A}=\mathcal{B}) \;=\; \int dU{\rm tr}\left[\varrho\mathcal{A}_j^U\otimes\mathcal{A}_k^U\right]\,. \tag{5.2}$$

Our a prior information manifested in the integration $\int dU$ causes that probabilities $p_{jk}(\mathcal{A}\neq\mathcal{B})$ and $p_{jk}(\mathcal{A}=\mathcal{B})$ do not depend on particular values of j,k, but only on their mutual relation $j=k$, or $j\neq k$. That is, whatever test state is used, we can split the outcomes at most into two classes, hence only two out of three conclusions can be made.

In general, conclusion y based on the observation of an outcome x is unambiguous, if for all possible options except y the conditional probability $p_x(z\neq y)$ vanishes. Since in our case the outcomes (j,k) are divided into two subsets, $x\in\{\text{same},\text{diff}\}$, in order to conclude that the observables are different the condition $\overline{q}_x(\mathcal{A}=\mathcal{B})=0$ must hold for some outcome x. Similarly, if we can conclude that $\mathcal{A}=\mathcal{B}$, then there must exist an outcome x such that $\overline{q}_x(\mathcal{A}\neq\mathcal{B})=0$. We refer to such conditions as the *unambiguous no-error conditions*. Their validity is necessary in order to call formulation and solution of the problem unambiguous. Outcomes not associated with unambiguous conclusions lead to an inconclusive result. The smaller is the probability of the inconclusive outcome the better is the solution.

Let us note that

$$\int d\psi\,\psi^{\otimes k}=\frac{(d-1)!k!}{(d+k-1)!}P^+_{1\dots k}\,, \tag{5.3}$$

where $P^+_{1\dots k}$ is the projection onto the completely symmetric subspace of $\mathcal{H}^{\otimes k}$ and $d_k=\mathrm{tr}\left[P^+_{1\dots k}\right]=\frac{(d+k-1)!}{(d-1)!k!}$ the dimension of that subspace. For a fixed vector ψ

$$\begin{aligned}\int d\psi_\perp\,\psi_\perp^{\otimes k} &= \int_{\mathcal{H}_\psi^\perp} d\varphi\,\varphi^{\otimes k}\\ &= \frac{(d-2)!k!}{(d+k-2)!}(I-\psi)^{\otimes k}P^+_{1\dots k}\,,\end{aligned} \tag{5.4}$$

where we used $\mathcal{H}_\psi^\perp$ to denote the subspace of $\mathcal{H}$ orthogonal to $|\psi\rangle\in\mathcal{H}$.

We will use these identities in the evaluation of the probabilities p_{jj} and p_{jk}. In particular,

$$\overline{q}_{jj}(\mathcal{A}\neq\mathcal{B}) = \int d\psi d\varphi\,\mathrm{tr}\left[\varrho\psi\otimes\varphi\right]=\frac{1}{d^2}\mathrm{tr}\left[\varrho\right]\,, \tag{5.5}$$

$$\overline{q}_{jk}(\mathcal{A}\neq\mathcal{B}) = \int d\psi d\varphi\,\mathrm{tr}\left[\varrho\psi\otimes\varphi\right]=\frac{1}{d^2}\mathrm{tr}\left[\varrho\right]\,, \tag{5.6}$$

$$\overline{q}_{jj}(\mathcal{A}=\mathcal{B}) = \int d\psi\,\mathrm{tr}\left[\varrho\psi\otimes\psi\right]=\frac{1}{d_2}\mathrm{tr}\left[\varrho P^+_{12}\right]\,, \tag{5.7}$$

$$\begin{aligned}\overline{q}_{jk}(\mathcal{A}=\mathcal{B}) &= \int d\psi\,\mathrm{tr}\left[\varrho\psi\otimes\psi_\perp\right]\\ &= \frac{1}{d-1}\mathrm{tr}\left[\varrho(\frac{1}{d}I\otimes I-\frac{1}{d_2}P^+_{12})\right]\,.\end{aligned} \tag{5.8}$$

We see that if the measurement devices are different ($\mathcal{A}\neq\mathcal{B}$), then for all test states ϱ the probabilities $\overline{q}_{jj}(\mathcal{A}\neq\mathcal{B})$ and $\overline{q}_{jk}(\mathcal{A}\neq\mathcal{B})$ do not vanish for any outcome. Because of that the equality of the observables cannot be concluded unambiguously.

Denoting by $P^-_{12}=I\otimes I-P^+_{12}$ the projection onto the antisymmetric subspace of $\mathcal{H}\otimes\mathcal{H}$ we can rewrite

$$\frac{1}{d}I\otimes I-\frac{1}{d_2}P^+_{12}=\frac{1}{d}P^-_{12}+\frac{d-1}{d(d+1)}P^+_{12}\,.$$

Since this is positive full-rank operator it follows that also $\overline{q}_{jk}(\mathcal{A}=\mathcal{B})>0$ for all test states. Therefore, the occurence of different outcomes cannot be used to unambiguously conclude that the measurements are different. However, $\overline{q}_{jj}(\mathcal{A}=\mathcal{B})=0$ if $\Pi_\varrho\leq P^-_{12}$, hence using a test state supported on the antisymmetric subspace and observing the same outcomes implies that $\mathcal{A}\neq\mathcal{B}$ with certainty.

In summary, the identicality of unknown sharp non-degenerate observables cannot be unambiguously confirmed if each of the labeled apparatuses is used only once. Using an antisymmetric test state ϱ and observing the same outcomes on both apparatuses lead us to unambiguous conclusion that the apparatuses are different. For fixed observables $\mathcal{A}\neq\mathcal{B}$ the conditional probability of unambiguous conclusion reads

$$q_{\text{same}}(\mathcal{A},\mathcal{B})=\sum_j\mathrm{tr}\left[\varrho\mathcal{A}_j\otimes\mathcal{B}_j\right]\,. \tag{5.9}$$

On average

$$\overline{q}_{\text{same}}(\mathcal{A}\neq\mathcal{B})=d\overline{q}_{jj}(\mathcal{A}\neq\mathcal{B})=\frac{1}{d}\,.$$

This value gives the average conditional success probability for revealing the difference of the compared labeled non-degenerate observables. It is independent of the used test state, however, the unambiguous no-error conditions restricts the possible test states to so-called antisymmetric states, i.e. those supported only in the antisymmetric subspace spanned by P^-_{12}. Let us stress that if we choose a test state $\varrho=\frac{1}{d_-}P^-$, then $q_{\text{same}}(\mathcal{A},\mathcal{B})>0$ whenever $\mathcal{A}\neq\mathcal{B}$.

VI. COMPARISON OF UNLABELED MEASUREMENTS

In this section we assume that the outcomes of measurement devices are not labeled. As previously, our goal is to design an experiment from which we are able to unambiguously conclude whether these apparatuses are same or not. But same now means that the observables are equivalent in the unlabeled sense.

