Repeatable quantum memory channels

Tomáš Rybár[1] and Mário Ziman[1,2]
[1]Research Center for Quantum Information, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia
[2] Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic

Within the framework of quantum memory channels we introduce the notion of repeatability of quantum channels. In particular, a quantum channel is called repeatable if there exist a memory device implementing the same channel on each individual input. We show that random unitary channels can be implemented in a repeatable fashion, whereas the nonunital channels cannot.

PACS numbers: 03.65.Ta,03.65.Yz,03.67.Hk

arXiv:0808.3851v1 [quant-ph] 28 Aug 2008

I. INTRODUCTION

A quantum channel is any transformation taking a state ϱ of a quantum system as an input and transforming it to some state ϱ' on the output. Within the standard model [1, 2] of quantum dynamics the channels are represented by completely positive trace-preserving linear maps acting on the set of (trace-class) Hilbert space operators $\mathcal{T}(\mathcal{H})$. Let us note that quantum states are represented by so-called density operators, i.e. positive trace-class operators with unit trace. The physical picture of quantum channels as the correct description of the evolutions of open quantum systems follows from the Stinespring theorem [3]. According to this theorem each quantum channel can be understood as the unitary evolution of the isolated "supersystem" composed of the system and its environment, where the unitary evolution is governed by the Schrödinger equation (see Fig.1). In other words quantum channels are implemented by suitable *quantum devices* consisting of intrinsic degrees of freedom (associated with the environment) and acting on the system via particular interactions between the system and the environment.

The statistical nature of quantum physics requires that the experiments must be typically repeated large number of times in order to make some relevant conclusions about the properties of quantum devices. A typical example is the problem of quantum channel tomography, in which the goal is to identify which channel the given device is implementing. Any estimation procedure is based on repeated use of the device. Considering the above model of the quantum device implementing some quantum channel we encounter the following problem. Although the concept of the quantum channel itself does not need any particular specification of the environment properties, for the repeated use of the same device the details about the environment can play a significant role. In particular, let us consider the following (not entirely realistic) example. Consider an optical device "storing" a single photon in some polarization state ξ. After inserting another photon (in the polarization state ϱ_1) this device will output the photon that was originally stored in it and the new photon will remain stored in the device. From the theory point of view the device implements the transformation mapping the whole state space onto the state ξ, i.e. $\mathcal{S}(\mathcal{H}) \mapsto \xi$, where by $\mathcal{S}(\mathcal{H})$ we denote the system's state space. However, using the same fiber once more we get the transformation $\mathcal{S}(\mathcal{H}) \mapsto \varrho_1$, i.e. the channel action is completely different (unless $\varrho_1 = \xi$). The main aim of this paper is to analyze and characterize the situations, in which the device can be reused infinitely many times and still implementing the same quantum channel. As we shall see such reusable devices would be good for saving resources. Instead of infinite amount of resources, needed to provide the channel transformation forever, finite resources would be sufficient.

The paper is organized as follows: In the Section II we shall recall the basics of quantum memory channels, after that in Section III we shall define the problems of reusability of quantum channels and prove the main theorems. In the last Section we shall discuss the derived results.

II. THE EFFECT OF MEMORY

In the case when the subsequent actions of the device are independent of the previous ones we say that the device implements a *memoryless channel*. If the output does not depend on future inputs we say that the channel is *causal*. Let us note that memoryless channels are automatically causal. In what follows we shall assume that all physically relevant channels are causal. Under such condition it was shown in the seminal paper by Kretschmann and Werner [4] that each causal memory channel can be understood as a sequence of collisions between the system and its environment playing the role of the memory. In the last few years different aspects of quantum memory channels attracted researchers [4, 5, 6, 7, 8, 9] and many interesting results have been achieved concerning capacity, structure and physical implementations for memory channels.

