COEXISTENCE OF QUBIT EFFECTS

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Abstract. We characterize all coexistent pairs of qubit effects. The characterization consists of three disjoint conditions which are easy to check for a given pair of effects. Known special cases are shown to follow from our general characterization theorem.

1. Introduction

Optimal solutions for quantum information processing tasks require typically observables (POVMs) which do not have projections as their elements. Generally, an element of an observable, called an effect, can be any operator bounded by the zero and the identity operators. For instance, an optimal observable for unambiguous discrimination of two non-orthogonal pure states has three elements and none of them is a projection [4]. Another example is provided by informationally complete observables, which do not have any non-trivial projections as their elements [2].

A measurement of an observable gives information of all effects which are in its range. For instance, a symmetric informationally complete qubit observable has four outcomes. We can sum two effects corresponding to two different outcomes, which simply means collecting the measurement results of these two outcomes together. Different pairings produce information of expectation values of all σ_x, σ_y and σ_z, and this therefore proves the informational completeness of the observable.

This arises the following question on the structure of observables: when two effects A and B can be contained in a single observable? Two effects are called coexistent if they are in this kind of relation. For two projections coexistence is simply equivalent to their commutativity. For generic effects, such a simple characterization is not known.

In this work we give a complete characterization of the previously stated coexistence problem in the case of two qubit effects. In Section 2 we recall the coexistent problem in a precise formulation. In Section 3 we present the main result of this paper - a characterization theorem of coexistent pairs of qubit effects. We also show that already known special cases are easily recovered from the theorem. A detailed proof of the characterization theorem is given in appendices. In Appendix

1 we recall some general facts on coexistence which are needed in our investigation. Appendix 2 then concentrates on the details of the proof.

2. The coexistence problem

Let $\mathcal{H}$ be a complex separable Hilbert space. An operator A on $\mathcal{H}$ is an *effect* if

$$0 \leq \langle \psi \,|\, A\psi \rangle \leq 1$$

for all $\psi \in \mathcal{H}$. In terms of operator inequalities this reads

$$O \leq A \leq I \,,$$

where O and I are the zero operator and the identity operator, respectively. We denote by $\mathcal{E}(\mathcal{H})$ the set of effects.

An *observable* G is a normalized effect valued measure, also called positive operator valued measure (POVM). It is defined on a measurable space $(\Omega, \mathcal{F})$, where Ω is the set of measurement outcomes and $\mathcal{F} \subseteq 2^{\Omega}$ is the σ-algebra of possible events. If the system is in a vector state ψ and a measurement of G is performed, the probability of getting a measurement outcome belonging to an event X is $\langle \psi \,|\, \mathsf{G}(X)\psi \rangle$.

For a singleton set $\{\omega\}$, we denote $\mathsf{G}_\omega \equiv \mathsf{G}(\{\omega\})$. Clearly, if Ω is a discrete set, then G is determined by the effects G_ω, $\omega \in \Omega$. A generic effect $\mathsf{G}(X)$ is recovered by formula

$$\mathsf{G}(X) = \sum_{\omega \in X} \mathsf{G}_\omega \,.$$

We can also look on the structure of observables from the other side: given two effects, we can ask whether they originate from a single observable. This concept, called coexistence, was first studied by Ludwig [7].

Definition 1. Effects $A, B \in \mathcal{E}(\mathcal{H})$ are *coexistent*, denoted $A \,\infty\, B$, if there exists an observable $\mathsf{G} : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ and events $X, Y \in \mathcal{F}$ such that

$$A = \mathsf{G}(X), \quad B = \mathsf{G}(Y) \,. \tag{1}$$

An observable G giving A and B as in (1) is called a *joint observable* for A and B. The coexistence problem can be now formulated in the following way.

Coexistence problem:

> Given an effect A, characterize all effects B which are coexistent with it.

The following simple observation shows that when we are studying the coexistence of two effects, we can resctrict to four outcome joint observables.