Consider a pair of known but unlabeled measurements $\mathcal{A}$ and $\mathcal{B}$. A single usage of each of the apparatuses leads us to outcome j on $\mathcal{A}$-apparatus and a on $\mathcal{B}$-apparatus with probability

$$p_{j,a}=\frac{1}{n^2}\sum_{j',a'}\mathrm{tr}\left[\varrho\mathcal{A}_{j'}\otimes\mathcal{B}_{a'}\right]=\frac{1}{n^2}\mathrm{tr}\left[\varrho\right]\,. \tag{6.1}$$

Since this probability is independent on whether $\mathcal{A}=\mathcal{B}$ or $\mathcal{A}\neq\mathcal{B}$ none of the outcomes can be used to make a conclusion. In fact $p_{j,a}$ is independent of particular observables at all. Hence, we need to use the unlabeled apparatuses more times. In particular, if each of them is used twice, then

$$\begin{aligned} p_{jk,ab} &= \tfrac{1}{n!n!}\textstyle\sum_{\pi,\pi'} \operatorname{tr}\left[\varrho\mathcal{A}_{\pi(j)}\otimes\mathcal{A}_{\pi(k)}\otimes\mathcal{B}_{\pi'(a)}\otimes\mathcal{B}_{\pi'(b)}\right] \\ &= \begin{cases} \frac{1}{d^2}\operatorname{tr}\left[\varrho\mathcal{A}_{\rm same}\otimes\mathcal{B}_{\rm same}\right] & \text{if } j=k, a=b \\ \frac{1}{d^2(d-1)}\operatorname{tr}\left[\varrho\mathcal{A}_{\rm same}\otimes\mathcal{B}_{\rm diff}\right] & \text{if } j=k, a\neq b \\ \frac{1}{d^2(d-1)}\operatorname{tr}\left[\varrho\mathcal{A}_{\rm diff}\otimes\mathcal{B}_{\rm same}\right] & \text{if } j\neq k, a=b \\ \frac{1}{d^2(d-1)^2}\operatorname{tr}\left[\varrho\mathcal{A}_{\rm diff}\otimes\mathcal{B}_{\rm diff}\right] & \text{if } j\neq k, a\neq b\,, \end{cases} \end{aligned}$$

where

$$\mathcal{A}_{\rm same} = \sum_j \mathcal{A}_j\otimes\mathcal{A}_j\,, \quad \mathcal{A}_{\rm diff} = \sum_{j\neq k}\mathcal{A}_j\otimes\mathcal{A}_k\,,$$

and similarly for $\mathcal{B}_{\rm same}$ and $\mathcal{B}_{\rm diff}$. We see that irrespectively whether $\mathcal{A}=\mathcal{B}$ or $\mathcal{A}\neq\mathcal{B}$ probability $p_{jk,ab}$ depends only on the mutual relation of the outcomes j,k and a,b of the two usages of the measurement $\mathcal{A}$ respectively $\mathcal{B}$. Hence, it is meaningful to distinguish at most four corresponding classes of outcomes.

For unknown $\mathcal{A}$ and $\mathcal{B}$ ($\mathcal{A}\neq\mathcal{B}$) restricted to be non-degenerate sharp observables the probability to find the same outcomes on apparatus $\mathcal{A}$ and the same outcomes on apparatus $\mathcal{B}$, respectively, can be expressed as $p_{\rm same,same} = \operatorname{tr}\left[\varrho\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same}\right]$ with

$$\begin{aligned} \mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same} &= d^2\frac{1}{d^2}\int dU dV \mathcal{A}^U_{\rm same}\otimes\mathcal{B}^V_{\rm same} \\ &= d^2\int d\psi d\varphi\, \psi\otimes\psi\otimes\varphi\otimes\varphi \\ &= d^2\overline{\mathcal{R}}_{\rm same}\otimes\overline{\mathcal{R}}_{\rm same}\,, \end{aligned} \tag{6.2}$$

where the factor d^2 stands for the number of same outcome pairs that can be observed on individual apparatuses and we used the definitions

$$\begin{aligned} \overline{\mathcal{R}}_{\rm same} &= \int d\psi\psi\otimes\psi = \frac{1}{d_2}P^+\,, \\ \overline{\mathcal{R}}_{\rm diff} &= \int d\psi d\psi_\perp\psi\otimes\psi_\perp = \frac{1}{d}I - \frac{1}{d_2}P^+\,. \end{aligned}$$

Similarly, for other outcomes we find that

$$\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,diff} = d^2(d-1)^2\overline{\mathcal{R}}_{\rm diff}\otimes\overline{\mathcal{R}}_{\rm diff} \tag{6.3}$$

$$\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,same} = d^2(d-1)\overline{\mathcal{R}}_{\rm diff}\otimes\overline{\mathcal{R}}_{\rm same} \tag{6.4}$$

$$\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm same,diff} = d^2(d-1)\overline{\mathcal{R}}_{\rm same}\otimes\overline{\mathcal{R}}_{\rm diff}\,, \tag{6.5}$$

providing $\mathcal{A}\neq\mathcal{B}$. Let us define operators

$$\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same} = P^+_{12}\otimes P^+_{34}\,, \tag{6.6}$$

$$\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm same,diff} = P^+_{12}\otimes I_{34}\,, \tag{6.7}$$

$$\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,same} = I_{12}\otimes P^+_{34}\,, \tag{6.8}$$

$$\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,diff} = I_{12}\otimes I_{34}\,, \tag{6.9}$$

that project onto the supports of operators $\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same}$, $\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm same,diff}$, $\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,same}$, $\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,diff}$, respectively.

If $\mathcal{A}=\mathcal{B}$, then

$$\begin{aligned} \mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm same,same} &= d^2\frac{1}{d^2}\int dU \mathcal{A}^U_{\rm same}\otimes\mathcal{A}^U_{\rm same} \\ &= d\int d\psi\psi\otimes\psi\otimes\psi\otimes\psi \\ &\quad + d(d-1)\int d\psi d\psi_\perp\,\psi\otimes\psi\otimes\psi_\perp\otimes\psi_\perp\,, \end{aligned}$$

where, in the second term the integration over $d\psi_\perp$ runs over all vectors orthogonal to a fixed ψ. In a general case the operators $\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{x,y} = \int dU \mathcal{A}^U_x\otimes\mathcal{A}^U_y$ read