A personification of the Stinespring's theorem describing one usage of a device implementing a quantum channel is depicted in Fig.1. According to this picture the device is consisting of some internal degrees of freedom forming the effective environment affecting the system transferred through the channel. We shall refer to this internal degrees of freedom as to channel's memory associated with the Hilbert space $\mathcal{H}_{\rm M}$. The interaction be-

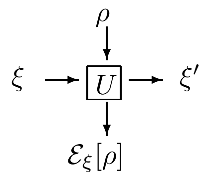

FIG. 1: One use of quantum channel

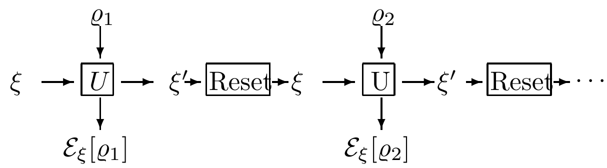

FIG. 2: Standard (memoryless) model.

tween the system and the memory is described by a unitary transformation $U : \mathcal{H}_{\rm M} \otimes \mathcal{H}_{\rm S} \to \mathcal{H}_{\rm M} \otimes \mathcal{H}_{\rm S}$, where $\mathcal{H}_{\rm S}$ is the system's Hilbert space. Assuming that the memory is initialized to state ξ the channel reads

$$\mathcal{E}_\xi[\varrho] = {\rm Tr}_{\rm M}[U(\xi \otimes \varrho)U^\dagger] , \tag{2.1}$$

where ${\rm Tr}_{\rm M}$ denotes the partial trace over the memory system.

We see that due to the interaction the state of the memory has changed. In particular

$$\xi' = {\rm Tr}_{\rm S}[U(\xi \otimes \varrho)U^\dagger] \equiv \mathcal{F}_\varrho[\xi] . \tag{2.2}$$

So how to reuse the same device again and implement the same quantum channel? Clearly, the only way is to apply some *reset* operation applied after each usage of the device and always initializing the memory into the fixed state ξ (see Fig.2). Let us note that this memory state might be unknown for us. The reset operation can be achieved by using different procedures that are known as *relaxation processes*. However, let us note that the action of the reset transformation is a bit cheating, because it is not unitary and therefore, it can be implemented only by employing some additional environment/memory system. Thus, resetting operation increases the total cost of the channel's implementation measured in the size of needed quantum resources. We shall see an example of the implementation of the reset operation at the end of Section III.

Nevertheless, within the framework of quantum (causal) memory channels [4] the role of the reset operation is to suppress the effect of memory on subsequent channel actions, i.e. to get device implementing a memoryless channel. In the problems dealing with channel estimation such reset procedures are implicitly assumed. In fact, for many systems the (approximate) relaxation is experimentally justified. Our goal here is not to analyze the effect of approximate relaxations, or verify the validity of this model in particular physical situations. We shall address more general question whether the perfect relaxation is possible, or needed, within the unitary model of quantum memory channel (see Fig.3). In other words we are asking under which conditions the model depicted on Fig.3 can embed the reset model depicted in Fig.2.

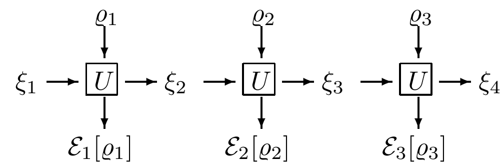

FIG. 3: Unitary memory model.

Let us now define the model and formulate the problem in an intuitive way. According to Fig.3 after the nth usage of the device the memory is described by the state

$$\begin{aligned} \xi_{n+1} &= {\rm Tr}_{{\rm S}^{\otimes n}}[U_n(\xi \otimes \varrho_1 \otimes \cdots \otimes \varrho_n)U_n^\dagger)] \\ &= \mathcal{F}_n \circ \cdots \circ \mathcal{F}_1[\xi_1] , \end{aligned} \tag{2.3}$$

where $U_n = U_{{\rm MS}_n} \cdots U_{{\rm MS}_1}$, $U_{{\rm MS}_j}$ acts on the memory and jth input system, $\mathcal{F}_j \equiv \mathcal{F}_{\varrho_j}$ and ${\rm S}^{\otimes n}$ denotes the composite system of n input systems. In this model we assume that the input states $\varrho_1, \ldots, \varrho_n$ are uncorrelated. On one hand this assumption is motivated by standard algorithms for channel testing. On the other hand if we allow correlations, then we are getting outside of the validity of the mathematical model for quantum channel as depicted in Fig.1. In fact, after the first usage of the device the subsequent inputs become to be correlated with the memory system. These issues extending the standard quantum channel framework are studied in Refs. [10, 11, 12, 13, 14].

The induced channel transformation $\mathcal{E}_j$ on the jth trial depends on the state of the memory ξ_{j-1} which is dependent on the particular choice of the input states $\varrho_1, \ldots, \varrho_{j-1}$. We shall investigate in which cases the particular choice of the input states does not matter. But before that, let us consider the following example.