Proposition 1. *Effects A and B are coexistent if and only if there exists an observable G with four outcomes $\{1,2,3,4\}$ such that*

$$A = \mathsf{G}_1 + \mathsf{G}_2, \quad B = \mathsf{G}_1 + \mathsf{G}_3 . \tag{2}$$

Proof. Trivially, if a four outcome observable G satisfying (2) exists, then A and B are coexistent. Assume then that A and B are coexistent and let $\mathsf{G} : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ be an observable such that $A = \mathsf{G}(X)$, $B = \mathsf{G}(Y)$ for some $X, Y \in \mathcal{F}$. We denote $X' = \Omega \setminus X$ and $Y' = \Omega \setminus Y$, and we set $\widetilde{G}_1 = \mathsf{G}(X \cap Y)$, $\widetilde{G}_2 = \mathsf{G}(X \cap Y')$, $\widetilde{G}_3 = \mathsf{G}(X' \cap Y)$, $\widetilde{G}_4 = \mathsf{G}(X' \cap Y')$. This defines an observable $\widetilde{G}$ with the required properties. □

For any nontrivial effect A (i.e. A is not of the form λI for some $0 \leq \lambda \leq 1$), there are effects which are not coexistent with it [8, Lemma 1]. If A is a projection (i.e. $A = A^2$), then the answer to the coexistence problem is simple and well-known: an effect B is coexistent with A exactly when $AB = BA$. Generally, however, a solution to the coexistence problem is not known. In the next section we present a full solution to the coexistence problem in the case of a qubit system, i.e. two dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$.

3. Qubit effects and their coexistence

Qubit effects A and B can be parametrized by vectors $(\alpha, \mathbf{a}), (\beta, \mathbf{b}) \in \mathbb{R}^4$ in the following way:

$$A = \frac{1}{2}(\alpha I + \mathbf{a} \cdot \boldsymbol{\sigma}), \qquad 0 \leq a \leq \alpha \leq 2 - a , \tag{3a}$$

$$B = \frac{1}{2}(\beta I + \mathbf{b} \cdot \boldsymbol{\sigma}), \qquad 0 \leq b \leq \beta \leq 2 - b . \tag{3b}$$

Here $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices, and we have denoted $a \equiv \|\mathbf{a}\|$, $b \equiv \|\mathbf{b}\|$.

We are now considering A to be fixed and we are looking for all effects B (hence all parameters β and $\mathbf{b}$), which are coexistent with A. In order to formulate the characterization theorem, we first introduce the following function $\mathfrak{S}$ from $\mathcal{E}(\mathcal{H})$ to $[0,1]$,

$$\mathfrak{S}(A) \equiv \mathfrak{S}(\alpha, a) := \frac{1}{2}\left(a^2 + \alpha(2-\alpha) - \sqrt{(\alpha^2 - a^2)[(2-\alpha)^2 - a^2]}\right) . \tag{4}$$

The following properties of $\mathfrak{S}$ are easy to confirm:

(a) $\mathfrak{S}$ is continuous;

(b) $\mathfrak{S}(I - A) = \mathfrak{S}(A)$;
(c) $\mathfrak{S}(UAU^*) = \mathfrak{S}(A)$ for every unitary operator U;
(d) $\mathfrak{S}(A) = 1$ iff A is a non-trivial projection (i.e. $A^2 = A$ and $O \neq A \neq I$);
(e) $\mathfrak{S}(A) = 0$ iff A is a trivial effect (i.e. $A = \lambda I$ for some $0 \leq \lambda \leq 1$).

Due to these properties, we interpret the number $\mathfrak{S}(A)$ as a quantification of the *sharpness* of A. The following facts follow directly from the definition of $\mathfrak{S}$,

(f) for a fixed α, the function $a \mapsto \mathfrak{S}(\alpha, a)$ is increasing;
(g) $\mathfrak{S}(1, a) = a^2$;
(h) $\mathfrak{S}(\alpha, \alpha) = \alpha$.

For simplicity, we formulate the main theorem below in the case of $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. We note that if A is defined by parameters α and $\mathbf{a}$, then $I - A$ corresponds to $2 - \alpha$ and $-\mathbf{a}$. Therefore, the other cases when $\alpha > 1$ or $\beta > 1$ can be recovered from this result by applying Proposition 2 in Appendix 1.

It follows from Proposition 3 in Appendix 1 that only the relative angle between $\mathbf{a}$ and $\mathbf{b}$ is relevant for their coexistence — not their absolute directions. In the following it is thus convenient to denote by $b_\parallel$ the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ and $b_\perp$ the length of the projection of $\mathbf{b}$ in the plane perpendicular to $\mathbf{a}$. The coexistence of B with A then depends on parameters $b_\parallel$, $b_\perp$ and β.