$$\begin{aligned} \mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm same,diff} &= d(d-1)\int\psi\otimes\psi\otimes[\psi\otimes\psi_\perp+\psi_\perp\otimes\psi] \\ &\quad+\frac{d!}{(d-3)!}\int\psi\otimes\psi\otimes\psi'\otimes\psi'_\perp\,, \\ \mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm diff,same} &= d(d-1)\int\psi\otimes\psi_\perp\otimes[\psi\otimes\psi+\psi_\perp\otimes\psi_\perp] \\ &\quad+\frac{d!}{(d-3)!}\int\psi'\otimes\psi'_\perp\otimes\psi\otimes\psi\,, \\ \mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm diff,diff} &= d(d-1)\int\psi\otimes\psi_\perp\otimes[\psi\otimes\psi_\perp+\psi_\perp\otimes\psi] \\ &\quad+\frac{d!}{(d-3)!}\int\psi\otimes\psi_\perp\otimes[\psi\otimes\psi'+\psi'\otimes\psi] \\ &\quad+\frac{d!}{(d-3)!}\int\psi\otimes\psi_\perp\otimes[\psi_\perp\otimes\psi'+\psi'\otimes\psi_\perp] \\ &\quad+\frac{d!}{(d-4)!}\int\psi\otimes\psi_\perp\otimes\psi'\otimes\psi'_\perp \end{aligned} \tag{6.10}$$

where for simplicity we do not write explicitly the Haar measures $d\psi$, $d\psi'$, $d\psi_\perp$, $d\psi'_\perp$ and $\psi',\psi'_\perp$ are vectors orthogonal to ψ and $\psi_\perp$. Of course, $\langle\,\psi\,|\,\psi_\perp\,\rangle = \langle\,\psi'\,|\,\psi'_\perp\,\rangle = 0$. Since for qubits the Hilbert space is two dimensional the terms containing $\psi'\otimes\psi'_\perp$ do not appear in these expressions for qubits. There are no two orthogonal vectors to a fixed ψ in such case.

Let us note that the integration leading to $\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{x,y}$ includes the integration covered in $\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{x,y}$. Therefore,

$$\Pi^{\mathcal{A}=\mathcal{B}}_{x,y} \leq \Pi^{\mathcal{A}\neq\mathcal{B}}_{x,y}\,, \tag{6.11}$$

which implies that whenever $p_{x,y}(\mathcal{A}\neq\mathcal{B}) = \operatorname{tr}\left[\varrho\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{x,y}\right]0$, then also $p_{x,y}(\mathcal{A}=\mathcal{B}) = \operatorname{tr}\left[\varrho\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{x,y}\right]=0$, hence, in two shots we cannot unambiguously conclude that the apparatuses are the same. We can only approve the difference of measurement devices.

In what follows we are going to specify for which test states and for which outcomes $x,y\in\{\text{same},\text{diff}\}$ the no-error conditions $\operatorname{tr}\left[\varrho\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{x,y}\right]=0$ are satisfied and simultaneuously, whether the associated conditional success probability rates $p_{\rm success} = p_{x,y} = \operatorname{tr}\left[\varrho\mathcal{O}^{\mathcal{A}\neq\mathcal{B}}_{x,y}\right] > 0$ are nonvanishing. We shall show that for qubits ($d=2$)

$$\begin{aligned}\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same} &= \Pi^{\mathcal{A}=\mathcal{B}}_{\rm same,same} + Q_{\rm same,same}\,, && (6.12)\\ \Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm same,diff} &= \Pi^{\mathcal{A}=\mathcal{B}}_{\rm same,diff} + Q_{\rm same,diff}\,, && (6.13)\\ \Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,same} &= \Pi^{\mathcal{A}=\mathcal{B}}_{\rm diff,same} + Q_{\rm diff,same}\,, && (6.14)\\ \Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm diff,diff} &= \Pi^{\mathcal{A}=\mathcal{B}}_{\rm diff,diff} + Q_{\rm diff,diff}\,, && (6.15)\end{aligned}$$

where $Q_{\rm same,same}=O$, $Q_{\rm diff,diff}\neq Q_{\rm same,diff}=Q_{\rm diff,same}$ are projections forming the relevant parts of the supports of potential test states ϱ enabling us to conclude the difference. That is, we shall see that three out of four outcomes can be used to make the unambiguous conclusion.

A. $\mathcal{O}_{\rm same,same}$

Evaluating the operator $\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm same,same}$ we obtain

$$\begin{aligned}\frac{1}{d(d-1)}\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm same,same} &= \int\psi^{\otimes 4} + (d-1)\int\psi^{\otimes 2}\otimes\psi_\perp^{\otimes 2}\\ &= \frac{1}{d_4}P^+_{1234} + \frac{2(d-1)}{d(d-1)}R_{12-34}P^+_{34}\,, \qquad (6.16)\end{aligned}$$

where

$$\begin{aligned}R_{12-34} &= \int\psi^{\otimes 2}\otimes(I-\psi)^{\otimes 2}\\ &= \frac{1}{d_2}P^+_{12} + \frac{1}{d_4}P^+_{1234} - \frac{1}{d_3}(P^+_{123}+P^+_{124})\,.\end{aligned}$$

Due to positivity of operators in Eq. (6.16) the unambiguous no-error conditions require that

$$\mathrm{tr}\left[\varrho P^+_{1234}\right]=0\,,\quad \mathrm{tr}\left[\varrho R_{12-34}P^+_{34}\right]=0\,,$$

hold simultaneously. Hence, the support of $R_{12-34}P^+_{34}$ is of interest for us and in particular we should decide whether it is different from $\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same}=P^+_{12}\otimes P^+_{34}$. If yes, then we can use this outcome for making the unambiguous conclusion.

Let us analyze properties of R_{12-34} and its terms. First of all by definition $R_{12-34}P^+_{34}$ is a positive operator, hence necessarily $[R_{12-34},P^+_{34}]=0$ and also $[P^+_{123}+P^+_{124},P^+_{34}]=0$. The support of the projections P^+_{12}, P^+_{1234}, P^+_{123}, and P^+_{124} contains the completely symmetric subspace spanned by P^+_{1234}. As it is shown in Appendix it is their greatest joint subspace and since $\frac{1}{d_2}+\frac{1}{d_4}-\frac{2}{d_3}>0$ the operator R_{12-34} is indeed supported on the whole P^+_{1234}.