A. Memory channel induced by SWAP interaction

Consider an experimentalist who would like to repeat his experiment aiming to describe the device schematically depicted on Fig.1. If he is able to set the initial conditions of the experiment to some initial values and repeat the experiment he is fine. This refers to the model on Fig.2. If not, then he is not repeating the same experiment (with the same channel) again. This has some severe consequences for channel tomography. As an illustrative example we will consider a quantum device with a two-dimensional memory implementing a single qubit channel. The interaction between the system (qubit) and the memory is described by the SWAP transformation

$U_{\rm SWAP}$ acting as follows

$$\varrho \otimes \xi \mapsto U_{\rm SWAP}(\varrho \otimes \xi) U_{\rm SWAP}^\dagger = \xi \otimes \varrho \,. \qquad (2.4)$$

In the memoryless settings this interaction induces completely contractive channels mapping the whole state space into the initial state of the memory ξ. However, if the reset operation is not applied the situation is completely different. Consequently, the result of the channel estimation will depend on the particular algorithm we choose. Let us note that SWAP transformation describes exactly the example mentioned in the introduction. In particular, the $(j-1)$th input is mapped into jth output, i.e. in jth run we observe the transformation $\varrho_j \mapsto \varrho_{j-1}$.

In order to estimate the qubit channel we can use six probe states: the eigenvectors of $\sigma_x, \sigma_y, \sigma_z$ operators. In the usual experiment we first insert N times state ψ_1, and after that N times the state ψ_2, etc. In such case we shall observe the transformations $\psi_1 \mapsto \frac{1}{N}\xi + \frac{N-1}{N}\psi_1 \approx \psi_1$ for large N and similarly for any other test state. Based on this we shall conclude that the channel is ideal, i.e. $\varrho \mapsto \varrho$. However, starting to use this ideal channel for communication we very quickly come into troubles. In the usual communication the states encoding the characters are used randomly. If we performed the channel tomography with randomly chosen test states we would find a completely different channel. In fact, each state would be mapped into the complete mixture, since the average test state is the total mixture. The differences between the ideal channel and completely noisy channel are obvious. As a result we see that the standard channel tomography loses its point and new methods must be developed for the estimation of memory channels, but this problem is beyond the scope of this paper.

III. REPEATABLE QUANTUM MEMORY CHANNELS

In order to avoid the problems mentioned in the previous Section we shall focus on existence of reusable quantum devices implementing in each trial the same channel. That is, our goal is to investigate the repeatability of the quantum memory channel induced by a fixed unitary transformation U as depicted in Fig.3. By *repeatable* quantum channels we understand linear trace-preserving completely positive maps for which there exists a unitary transformation U and some initial memory state ξ such that $\mathcal{E}_1 = \cdots = \mathcal{E}_n = \mathcal{E}$ for all $n > 0$. The key feature of repetable quantum memory channels is that the memory effects are suppressed. We call the triple $\langle \mathcal{E}, U, \xi \rangle$ a repeatable quantum memory channel of $\mathcal{E}$. We have two basic questions:

- Which channels are repeatable?
- Which channels are not repeatable?

Answering these two questions it is of interest to understand the consequences. What does it mean when a channel is not repeatable? Does it mean that it cannot be implemented at all? Let us remind that the considered model is in fact the most general one in which the concept of channel makes sense. We shall get back to this question later at the end of this section.

A partial answer to the first question is given in the following theorem.

Theorem 1. *If $\mathcal{E}$ is a random unitary channel, i.e. $\mathcal{E}[\varrho] = \sum_j p_j U_j \varrho U_j^\dagger$ and $0 \leq p_j \leq 1, \sum_j p_j = 1$, then it is repeatable.*