Theorem 1. *An effect B is coexistent with A if and only if it falls into one of the following (disjoint) cases:*

(C1) *if $\beta \leq 1 - \mathfrak{S}(A)$, $A \,\text{ⵙ}\, B$ irrespectively of $\mathbf{b}$;*
(C2) *if $\beta > 1 - \mathfrak{S}(A)$ and $|b_\parallel - b_0| \geq l$, then $A \,\text{ⵙ}\, B$;*
(C3) *if $\beta > 1 - \mathfrak{S}(A)$ and $|b_\parallel - b_0| < l$, then $A \,\text{ⵙ}\, B$ if and only if*

$$\begin{aligned} b_\perp \;\leq\; & \frac{1}{2a}\sqrt{[(2-\alpha)^2 - a^2]\{a^2 - [a(b_\parallel - b_0) + (1-\beta)]^2\}} \\ & + \frac{1}{2a}\sqrt{[\alpha^2 - a^2]\{a^2 - [a(b_\parallel - b_0) - (1-\beta)]^2\}}\,, \end{aligned} \tag{5}$$

where

$$b_0 = \frac{1}{a}(1-\alpha)(1-\beta)\,, \tag{6}$$

$$l = \frac{1}{a}\sqrt{(1-\alpha)^2 - \beta[(1-\alpha)^2 + 1 - a^2] + \beta^2}\,. \tag{7}$$

The case (C3) is the only one where (for fixed β) the coexistence of A and B imposes a nontrivial restriction on the length of vector $\mathbf{b}$. We also note that the shortest vector $\mathbf{b}$ for which (5) is equality is the one with $b_\parallel = b_0$.[1] Since for fixed β the shorter $\mathbf{b}$, the smaller the sharpness $\mathfrak{S}(B)$, we interpret the directions along these shortest vectors[2] to represent operators which are mostly restricted if we require coexistence with A.

With this interpretation the division of coexistence condition to three different situation (C1)–(C3) can be intuitively understood in the following way. The first class (C1) consists of those effects B for which β [and consequetively the sharpness $\mathfrak{S}(B)$] is so small that no choice of vector $\mathbf{b}$ can prevent the coexistence of A and B. If β is above the given treshold, then there exists an interval for $b_\parallel$, where there is a nontrivial restriction on the length of vectors $\mathbf{b}$ if A and B coexist. The center of the interval is b_0, which represent the most strict restriction, and the width of the interval is $2l$. The second class (C2) then consists of those effects B for which $b_\parallel$ is outside the interval and which are coexistent with A even if their sharpness is the highest possible (i.e. $\mathfrak{S}(B) = \beta$). The third class (C3) represents effects for which their sharpness is nontrivially restricted if they are to coexist with A.

In Fig. 1 we present four illustrative examples. Fig. 1a demonstrates the case (C1) where $\beta < 1 - \mathfrak{S}(A)$, and hence all effects with this β coexist with A. In Fig. 1b we keep the parameters α and a unchanged while β is enlarged such that $\beta > 1-\mathfrak{S}(A)$. The interval with nontrivial restriction on the length of vectors $\mathbf{b}$ appears – for $b_\parallel$ outside this interval (C2) applies, while for $b_\parallel$ inside (C3) applies. Note that the center of the interval is not zero. In Fig. 1c we have $\beta = 1$ and now the interval is centered at zero, meaning the restriction on the sharpnes of B is most strict if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. Furthermore, $l = 1$ and thus (C3) covers all the possible cases. In Fig. 1d we have $a = \alpha$ which means that A is a multiple of a projection. Nonzero b_0 results in a clearly visible asymmetry in the picture.

In the following examples we demonstrate that the known special cases follow easily from Theorem 1.

Example 1. Assume that $\alpha = \beta = 1$. Using property (e) of $\mathfrak{S}(A)$, we see that the coexistence condition (C1) holds if and only if $\mathbf{a} = \mathbf{0}$. We have $b_0 = 0$, $l = 1$ and therefore (C2) never occurs. The condition (C3) now gives

$$b_\perp^2 \leq (1-a^2)(1-b_\parallel^2)\,. \tag{8}$$

[1] The proof is analogous to the proof of (19) which we give later.

[2] These shortest vectors make angle θ with $\mathbf{a}$ so they lie on a cone. The angle θ is given by $\tan\theta = b_\perp / b_\parallel$.