It remains to analyze the properties of $R_{12-34}P^+_{34}$ on the subspace $Q^+_{12}=P^+_{12}\otimes P^+_{34}-P^+_{1234}$. In particular, we are interested whether

$$\langle\varphi|\frac{1}{d_2}Q^+_{12} - \frac{1}{d_3}(Q_{123}+Q_{124})|\varphi\rangle > 0 \qquad (6.17)$$

for all $|\varphi\rangle$ from the support of Q^+_{12}, where $Q_{123}=P^+_{123}-P^+_{1234}$, $Q_{124}=P^+_{124}-P^+_{1234}$. For qubits these subspaces are described in details in Appendix A, where it is shown that the operator $Q_{123}+Q_{124}$ have two nonzero eigenvalues 4/3 and 2/3. However, the eigenvectors associated with 4/3 are from the subspace spanned by $P^+_{12}\otimes P^-_{34}$, which is irrelevant due to multiplication of R_{12-34} by P^+_{34}. The eigenvectors associated with the eigenvalue 2/3 are from $P^+_{12}\otimes P^+_{34}$, thus $\langle\varphi|Q_{123}+Q_{124}|\varphi\rangle\le 2/3$ for all $|\varphi\rangle\in P^+_{12}\otimes P^+_{34}\ge Q^+_{12}$. Since $d_2=3$, $d_3=4$

$$\langle\varphi|\frac{1}{3}Q^+_{12} - \frac{1}{4}(Q_{123}+Q_{124})|\varphi\rangle \ge \frac{1}{3}-\frac{1}{6} > 0\,. \qquad (6.18)$$

As a result we have shown that support of $R_{12-34}P^+_{34}$ equals to support of $P^+_{12}\otimes P^+_{34}$, thus $\Pi^{\mathcal{A}=\mathcal{B}}_{\rm same,same}=P^+_{12}\otimes P^+_{34}=\Pi^{\mathcal{A}\neq\mathcal{B}}_{\rm same,same}$. In summary, an observation of pairs of same outcomes on both apparatuses cannot be used to make any unambiguous conclusion, because $Q_{\rm same,same}=O$.

B. $\mathcal{O}_{\rm diff,diff}$

In this case our aim is to show that $Q_{\rm diff,diff}\neq O$. For qubits there are at most two mutually orthogonal vectors, hence

$$\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm diff,diff} = d(d-1)\int\psi\otimes\psi_\perp\otimes(\psi\otimes\psi_\perp+\psi_\perp\otimes\psi)\,.$$

Let us remind that for larger systems, this expression contains additional terms. Using the operators R_{13-24}, R_{14-23} introduced in a similar way as R_{12-34} defined in the previous section we obtain

$$\mathcal{O}^{\mathcal{A}=\mathcal{B}}_{\rm diff,diff} = 2(R_{13-24}P^+_{24}+R_{14-23}P^+_{23}). \qquad (6.19)$$

Using the same arguments as for R_{12-34} we find that $R_{13-24}P^+_{24}$ is supported on $P^+_{13}\otimes P^+_{24}$ and $R_{14-23}P^+_{23}$ is supported on $P^+_{14}\otimes P^+_{23}$. Therefore, for the test state ϱ we can write the following no-error condition

$$0=\mathrm{tr}\left[\varrho(P^+_{13}\otimes P^+_{24}+P^+_{14}\otimes P^+_{23})\right]\,. \qquad (6.20)$$

The completely symmetric subspace P^+_{1234} is the greatest joint subspace of $P^+_{13}\otimes P^+_{24}$ and $P^+_{14}\otimes P^+_{23}$. According to Appendix A 1 the support of $P^+_{13}\otimes P^+_{24}+P^+_{14}\otimes P^+_{23}$ is 13 dimensional, because $d_4=5$ and $Q^+_{13}=P^+_{13}\otimes P^+_{24}-P^+_{1234}$ and $Q^+_{14}=P^+_{14}\otimes P^+_{23}-P^+_{1234}$ are both four dimensional. Since the total Hilbert space $\mathcal{H}^{\otimes 4}$ for qubits is 16-dimensional, it follows that test states satisfying the no-error conditions live in a three-dimensional subspace. In Appendix A 1 it is shown that this subspace is a linear span of vectors

$$\begin{aligned}|\kappa_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle|\psi^+\rangle-|\psi^+\rangle|00\rangle)\,,\\ |\kappa_2\rangle &= \frac{1}{\sqrt{2}}(|0011\rangle-|1100\rangle)\,,\\ |\kappa_3\rangle &= \frac{1}{\sqrt{2}}(|11\rangle|\psi^+\rangle-|\psi^+\rangle|11\rangle)\,,\end{aligned}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Thus, $Q_{\rm diff,diff} = \sum_j |\kappa_j\rangle\langle\kappa_j| \leq Q_{12}^+ \leq P_{12}^+ \otimes P_{34}^+$ and arbitrary test state $\varrho \leq Q_{\rm diff,diff}$ satisfies the no-error condition.

Let us optimize the conditional probability

$$\overline{p}_{\rm diff,diff}(\mathcal{A} \neq \mathcal{B}) = {\rm tr}\left[\varrho \mathcal{O}_{\rm diff,diff}^{\mathcal{A}\neq\mathcal{B}}\right] \qquad (6.21)$$

where

$$\begin{aligned}\mathcal{O}_{\rm diff,diff}^{\mathcal{A}\neq\mathcal{B}} &= 4(\frac{1}{2}I - \frac{1}{3}P_{12}^+) \otimes (\frac{1}{2}I - \frac{1}{3}P_{34}^+) \\ &= I - \frac{2}{3}(P_{12}^+ + P_{34}^+) + \frac{4}{9}P_{12}^+ \otimes P_{34}^+ .\end{aligned}$$

Arbitrary pure state $|\varphi\rangle \in Q_{\rm diff,diff}$ is an eigenvector of projections P_{12}^+, P_{34}^+ and $P_{12}^+ \otimes P_{34}^+$. Therefore, the probability is independent of the test states $\varrho \leq Q_{\rm diff,diff}$ and reads

$$\overline{p}_{\rm diff,diff}(\mathcal{A} \neq \mathcal{B}) = 1 - \frac{4}{3} + \frac{4}{9} = \frac{1}{9} . \qquad (6.22)$$

C. $\mathcal{O}_{\rm same,diff}$

For qubits

$$\begin{aligned}\mathcal{O}_{\rm same,diff}^{\mathcal{A}=\mathcal{B}} &= d(d-1)\int \psi^{\otimes 2} \otimes (\psi \otimes \psi_\perp + \psi_\perp \otimes \psi) \\ &= d\left(\frac{1}{d_3}(P_{123}^+ + P_{124}^+) - \frac{2}{d_4}P_{1234}^+\right) ,\end{aligned}$$

and since $P_{1234}^+ \leq P_{123}^+, P_{124}^+$; $1/d_3 > 1/d_4$ we can conclude that no-error unambiguous condition reads

$$\mathrm{tr}\left[\varrho(P_{123}^+ + P_{124}^+)\right] = 0 . \qquad (6.23)$$

Let us remind that $\Pi_{\rm same,diff}^{\mathcal{A}\neq\mathcal{B}} = P_{12}^+$ and $P_{123}^+, P_{124}^+ \leq P_{12}^+$. The question is whether $\Pi_{\rm same,diff}^{\mathcal{A}=\mathcal{B}} = P_{12}^+$, or not. We know (see Appendix A) that P_{123}^+, P_{124}^+ are not orthogonal, however, their greatest joint subspace is the completely symmetric one. The dimension of P_{12}^+ is 12, whereas the total support of $P_{123}^+ + P_{124}^+$ is 11 dimensional. It follows that there exist a unique vector such that $\Pi_{\rm same,diff}^{\mathcal{A}\neq\mathcal{B}}|\varphi_Q\rangle = |\varphi_Q\rangle$, and, simultaneuously, $\Pi_{\rm same,diff}^{\mathcal{A}=\mathcal{B}}|\varphi_Q\rangle = 0$, thus, $Q_{\rm same,diff} = |\varphi_Q\rangle\langle\varphi_Q|$. For such test state the observation of this outcome leads to unambiguous confirmation of the difference of the measurement devices.