Proof. We shall show that here exists a repeatable Stinespring dilation for any random unitary channel. Consider a random unitary channel $\mathcal{E}$. Define a unitary transformation $U = \sum_j |j\rangle\langle j| \otimes U_j$ acting on $\mathcal{H}_{\rm M} \otimes \mathcal{H}_{\rm S}$, with $\{|j\rangle\}$ being an orthonormal basis on $\mathcal{H}_{\rm M}$ and U_j are unitary transformation from the decomposition of the channel $\mathcal{E}$. The unitary transformations of such form are also called *controlled-U transformations*. The memory system plays the role of the controling system and the system itself is the target system. Consider a general factorized input $\xi \otimes \varrho$ and calculate the states of the system and the memory after the unitary transformation U is applied. We obtain

$$\begin{aligned} \mathcal{E}[\rho] &= {\rm Tr}_{\rm M}[U(\xi\otimes\varrho)U^\dagger] \\ &= \sum_{j,k} {\rm Tr}_{\rm M}[(|j\rangle\langle j| \otimes U_i)(\xi\otimes\varrho)(|k\rangle\langle k| \otimes U_k^\dagger)] \\ &= \sum_{j,k} {\rm Tr}[(|j\rangle\langle j|\xi|k\rangle\langle k|)]\,(U_j \varrho U_k^\dagger) \\ &= \sum_j \xi_{jj}\, U_j \varrho U_j^\dagger . \end{aligned} \qquad (3.1)$$

for the system's transformation and

$$\begin{aligned} \mathcal{F}[\xi] &= {\rm Tr}_{\rm S}[U(\xi\otimes\varrho)U^\dagger] \\ &= \sum_{j,k} |j\rangle\langle j|\xi|k\rangle\langle k|\,{\rm Tr}[U_j\varrho U_k^\dagger] \\ &= \sum_{j,k} \xi_{jk}\,{\rm Tr}[U_j \varrho U_k^\dagger]\,|j\rangle\langle k| \end{aligned} \qquad (3.2)$$

for the memory transformation.

In order to implement random unitary channel $\mathcal{E}[\varrho] = \sum_j p_j U_j \varrho U_j^\dagger$ it is sufficient to choose a state with diagonal elements $\xi_{jj} = \langle j|\xi|j\rangle = p_j$. Let us note that diagonal elements of density operator always form a probability distribution. From Eq.(3.2) it follows that diagonal elements of the memory state are preserved, because

$$\langle j|\mathcal{F}[\xi]|j\rangle = \sum_{j,k} \xi_{jk}\,{\rm Tr}[U_j\varrho U_k^\dagger]\,\delta_{jk} = \xi_{jj}\,. \qquad (3.3)$$

The last equality holds because ${\rm Tr}[U_j \varrho U_j^\dagger] = {\rm Tr}[\varrho] = 1$.

Since the diagonal elements of ξ defines the random unitary channels and, moreover, they are preserved, it follows that random unitary channels are indeed repeatable. In particular, ${\rm diag}[\xi_1] = {\rm diag}[\xi_2] = \cdots = {\rm diag}[\xi_n]$ and therefore $\mathcal{E}_1 = \mathcal{E}_2 = \cdots = \mathcal{E}_n$ for all $n > 0$. □

As a result we get that a particular Stinespring's dilation of any random unitary transformation forms a reusable quantum device, meaning that random unitary transformations are repeatable. Could it be that all transformations have such dilation? The following theorem gives a negative answer saying that nonrepeatable channels do exist.

Theorem 2. *If $\mathcal{E}$ is a nonunital channel, i.e. $\mathcal{E}[I] \neq I$, then it is not repeatable with finite memory.*

Proof. We shall prove that repeatability of quantum memory channel (specified by U) implies unitality of the induced channels $\mathcal{E}$. Let us start with the entropy analysis of the memory channel. Becuase of the unitarity it follows that

$$S(\varrho_1) + S(\xi_1) = S(U(\varrho_1 \otimes \xi_1)U^\dagger)\,, \tag{3.4}$$

where $S(\varrho) = -\mathrm{Tr}[\varrho \log \varrho]$ is the von Neumann entropy of state ϱ. The entropy is subaditive, i.e. $S(\omega_{AB}) \leq S(\omega_A) + S(\omega_B)$, where $\omega_A = \mathrm{Tr}_B \omega_{AB}$ and $\omega_B = \mathrm{Tr}_A \omega_{AB}$ are the states of the subsytems A, B, respectively. Applying this inequality for our situation we obtain

$$S(\varrho_1) + S(\xi_1) \leq S(\mathcal{E}[\varrho_1]) + S(\xi_2)\,. \tag{3.5}$$

Let us repeat the quantum memory channel n times by using the same input state, i.e. $\varrho_1 = \varrho_2 = \cdots = \varrho_n = \varrho$. The repeatability of the channel $\mathcal{E}$ implies that

$$nS(\varrho_1) + S(\xi_1) \leq nS(\mathcal{E}[\varrho_1]) + S(\xi_{n+1})\,. \tag{3.6}$$

From this immediately follows the inequality

$$n\Delta(\varrho_1) \leq S(\xi_{n+1}) - S(\xi_1) \leq \log(\dim \mathcal{H}_{\mathrm{M}}), \tag{3.7}$$

where $\Delta(\varrho_1) = S(\rho_1) - S(\mathcal{E}[\rho_1])$.