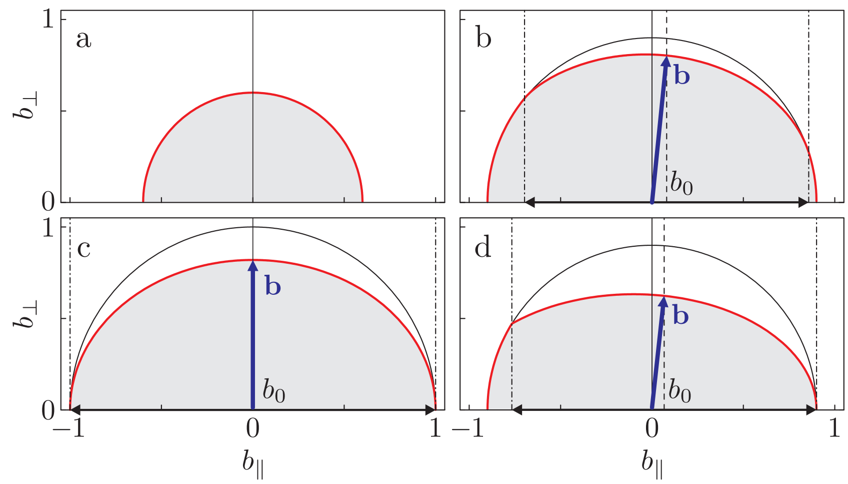

FIGURE 1. Examples of specification of effects B which coexist with given effect A. On each figure parameters α, a, β are fixed, while the **b** vector components, $b_\parallel$, and $b_\perp$ are on x and y axis, respectively. The thick red line denotes the boundary – A and B coexist if and only if the vector **b** is inside the shaded region. The thin black circle represents the condition on B to be an effect, $b \leq \beta$. The blue vector represents a shortest vector **b** laying on the boundary – its projection to x axis equals b_0. The interval $[b_0 - l, b_0 + l]$ where a nontrivial restriction on the length of the allowed vectors **b** exist is denoted by the vertical dot-dashed lines. The following parameters are used: $\alpha = 0.6$ in every figure and a) $a = 0.5$, $\beta = 0.6$. b) $a = 0.5$, $\beta = 0.9$. c) $a = 0.5$, $\beta = 1$. d) $a = 0.6$, $\beta = 0.9$.

Clearly, this inequality also covers the case $\mathbf{a} = \mathbf{0}$. This result has been derived by Busch [1, Theorem 4.5] in an equivalent form

$$\|\mathbf{a} + \mathbf{b}\| + \|\mathbf{a} - \mathbf{b}\| \leq 2.$$

Example 2. Assume that $\beta = 1$ and $\mathbf{a} \perp \mathbf{b}$. The coexistence condition (C1) holds if and only if $\mathbf{a} = \mathbf{0}$. Since $\mathbf{a} \perp \mathbf{b}$, we have $b_\parallel = 0$ and $b_\perp = b$, while $b_0 = 0$ due to $\beta = 1$. Therefore (C2) does not occur and $\mathbf{a} \neq \mathbf{0}$ leads to the case (C3) which now reads

$$b \leq \frac{1}{2}\sqrt{(2-\alpha)^2 - a^2} + \frac{1}{2}\sqrt{\alpha^2 - a^2}\,. \tag{9}$$

Clearly, this inequality also covers the case $\mathbf{a} = \mathbf{0}$. This result has been derived by Liu et al. [6, Theorem 1].

Example 3. Assume that $\|\mathbf{a}\| = \alpha$ and $\|\mathbf{b}\| = \beta$. We now have $b_0 = (1-\alpha)(1-\beta)/\alpha$ and $l^2 = (\alpha+\beta^2-1)/\alpha^2$. Since $b = \beta$, A and B coexist if (C1) or (C2) are satisfied. (C1) holds if and only if $\alpha+\beta \leq 1$, while in the case $\alpha+\beta > 1$ condition (C2) gives

$$\alpha b_{\|} \leq 2 - 2\alpha - 2\beta + \alpha\beta \,. \tag{10}$$

However, $\alpha+\beta \leq 1$ implies (10) and therefore (10) characterizes all solutions. This inequality is also easily obtained from a result of Molnar [8, Lemma 2].

In conclusions, we have studied the coexistence of two qubit effects, that is, when the effects originate from a single observable. We have solved the general case by providing simple criteria for the coexistence. We have shown that known special cases follow straightforwardly from our result.

Appendix 1: General observations on coexistence

In this section we list some general observations which are needed in the proof of Theorem 1.