D. $\mathcal{O}_{\rm diff,same}$

There is no substantial difference in the analysis of this case and the previous one. We only need to exchange the role of pairs of indexes 12 and 34. Therefore, there exists a unique vector $|\varphi_Q'\rangle$ such that $\Pi_{\rm diff,same}^{\mathcal{A}=\mathcal{B}}|\varphi_Q'\rangle = 0$, but $\Pi_{\rm diff,same}^{\mathcal{A}\neq\mathcal{B}}|\varphi_Q'\rangle = P_{34}^+|\varphi_Q'\rangle = |\varphi_Q'\rangle$. Surprisingly, we shall see that $|\varphi_Q'\rangle \equiv |\varphi_Q\rangle$, which means that the same test state $|\varrho_Q\rangle$ guarantees the unambiguity of both outcomes $\mathcal{O}_{\rm same,diff}, \mathcal{O}_{\rm diff,same}$.

On the systems j and k we define a singlet vector as $|\psi_{jk}^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{jk} - |10\rangle_{jk})$. After a short calculation one can verify that the vector

$$|\varphi_Q\rangle = \frac{1}{\sqrt{3}}(|\psi_{13}^- \otimes \psi_{24}^-\rangle + |\psi_{14}^- \otimes \psi_{23}^-\rangle) \qquad (6.24)$$

satisfies all the required properties, i.e. it is symmetric with respect to $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ exchanges, i.e. $\Pi_{\rm same,diff}^{\mathcal{A}\neq\mathcal{B}}|\varphi_Q\rangle = \Pi_{\rm diff,same}^{\mathcal{A}\neq\mathcal{B}}|\varphi_Q\rangle = |\varphi_Q\rangle$, and $P_{123}^+|\varphi_Q\rangle = P_{124}^+|\varphi_Q\rangle = P_{134}^+|\varphi_Q\rangle = P_{234}^+|\varphi_Q\rangle = 0$, because both terms of $|\varphi_Q\rangle$ are antisymmetric exactly in one pair of all considered triples of indexes.

Using $|\varphi_Q\rangle$ as the test state we get

$$\begin{aligned}\overline{p}_{\rm same,diff}(\mathcal{A} \neq \mathcal{B}) &= \langle\varphi_Q|\mathcal{O}_{\rm same,diff}^{\mathcal{A}\neq\mathcal{B}}|\varphi_Q\rangle \\ &= \frac{4}{3}\langle\varphi_Q|\frac{1}{6}P_{12}^+ \otimes P_{34}^+ + \frac{1}{2}P_{12}^+ \otimes P_{34}^-|\varphi_Q\rangle .\end{aligned}$$

Similarly, we find

$$p_{\rm diff,same}(\mathcal{A} \neq \mathcal{B}) = \frac{4}{3}\langle\varphi_Q|\frac{1}{6}P_{12}^+ \otimes P_{34}^+ + \frac{1}{2}P_{12}^- \otimes P_{34}^+|\varphi_Q\rangle .$$

Since $P_{12}^+|\varphi_Q\rangle = P_{34}^+|\varphi_Q\rangle = |\varphi_Q\rangle$ implies $P_{12}^+ \otimes P_{34}^+|\varphi_Q\rangle = |\varphi_Q\rangle$ and

$$\langle\varphi_Q|P_{12}^- \otimes P_{34}^+|\varphi_Q\rangle = \langle\varphi_Q|P_{12}^+ \otimes P_{34}^-|\varphi_Q\rangle = 0 ,$$

we obtain

$$\overline{p}_{\rm same,diff}(\mathcal{A} \neq \mathcal{B}) + \overline{p}_{\rm diff,same}(\mathcal{A} \neq \mathcal{B}) = \frac{4}{9} . \qquad (6.25)$$

This gives a better success rate than we achieved for the outcome $\mathcal{O}_{\rm diff,diff}$. Unfortunately, $|\varphi_Q\rangle \notin Q_{\rm diff,diff}$. In conclusion, $\overline{p} = 4/9$ is the optimal value of the average success rate for unambiguous comparison of unlabeled qubit non-degenerate sharp observables in two shots.

Consider a pair of observables $\mathcal{A} = \{\psi, \psi_\perp\}$, $\mathcal{B} = \{\varphi, \varphi_\perp\}$ such that $\psi \neq \varphi$. Then the projections

$$\begin{aligned}\mathcal{O}_{\rm diff,same} &= (\psi \otimes \psi_\perp + \psi_\perp \otimes \psi) \otimes (\varphi \otimes \varphi + \varphi_\perp \otimes \varphi_\perp) , \\ \mathcal{O}_{\rm same,diff} &= (\psi \otimes \psi + \psi_\perp \otimes \psi_\perp) \otimes (\varphi \otimes \varphi_\perp + \varphi_\perp \otimes \varphi) .\end{aligned}$$

The success probability of revealing their difference using the test state $|\varphi_Q\rangle$ reads

$$p_{\rm success}(\psi, \varphi) = \langle\varphi_Q|\mathcal{O}_{\rm same,diff} + \mathcal{O}_{\rm diff,same}|\varphi_Q\rangle . \qquad (6.26)$$

Let us note that in a fixed orthonormal basis $|\psi\rangle, |\psi_\perp\rangle$ the test state $|\varphi_Q\rangle$ takes the form

$$|\varphi_Q\rangle = \frac{1}{\sqrt{3}}\left(|\psi^{\otimes 2} \otimes \psi_\perp^{\otimes 2}\rangle + |\psi_\perp^{\otimes 2} \otimes \psi^{\otimes 2}\rangle - |\psi^+ \otimes \psi^+\rangle\right) ,$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|\psi \otimes \psi_\perp\rangle + |\psi_\perp \otimes \psi\rangle)$. Using the identities $|\langle \psi | \varphi \rangle| = |\langle \psi_\perp | \varphi_\perp \rangle| = \cos\theta$, $|\langle \psi | \varphi_\perp \rangle| = |\langle \psi_\perp | \varphi \rangle| = \sin\theta$ a direct calculation gives

$$\begin{aligned}\langle\varphi_Q|\mathcal{O}_{\mathrm{same,diff}}|\varphi_Q\rangle &= \frac{1}{3}\langle\psi_\perp^{\otimes 2}|\varphi\otimes\varphi_\perp+\varphi_\perp\otimes\varphi|\psi_\perp^{\otimes 2}\rangle \\ &\quad +\frac{1}{3}\langle\psi^{\otimes 2}|\varphi\otimes\varphi_\perp+\varphi_\perp\otimes\varphi|\psi^{\otimes 2}\rangle \\ &= \frac{4}{3}|\langle \psi | \varphi \rangle|^2|\langle \psi_\perp | \varphi \rangle|^2 .\end{aligned}$$