For unital channels the entropy cannot decrease, i.e. $\Delta(\varrho_1) \leq 0$ for all states ϱ_1 (see Appendix). Consequently, the above inequality is satisfied by all unital channels. For nonunital channels the complete mixture decreases its entropy, i.e. $\Delta(\frac{1}{d}I) > 0$. The right hand side is bounded by the dimension of the memory system. However, since n is arbitrarily large, the left hand side goes to infinity, hence necessarily also the dimension of the memory system must be infinite. Thus repeatability requires unitality as it is stated in the theorem □

We say that a channel is n-repeatable if its action can be repeated n-times. For nonunital channels and finite memory there exist n ($n > \frac{\log \dim(\mathcal{H}_{\mathrm{M}})}{\Delta_{\max}}$) such that the channel cannot be n-repeatable. Let us now discuss the power of infinite memory systems. Consider a memory consisting of n systems in the same state and of the same dimension as is the system under consideration, i.e. $\xi_1 = \xi^{\otimes n}$ is the initial state of the memory. The action of the memory channel can be decomposed into a unitary interaction implementing the desired channel $\mathcal{E}$ (encoded in the state ξ) by acting only on one subsystem of the memory and the input, and an operation permuting the memory subsystems by one to the left, i.e. the active memory subsystem is shifted to the end. For such quantum memory channel the jth input is effectively interacting with the jth subsystem in the state ξ, hence each input is transformed by the same mapping $\mathcal{E}_\xi[\varrho_j] = \mathrm{Tr}_{\mathrm{M}}[U(\xi \otimes \varrho_j)U^\dagger]$. In this way arbitrary channel has n-repeatable implementation for all $n < \infty$. The limiting case $n \to \infty$ is physically senseless, because the Hilbert space of infinitely many subystems is not separable. On the other hand no one is probably interested in infinitely many repetitions of the same channel. Let us note that this type of implementation is essentially based on the complete replacement of the device by a new one with the same properties.

IV. CONCLUSION

We investigated the problem of reusability of quantum devices implementing (in each single use) state transformations described by quantum channels. Due to interaction of the system with the device both, the system and the device, are affected by some noise, hence the original settings of the device have changed. Consequently the repeated usage of the same quantum device can result in a different noise, i.e. different quantum channel. This picture leads to an emergence of memory effects in the description of quantum channels. If the channel can be repeated infinitely many times without resetting the memory we say it is repeatable. For such type of channels the memory effects are supressed although the memory itself undergoes a nontrivial dynamics. It was shown in this paper that any random unitary channel is repeatable with a finite memory, whereas the repeatable implementation of nonunital channels requires infinite resources. For qubit channels we can make even stronger statement that unitality is equivalent to repeatability, because each unital channel can be expressed as a random unitary channel [15]. For general systems we leave the question of repetable implementation of unital, but not random unitary channels open.

One possible way how to tackle the problem is to investigate the channels that can be implemented by a quantum device with the memory initialized in the total mixture. For such channels the reset operation can be implemented in a repeatable way, since the channel $\mathcal{A}$ transforming the whole state space into the total mixture is random unitary and therefore is repeatable. That is, whatever is the output memory state, it can be reset to the total mixture by using only finite resources. Interestingly, since the entropy of the total system is preserved, it follows that if the memory is initially in the total mixture, then the implemented channel is necessarily unital. In fact, if the system is initially in the total mixture, then necessarily also output must be in the total mixture, because the entropy achieves its maximum for a unique state being the total mixture. But, this is nothing else as the unitality of the channel. It is an open problem whether there are some unital but not random unitary channels that are implementable in the described repeatable way.