Proposition 2. *Let $A, B \in \mathcal{E}(\mathcal{H})$. The following conditions are equivalent:*

(i) *A and B are coexistent;*
(ii) *A and $I-B$ are coexistent;*
(iii) *$I-A$ and B are coexistent;*
(iv) *$I-A$ and $I-B$ are coexistent.*

Proof. It is enough to prove that (i) implies (ii). The other implications follow by applying this to different combinations of A and $I-A$ with B and $I-B$.

Assume that A and B are coexistent and that G is a four outcome observable satisfying (2). We define another four outcome observable $\widetilde{\mathsf{G}}$ by

$$\widetilde{\mathsf{G}}_1 := \mathsf{G}_2, \quad \widetilde{\mathsf{G}}_2 := \mathsf{G}_1, \quad \widetilde{\mathsf{G}}_3 := \mathsf{G}_4, \quad \widetilde{\mathsf{G}}_4 := \mathsf{G}_3 \,.$$

Then

$$\widetilde{\mathsf{G}}_1 + \widetilde{\mathsf{G}}_2 = \mathsf{G}_2 + \mathsf{G}_1 = A$$

and

$$\widetilde{\mathsf{G}}_1 + \widetilde{\mathsf{G}}_3 = \mathsf{G}_2 + \mathsf{G}_4 = \mathsf{G}_2 + I - \mathsf{G}_1 - \mathsf{G}_2 - \mathsf{G}_3 = I - B \,.$$

Thus, A and $I-B$ are coexistent. □

Proposition 3. *Let $A, B \in \mathcal{E}(\mathcal{H})$ and U a unitary operator on $\mathcal{H}$. The following conditions are equivalent:*

(i) *A and B are coexistent;*
(ii) *UAU^* and UBU^* are coexistent.*

Proof. It is enough to prove that (i) implies (ii) as the other implication is obtained by applying this to a unitary operator U^*. Assume that A and B are coexistent and that G is their joint observable. Then $U\mathsf{G}(\cdot)U^*$ is a joint observable of UAU^* and UBU^*. □

Proposition 4. *Let A, B_1, B_2 be effects such that A is coexistent with both B_1 and B_2. Then for any $0 \leq \lambda \leq 1$, the effects A and $\lambda B_1 + (1 - \lambda)B_2$ are coexistent.*

Proof. Let G^1 be a joint observable of A and B_1 and G^2 a joint observable of A and B_2. By Proposition 1 we can assume that both G^1 and G^2 have four outcomes: $1, 2, 3, 4$. An observable G defined as $\mathsf{G}_j = \lambda \mathsf{G}^1_j + (1 - \lambda)\mathsf{G}^2_j$ for $j = 1, 2, 3, 4$, is a joint observable of A and $\lambda B_1 + (1 - \lambda)B_2$. □

Corollary 1. *If A and B be coexistent effects. Then A is coexistent with effect λB for any $0 \leq \lambda \leq 1$.*

Proof. Choose $B_2 = O$ in Proposition 4. □

Appendix 2: Proof of the characterization theorem

3.1. **Step 1 - Formulation of the coexistence condition as an intersection requirement for four circles.** We first shortly recall the formulation of the coexistence condition as an intersection requirement for four circles [1], [3]. As shown in Proposition 1, the coexistence of A and B is equivalent to the existence of a four outcome observable G. This, in turn, is equivalent to the existence of a single effect G_1 satisfying the following operator inequalities [5]:

$$\mathsf{G}_1 \geq O, \quad \mathsf{G}_1 \leq A, \quad \mathsf{G}_1 \leq B, \quad I + \mathsf{G}_1 \geq A + B\,. \tag{11}$$

We parametrize G_1 in the same way as A and B,

$$\mathsf{G}_1 = \frac{1}{2}(\gamma I + \mathbf{g} \cdot \boldsymbol{\sigma}), \qquad 0 \leq \|\mathbf{g}\| \leq \gamma \leq 2 - \|\mathbf{g}\|\,. \tag{12}$$

Conditions (11) can be recast into the following four inequalities,

$$\|\mathbf{g}\| \leq \gamma, \tag{13}$$
$$\|\mathbf{a} - \mathbf{g}\| \leq \alpha - \gamma, \tag{14}$$
$$\|\mathbf{b} - \mathbf{g}\| \leq \beta - \gamma, \tag{15}$$
$$\|\mathbf{a} + \mathbf{b} - \mathbf{g}\| \leq 2 + \gamma - \alpha - \beta. \tag{16}$$

The conclusion from the above is that effects A and B are coexistent if and only if there exist parameters γ and $\mathbf{g}$ such that the inequalities (13)-(16) are satisfied.