Since $\langle\varphi_Q|\mathcal{O}_{\mathrm{same,diff}}|\varphi_Q\rangle = \langle\varphi_Q|\mathcal{O}_{\mathrm{diff,same}}|\varphi_Q\rangle$ the success probability reads

$$p_{\mathrm{success}}(\psi,\varphi) = \frac{2}{3}(\sin 2\theta)^2 . \tag{6.27}$$

It vanishes only if $\theta = 0$, or $\theta = \pi/2$, hence $\psi \equiv \varphi$, or $\psi \equiv \varphi_\perp$, respectively. As a result we get that the optimal test state detects unambiguously the difference for any pair of non-equivalent sharp qubit observables with strictly nonzero success probability. The actual probability depends on the angle between the observables. In fact, if sharp qubit POVMs are understood as ideal Stern-Gerlach apparatuses, then $\alpha = 2\theta$ is the angle between the measured spin directions. The probability achieves its maximum for orthogonal spin directions as one would expect.

VII. SUMMARY

We have investigated the problem of unambiguous comparison of quantum measurements. We restricted our analysis to subset of sharp non-degenerate observables that can be associated with non-degenerate selfadjoint operators. Let us note that without any restriction the comparison problem has only a trivial solution.

We distinguished two different types of measurement apparatuses depending whether the labels of their outcomes are apriori given, or not. We give solution to single shot comparison of labeled measurements in arbitrary dimension. For unlabeled measurements the single usage of each of the apparatuses is not sufficient. In the two shots scenario we find solution for unlabeled qubit measurement apparatuses. In both cases, the unambiguous confirmation of the equivalence of measurements is not possible. Similarly, as in the case of pure states and unitary channels, also for sharp non-degenerate observables only the difference can be unambiguously concluded.

In summary, for the measurement apparatuses with labeled outcomes the optimal procedure exploits the so-called antisymmetric test states. For any such test state ϱ the success is associated with the observation of the same outcomes. The difference of observables can be concluded with the average conditional probability

$$\overline{q}_{\mathrm{success}}(\mathcal{A} \neq \mathcal{B}) = 1/d . \tag{7.1}$$

In the case of unlabeled measurements individual outcomes can be associated with an unambiguous conclusion

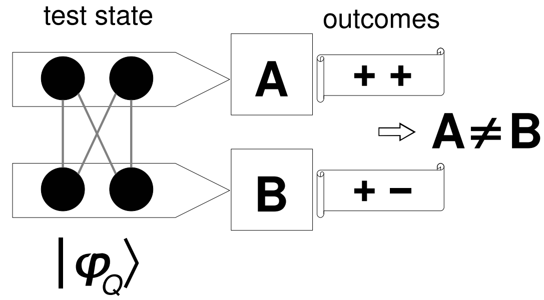

FIG. 1: Illustration of the optimal scheme for unambiguous comparison of qubit apparatuses leading to unambiguous conclusion $A \neq B$ with average conditional probability 4/9.

only if the support of the test state belongs to at least one of the subspaces spanned by projections $1-\Pi_{x,y}^{\mathcal{A}=\mathcal{B}}$, $x,y \in \{\mathrm{same},\mathrm{diff}\}$. We showed that only part of the test state acting on the support of the projections $Q_{\mathrm{same,same}} = O$, $Q_{\mathrm{diff,diff}}$ and $Q_{\mathrm{same,diff}} = Q_{\mathrm{diff,same}} = |\varphi_Q\rangle\langle\varphi_Q|$ may contribute to the success probability. Out of these possibilities, it turns out that the optimal test state is

$$|\varphi_Q\rangle = \frac{1}{\sqrt{3}}(|\psi_{13}^- \otimes \psi_{24}^-\rangle + |\psi_{14}^- \otimes \psi_{23}^-\rangle) , \tag{7.2}$$

for which the average conditional probability of the unambiguous conclusion equals

$$\overline{p}_{\mathrm{success}}(\mathcal{A} \neq \mathcal{B}) = 4/9 . \tag{7.3}$$

Using such test state and finding on one of the measurement different outcomes, whereas on the second the same outcomes, we can conclude with certainty that the apparatuses are different. This scheme is illustrated on Fig. 1.

Let us compare these success probabilities with the comparison problem for pure states and unitary channels. In particular, for single shot comparisons

$$\overline{p}_{\mathrm{state}} = (d-1)/2d , \tag{7.4}$$
$$\overline{p}_{\mathrm{unitary}} = (d+1)/2d . \tag{7.5}$$

We see that unlike for states and channels the success rate for comparison of labeled measurements vanishes as the dimension is increasing. Unfortunately, for unlabeled measurements on systems of larger dimensions the situation is more complex and two shots are not sufficient to make any unambiguous conclusion. The problem is still open and will be analyzed elsewhere.

APPENDIX A: SUBSPACES

In this appendix we shall analyze the subspaces of four quantum systems $\mathcal{H}^{\otimes 4}$, especially four qubits. Let us start with the simpler case of $\mathcal{H}\otimes\mathcal{H}$. Denote by $|j\rangle$ the basis of $\mathcal{H}$ and define

$$|\varphi_{jk}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|j\otimes k\rangle \pm |k\otimes j\rangle) . \tag{A1}$$

for $j < k$. For $j = k$

$$|\varphi_{jj}^{+}\rangle = |j \otimes j\rangle\,. \tag{A2}$$

These vectors form an orthonormal bases of symmetric and antisymmetric subspaces of $\mathcal{H}\otimes\mathcal{H}$, i.e. they define the projections $P_{12}^{\pm} = \sum_{j\leq k} |\varphi_{jk}^{\pm}\rangle\langle\varphi_{jk}^{\pm}|$.

We shall use the notation $P_{12}^{\pm} = P_{12}^{\pm}\otimes I_{34} = P_{12}^{\pm}\otimes(P_{34}^{+}+P_{34}^{-})$. Let us stress that $P_{1234}^{+} \leq P_{123}^{+} \leq P_{12}^{+}$. We shall be interested in properties of projections that are substracted from other projections to create the projections onto the completely symmetric subspace, for example, operators $Q_{12} = P_{12}^{+} - P_{1234}^{+}$ and $Q_{123} = P_{123}^{+} - P_{1234}^{+}$. Similar notations, definitions and relations hold also for other combination of indexes.