Let us note that the concept of repeatability is similar to the concept of quantum cloning [16] in a sense that the channels (just like copies in quantum cloning) are not completely independent if measurements are taken into account. In fact, the memory system may act as a mediator of correlations between the channel outputs although the inputs are factorized. For sure, the impact of measurements on repeatability of quantum memory channels deserves further investigation. The presented analysis of the repeatability of quantum channels is a part of the research program aiming to understand and develop realistic models of quantum dynamics of open systems including the memory effects.

Acknowledgments

This work was supported in part by the European Union projects QAP and the APVV project QIAM. TR acknowledges the support of the project APVV LPP-0264-07.

APPENDIX A: MONOTICITY OF VON NEUMANN ENTROPY UNDER UNITAL CHANNELS

Lemma 1. *If $\mathcal{E}$ is a unital channel, then $S(\mathcal{E}[\varrho]) \geq S(\varrho)$ for all states ϱ.*

Proof. The proof of entropy monoticity for unital channels is a consequence of the monoticity of the relative entropy [17]. In particular, for arbitrary quantum channel $\mathcal{E}$

$$S(\mathcal{E}[\varrho]||\mathcal{E}[\omega]) \leq S(\varrho||\omega)\,, \tag{A1}$$

where $S(\varrho||\omega) = \mathrm{Tr}[\varrho(\log\varrho - \log\omega)]$ is the quantum relative entropy. Setting $\omega = \frac{1}{d}I$ we get $S(\varrho||I/d) = -S(\varrho) + \log d$. Using this fact and assuming that $\mathcal{E}$ is unital the above inequality can be rewritten as

$$\begin{aligned} S(\mathcal{E}[\varrho]||I/d) &\leq S(\varrho||I/d) \\ -S(\mathcal{E}[\varrho]) &\leq -S(\varrho)\,, \end{aligned}$$

from which the lemma follows. □

[1] E. B. Davies, *Quantum theory of open systems*, (Academic Press, 1976)

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2000)

[3] W. F. Stinespring, *Positive Functions on C*-algebras*, Proceedings of the American Mathematical Society 6, 211-216 (1955)

[4] D. Kretschmann, and R. Werner, *Quantum channels with memory*, Phys. Rev. A (2005)

[5] C. Macchiavello, G. M. Palma and S. Virmani, *Entangled states maximize the two qubit channel capacity for some Pauli channels with memory*, Phys. Rev. A 69, 010303(R), (2004).

[6] V. Giovannetti, *A dynamical model for quantum memory channels*, J. Phys. A: Math. Gen. 38, 10989 (2005).

[7] G. Ruggeri, G. Soliani, V. Giovannetti and S. Mancini Information transmission through lossy bosonic memory channels Europhys. Lett. 70, 719 (2005)

[8] E. Karpov, D. Daems and N. J. Cerf, *Entanglement enhanced classical capacity of quantum communication channels with correlated noise in arbitrary dimensions*, Phys. Rev. A 74, 032320 (2006)

[9] M. B. Plenio and S. Virmani, *Spin chains and channels with memory*, e-print arXiv quant-ph/0702059

[10] P. Pechukas, *Reduced Dynamics Need Not Be Completely Positive*, Phys.Rev.Lett. 73, 1060 (1994)

[11] P. Štelmachovič, V. Bužek, *Dynamics of open systems initialy entangled with environment: Beyond the Kraus representation*, Phys.Rev.A 64, (2001)

[12] T. Jordan, A. Shaji, E. C. G. Sudarshan, *The dynamics of initially entangled open quantum systems*, Phys.Rev.A 70, 052110 (2004)

[13] H. Carteret, D. R. Terno, K. Zyckowski, *Physical accessibility of non-completely positive maps*, e-print arXiv quant-ph/0512167.

[14] M. Ziman, *Quantum process tomography: the role of initial correlations*, e-print arXiv quant-ph/0603166.

[15] M. B. Ruskai, S. Szarek, and E. Werner, *A characterizarion of completely positive tracepreserving maps on $\mathcal{M}_2$*, Lin. Alg. Appl. 347, 159 (2002)

[16] V. Scarani, S. Iblisdir, N. Gisin, and A. Acin, *Quantum cloning*, Rev.Mod.Phys. 77, 1225 (2005)

[17] D. Petz, *Monotonicity of quantum relative entropy revisited*, Rev. Math. Phys. 15, 79-91 (2003).