In the three dimensional space, each inequality can be viewed as a ball of allowed vectors $\mathbf{g}$. These four balls are centered in points $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$, respectively, with radii given by the right hand side of the corresponding inequality. A joint observable of A and B therefore exists if and only if there is a γ such that the intersection of the four balls is non-empty. Important is that the radii change with γ, which is a free parameter. To find out whether a joint observable exists, we have to check the intersection for all γ. The intersection also shows the freedom in choosing different vectors $\mathbf{g}$ – for each γ when the intersection is non-empty, all points in the intersection can be chosen as $\mathbf{g}$. From this also follows that a unique effect G_1 exists if and only if there is only one γ such that the four balls intersect and for this particular γ, they intersect only in one point. (We will see that the requirement that there is only a single γ where the intersection is non-empty implies that the intersection is formed by one point only.)

By Proposition 3 in Appendix 1, the coexistence of A and B depends only on the numbers α, β, a, b and on the relative angle between $\mathbf{a}$ and $\mathbf{b}$. Without any loss of generality, we choose a coordinate system such that the vector $\mathbf{a}$ lies along the x-axis and vector $\mathbf{b}$ is in the x-y plane. Then, if there is a point $\mathbf{g}$ in the intersection, its projection to the x-y plane is also in the intersection, because the projection is closer then $\mathbf{g}$ to each of the centers of the four balls. As we are interested whether the intersection is empty or not, it is thus enough to study the intersection in the x-y plane only. Projecting on the x-y plane we obtain four circles centered in the corners of parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, which have the radii given in Eq. (13)-(16). The geometry is summarized in Fig. 2.

3.2. **Step 2 – Restriction to the boundary cases.** We will answer the question whether A and B are coexistent by fixing the parameters α, a, and β and specifying the region in the two dimensional x-y plane – if, for given α, a, and β, the vector $\mathbf{b}$ lies inside the region, then there exists at least one γ such that the four circles have at least one common point and then A and B are coexistent.

By Proposition 4, the set of effects B coexistent with A is convex. Thus, also the allowed region of vectors $\mathbf{b}$ is convex. We can therefore restrict our study to the boundary cases.

The border of the intersection is formed by the borders of the four circles. If there is γ such that the intersection has nonzero volume,

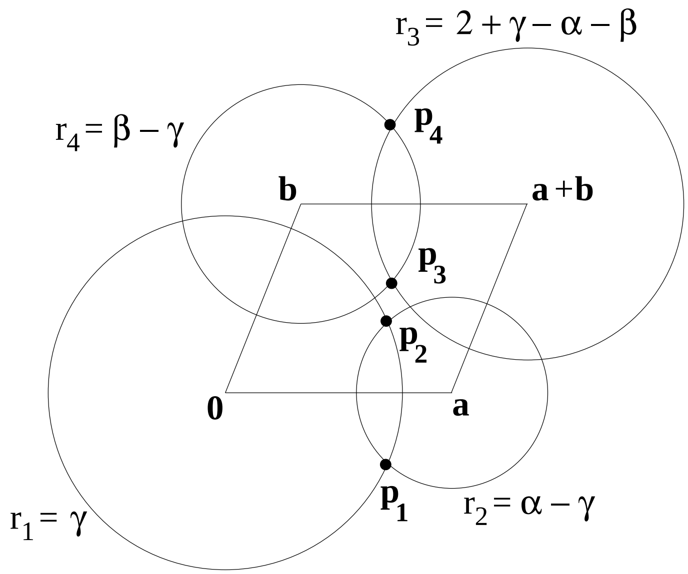

FIGURE 2. A joint observable exists if the intersection of the four circles is non-empty. The four circles are centered at corners of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. The circles' radii are given in the Figure and depend on γ. For the particular γ used in the figure, the intersection is empty. Later we will also need the common points of circles 1 and 2 denoted by $\mathbf{p_1}$ and $\mathbf{p_2}$ and circles 3 and 4, denoted by $\mathbf{p_3}$ and $\mathbf{p_4}$.

then there is $\epsilon > 0$ such that for all $\mathbf{b}' : \|\mathbf{b}' - \mathbf{b}\| < \epsilon$ changing vector $\mathbf{b}$ to $\mathbf{b}'$ does not make the nonzero area to disappear; see Fig. 3 for an illustration. Boundary is therefore formed by such $\mathbf{b}$'s that only such γ's exist that the intersection is non-empty but has zero volume.