For qubits $\dim P_{12}^{+} = d^2\cdot d_2 = 12$, $\dim P_{12}^{+}\otimes P_{34}^{+} = d_2^2 = 9$, $\dim P_{123}^{+} = \dim P_{124}^{+} = d\cdot d_3 = 8$ and $\dim P_{1234}^{+} = d_4 = 5$, thus, $\dim Q_{123} = \dim Q_{124} = 3$ and $Q_{12} = 7.$, etc.

1. $P_{12}^{+}\otimes P_{34}^{+}$ and P_{1234}^{+}

Let us start with the analysis of the subspace of P_{12}^{+} not contained in P_{1234}^{+}, i.e. with Q_{12}. In the first step, let us split Q_{12} into $Q_{12} = Q_{12}^{-} + Q_{12}^{+}$, where $Q_{12}^{\pm} = P_{12}^{+}\otimes P_{34}^{\pm} - P_{1234}^{+}$, $Q_{12}^{-} = P_{12}^{+}\otimes P_{34}^{-}$. Due to asymmetry of $P_{12}^{+}\otimes P_{34}^{-}$ in $3\leftrightarrow 4$ exchange the projections P_{1234}^{+} **and** Q_{12}^{-} are orthogonal. For Q_{12}^{+} the situation is more tricky. Our goal is to design a basis of the support of Q_{12}^{+}. The completely symmetric subspace P_{1234}^{+} is spanned by the following orthonormal basis

$$\begin{aligned}
|\eta_0\rangle &= |\varphi_{00}^{+}\otimes\varphi_{00}^{+}\rangle\\
|\eta_1\rangle &= \frac{1}{\sqrt{2}}(|\varphi_{00}^{+}\otimes\varphi_{01}^{+}\rangle + |\varphi_{01}^{+}\otimes\varphi_{00}^{+}\rangle)\\
|\eta_2\rangle &= \sqrt{\frac{2}{3}}|\varphi_{01}^{+}\otimes\varphi_{01}^{+}\rangle + \sqrt{\frac{1}{6}}(|\varphi_{00}^{+}\otimes\varphi_{11}^{+}\rangle + |\varphi_{11}^{+}\otimes\varphi_{00}^{+}\rangle)\\
|\eta_3\rangle &= \frac{1}{\sqrt{2}}(|\varphi_{11}^{+}\otimes\varphi_{01}^{+}\rangle + |\varphi_{01}^{+}\otimes\varphi_{11}^{+}\rangle)\\
|\eta_4\rangle &= |\varphi_{11}^{+}\otimes\varphi_{11}^{+}\rangle\,.
\end{aligned}$$

Our aim is to specify a basis spanning the support of Q_{12}^{+}. Since $\dim P_{1234}^{+} = 5$ and $\dim P_{12}^{+}\otimes P_{34}^{+} = 9$ it follows we need to find four mutually orthogonal vectors in $P_{12}^{+}\otimes P_{34}^{+}$ that are also orthogonal to vectors $|\eta_j\rangle$. It is straightforward to verify that the following vectors

$$\begin{aligned}
|\kappa_1\rangle &= \frac{1}{\sqrt{2}}(|\varphi_{00}^{+}\otimes\varphi_{01}^{+}\rangle - |\varphi_{01}^{+}\otimes\varphi_{00}^{+}\rangle)\\
|\kappa_2\rangle &= \frac{1}{\sqrt{2}}(|\varphi_{00}^{+}\otimes\varphi_{11}^{+}\rangle - |\varphi_{11}^{+}\otimes\varphi_{00}^{+}\rangle)\\
|\kappa_2'\rangle &= \sqrt{\frac{1}{3}}(|\varphi_{01}^{+}\otimes\varphi_{01}^{+}\rangle - |\varphi_{00}^{+}\otimes\varphi_{11}^{+}\rangle - |\varphi_{11}^{+}\otimes\varphi_{00}^{+}\rangle)\\
|\kappa_3\rangle &= \frac{1}{\sqrt{2}}(|\varphi_{11}^{+}\otimes\varphi_{01}^{+}\rangle - |\varphi_{01}^{+}\otimes\varphi_{11}^{+}\rangle)
\end{aligned}$$

form such a basis.

Let us define a swap operator $S_{ab} = P_{ab}^{+} - P_{ab}^{-}$ implementing the exchange of the subsystems a, b. This operation is unitary and arbitrary permutation can be written as a composition of swap operations. The following identities hold

$$\begin{aligned}
P_{13}^{+}\otimes P_{24}^{+} &= S_{23}(P_{12}^{+}\otimes P_{34}^{+})S_{23}\,,\\
P_{14}^{+}\otimes P_{23}^{+} &= S_{34}(P_{13}^{+}\otimes P_{24}^{+})S_{34}\,,\\
P_{12}^{+}\otimes P_{34}^{+} &= S_{24}(P_{14}^{+}\otimes P_{23}^{+})S_{24}\,.
\end{aligned}$$

The vectors $|\kappa_1\rangle, |\kappa_2\rangle, |\kappa_3\rangle$ defined with respect to division $P_{12}^{+}\otimes P_{34}^{+}$ are orthogonal to all vectors $|\kappa_j\rangle, |\kappa_2'\rangle$ defined with respect to splittings $P_{13}^{+}\otimes P_{24}^{+}$ and $P_{14}^{+}\otimes P_{23}^{+}$, i.e. $P_{13}^{+}\otimes P_{24}^{+}|\kappa_j\rangle = P_{14}^{+}\otimes P_{23}^{+}|\kappa_j\rangle = 0$. However, $\langle\kappa_2'|P_{13}^{+}\otimes P_{24}^{+}|\kappa_2'\rangle = \langle\kappa_2'|P_{14}^{+}\otimes P_{23}^{+}|\kappa_2'\rangle = 1/4$, because the vectors $|\kappa_2'\rangle$ defined with respect to different splittings are mutually nonorthogonal. This means that the 4 dimensional projections $Q_{12}^{+}, Q_{13}^{+}, Q_{14}^{+}$ are not orthogonal, however, there is a three-dimensional subspace of Q_{12}^{+} (spanned by vectors $|\kappa_j\rangle$) orthogonal to both Q_{13}^{+} and Q_{14}^{+}.

2. $P_{123}^{+} + P_{124}^{+}$

For the purposes of this paper it is of interest to analyze the relation of the supports of projections P_{123}^{+} and P_{124}^{+}. The swap operator S_{34} can be written as a composition $S_{34} = S_{24}S_{23}S_{24}$. Consider a vector $|\varphi\rangle$ belonging to both subspaces, i.e. $P_{123}^{+}\varphi = P_{124}^{+}|\varphi\rangle = |\varphi\rangle$. For such vector $S_{12}|\varphi\rangle = S_{13}|\varphi\rangle = S_{14}|\varphi\rangle = S_{23}|\varphi\rangle = S_{24}|\varphi\rangle = |\varphi\rangle$ and therefore also $S_{34}|\varphi\rangle = S_{24}S_{23}S_{24}|\varphi\rangle = |\varphi\rangle$, hence the state φ is symmetric also with respect to exchange $3\leftrightarrow 4$. Consequently, it is invariant under the swap of arbitrary subsystems, i.e. it belongs to the completely symmetric subspace. Therefore, the greatest joint subspace of supports of P_{123}^{+} and P_{124}^{+} corresponds to the projection P_{1234}^{+}.

Further we shall prove that the projections $Q_{123} = P_{123}^{+} - P_{1234}^{+}$ and $Q_{124} = P_{124}^{+} - P_{1234}^{+}$ are not mutually orthogonal and we shall specify the support of $Q_{123} + Q_{124}$. It is relatively stragihtforward to verify that the following unnormalized vectors

$$\begin{aligned}
|\omega_1\rangle &= |\varphi_{00}^{+}\rangle_{12}|\varphi_{01}^{-}\rangle_{34} + |\varphi_{00}^{+}\rangle_{13}|\varphi_{01}^{-}\rangle_{24} + |\varphi_{00}^{+}\rangle_{23}|\varphi_{01}^{-}\rangle_{14}\,,\\
|\omega_2\rangle &= |\varphi_{00}^{+}\otimes\varphi_{11}^{+}\rangle - |\varphi_{11}^{+}\otimes\varphi_{00}^{+}\rangle + 2|\varphi_{01}^{+}\otimes\varphi_{01}^{-}\rangle\,,\\
|\omega_3\rangle &= |\varphi_{11}^{+}\rangle_{12}|\varphi_{01}^{-}\rangle_{34} + |\varphi_{11}^{+}\rangle_{13}|\varphi_{01}^{-}\rangle_{24} + |\varphi_{11}^{+}\rangle_{23}|\varphi_{01}^{-}\rangle_{14}\,,
\end{aligned}$$

form an orthogonal basis of the support of Q_{123}. These vectors are orthogonal to vectors $|\eta_j\rangle$ forming the completely symmetric subspace. In fact, they are completely symmetric only with respect to three indexes (123), but they not with respect to exchanges with the fourth qubit, hence, $P_{12}^{+}\otimes P_{34}^{+}|\omega_j\rangle$ is not proportional to $|\omega_j\rangle$. In the same way we can design a basis for each Q_{jkl}, in particular, for Q_{124}

$$\begin{aligned}|\omega_1'\rangle &= -|\varphi_{00}^+\rangle_{12}|\varphi_{01}^-\rangle_{34} + |\varphi_{00}^+\rangle_{14}|\varphi_{01}^-\rangle_{23} + |\varphi_{00}^+\rangle_{24}|\varphi_{01}^-\rangle_{13}\,,\\ |\omega_2'\rangle &= |\varphi_{00}^+ \otimes \varphi_{11}^+\rangle - |\varphi_{11}^+ \otimes \varphi_{00}^+\rangle - 2|\varphi_{01}^+ \otimes \varphi_{01}^-\rangle\,,\\ |\omega_3'\rangle &= -|\varphi_{11}^+\rangle_{12}|\varphi_{01}^-\rangle_{34} + |\varphi_{11}^+\rangle_{14}|\varphi_{01}^-\rangle_{23} + |\varphi_{11}^+\rangle_{24}|\varphi_{01}^-\rangle_{13}\,.\end{aligned}$$

Since $\langle\, \omega_j \,|\, \omega_k' \,\rangle = -2\delta_{jk}$ the pair of unnormalized vectors $|\omega_j\rangle, |\omega_j'\rangle$ forms a two-dimensional subspace orthogonal to remaining vectors. Equal superpositions $|\omega_j^+\rangle = |\omega_j\rangle + |\omega_j'\rangle$ are already symmetric in $3 \leftrightarrow 4$ exchange, hence $|\omega_j^+\rangle \in P_{12}^+ \otimes P_{34}^+$. On the other hand, the vectors $|\omega_j^-\rangle = |\omega_j\rangle - |\omega_j'\rangle$ are antisymmetric in $3 \leftrightarrow 4$, hence $|\omega_j^-\rangle \in P_{12}^+ \otimes P_{34}^-$. It is easy to verify that they are orthogonal, i.e. $\left\langle\, \omega_j^+ \,|\, \omega_j^- \,\right\rangle = 0$, because $\langle\, \omega_j \,|\, \omega_j \,\rangle = \left\langle\, \omega_j' \,|\, \omega_j' \,\right\rangle = 6$ and $\left\langle\, \omega_j \,|\, \omega_j' \,\right\rangle = \left\langle\, \omega_j' \,|\, \omega_j \,\right\rangle = -2$. Moreover, $\left\langle\, \omega_j^+ \,|\, \omega_j^+ \,\right\rangle = 8$ and $\left\langle\, \omega_j^- \,|\, \omega_j^- \,\right\rangle = 16$. Since $|\omega_j\rangle = \frac{1}{2}(|\omega_j^+\rangle + |\omega_j^-\rangle)$, $|\omega_j'\rangle = \frac{1}{2}(|\omega_j^+\rangle - |\omega_j^-\rangle)$ we have

$$\begin{aligned}Q_{123} + Q_{124} &= \frac{1}{6}\sum_j (|\omega_j\rangle\langle\omega_j| + |\omega_j'\rangle\langle\omega_j'|)\\ &= \sum_j \frac{1}{12}(|\omega_j^+\rangle\langle\omega_j^+| + |\omega_j^-\rangle\langle\omega_j^-|)\\ &= \sum_j \left(\frac{4}{3}\frac{1}{16}|\omega_j^-\rangle\langle\omega_j^-| + \frac{2}{3}\frac{1}{8}|\omega_j^+\rangle\langle\omega_j^+|\right)\,,\end{aligned}$$

where $\frac{1}{16}|\omega_j^-\rangle\langle\omega_j^-|$ and $\frac{1}{8}|\omega_j^+\rangle\langle\omega_j^+|$ are one-dimensional projections, hence, we get the spectral decomposition of $Q_{123} + Q_{124}$ with eigenvalues $2/3, 4/3$. For our purposes the relevant part is associated with vectors $|\omega_j^+\rangle$, because $|\omega_j^-\rangle$ are not from the support of $P_{12}^+ \otimes P_{34}^+$.

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