Two circles can intersect in an area of nonzero volume or in a point. Another circle can cut out from this either another area of a nonzero volume or a single point (or make the intersection empty). Again, also a four circle intersection can be either a nonzero volume area, or a single point, or empty.

Note that in the same way we can deduce that if the intersection is of a nonzero volume, there must be an interval of γ's leading to nonzero intersections. That means the following implication holds: unique γ implies single point intersection (and consequently unique $\mathbf{g}$). It is, however, not true, that the existence of a single point intersection implies unique G — for example, there are cases where there are only

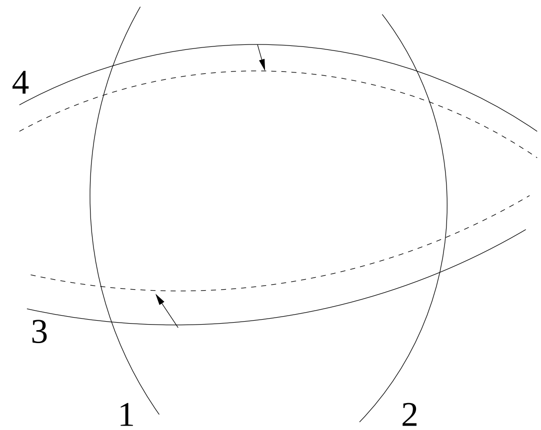

FIGURE 3. An illustrative nonzero volume intersection. The boundaries of the intersection are defined by the boundaries of the circles (solid lines). The boundaries 3 and 4 move smoothly when changing vector **b**. Therefore there is an upper limit on the change of **b** under which it is not possible to make the nonzero volume to disappear.

single points intersections, but γ can be chosen from an interval of positive length (and also vectors **g** differ for different γ).

From the previous follows that the boundary is formed by such vectors **b** that there exists γ for which the four circles intersection is a single point and no such γ exists that the intersection is a nonzero volume area. We do not say here that only one γ exists where the intersection is a single point. We will see that there are parts of the boundary, where an interval of γ's leads to single point intersections.

We also note here that certain transformations of the plane preserve also the four circles intersection. Rotation of vectors **a** and **b** by the same angle is one example of transformation, which allowed us to put vector **a** along the x-axis. Another useful transformation is the inversion with respect to the axis defined by vector **a**. In this case a vector **b** is transformed from $(b_\parallel, b_\perp)$ to $(b_\parallel, -b_\perp)$. This symmetry allows us to restrict to the case $b_\perp \geq 0$ – the allowed region for vectors **b** (and then also the boundary) is symmetric with respect to axis x.

We conclude that it is enough to characterize the boundary curve of the allowed region. By convexity, all vectors **b** inside the boundary curve lead to coexistence.

3.3. **Step 3 - Division of single point intersections into three cases.** Four circles can intersect in one point in three distinct cases:

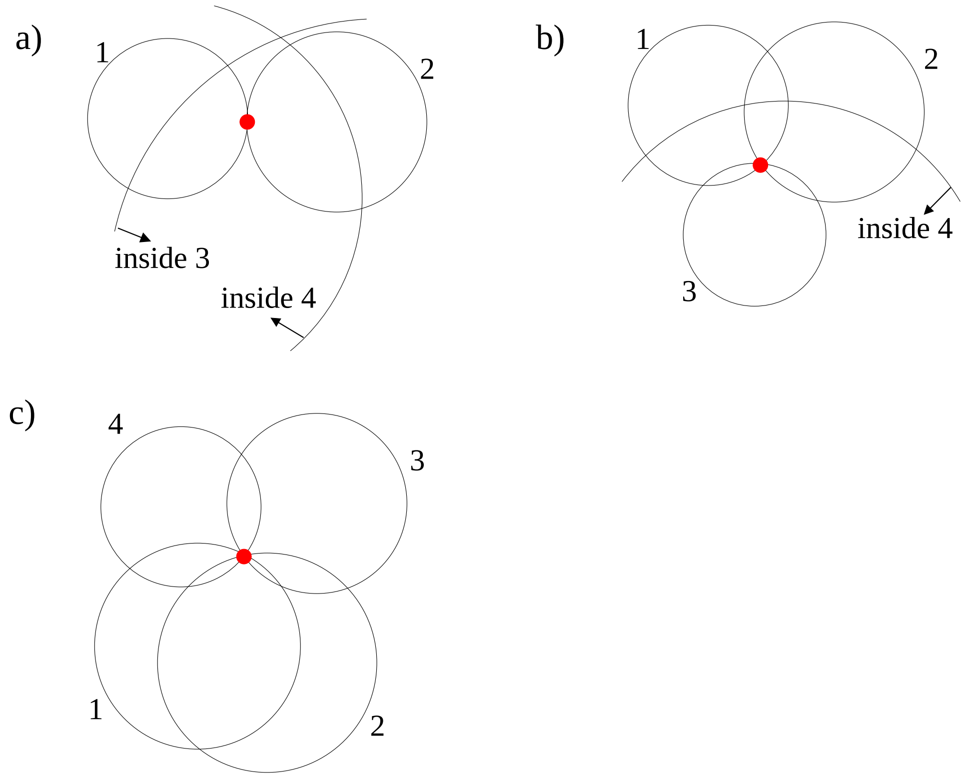

FIGURE 4. Three cases when the intersection is a single point one. a) Two circle intersect in a single point, which is inside the remaining two circles. b) Three circles intersect in a single point, it is not the case a, and the intersection is inside the fourth circle. c) Four circles intersect in a single point and it is not the case a, or b.

- 2CI – two circles intersect in one point and this point lies inside (that is within a positive distance from the boundary) of the two remaining circles
- 3CI – three circles intersect in one point, but out of these no two circles intersect in one point and this point lies inside of the fourth circle
- 4CI – there is a point laying on boundary of all four circles

See Fig. 4 for an illustration of these three cases.

3.3.1. *Boundary of the allowed region in the 2CI case.* If $\mathbf{b}$ is on the boundary of the allowed region and it is a 2CI case, then there is γ_0 such that two circles intersect in a point that is inside the remaining two circles and for no γ the intersection has a nonzero volume. Then the two circles making the single point intersection can not be circles 1 and 3. It is because both these circles' radii grow with γ – we could enlarge γ_0 by a small amount such that circles 1 and 3 would intersect in a nonzero volume and the two remaining circles would not move enough to leave this area. We would thus obtain a four circle intersection of a nonzero volume, what is in contradiction with the assumption that the $\mathbf{b}$ is on the boundary. Similarly, the boundary formed by 2CI can not be due to circles 2 and 4 – they both grow if we decrease γ. By decreasing γ we could make in an analogous way the four circle intersection with a nonzero volume. Single point intersection of circles 1 and 2 also can not make a boundary in the space of vectors $\mathbf{b}$, since they do not change with changing $\mathbf{b}$. Finally, circles 3 and 4 shift by the same amount when $\mathbf{b}$ is changed and therefore also can not define the boundary. What is left for a 2CI boundary case is a single point intersection of the circles 2 and 3, which leads to equation $b = 2 - \beta$, which is never the case since we have restricted to the case $\beta \leq 1$ and it must be $b \leq \beta$. The only possibility is that a single point intersection is formed by circles 1 and 4, leading to equation $b = \beta$. This equation indeed defines part of the boundary of the allowed region for vectors $\mathbf{b}$.

We will specify now when vectors $\mathbf{b}$ such that $b = \beta$ are in the allowed region [and then they are the boundary, since b can not be larger than β according to condition in Eq. (3b)]. If $b = \beta$, then at a given $\gamma \in [0, \beta]$, circles 1 and 4 intersect in point $\gamma\mathbf{b}/b$. This point lies on the boundary of circle 2 if $\|\gamma\mathbf{b}/b - \mathbf{a}\| = \alpha - \gamma$, that is if

$$\gamma = \gamma_M := \frac{\beta}{2}\frac{\alpha^2 - a^2}{\alpha\beta - \mathbf{a}\cdot\mathbf{b}} \tag{17}$$

Since $\alpha \geq a$ and $\beta \geq b$, the number γ_M is well defined and positive, unless $\mathbf{a}$ and $\mathbf{b}$ are collinear and $\alpha = a$ and $\beta = b$. We can, however, put aside this case, since if $\mathbf{a}$ and $\mathbf{b}$ are collinear, A and B are coexistent for any allowed $\mathbf{b}$ – this means that on the x-axis the allowed region is the interval $[-\beta, \beta]$. Since circles 2 and 4 diminish when γ increases, for $\gamma < \gamma_M$ ($\gamma > \gamma_M$) the intersection of circles 1 and 4 is inside (outside) circle 2. It can happen that $\gamma_M > \beta$ (circle 4 is not defined for $\gamma > \beta$) or $\gamma_M > \alpha$, when circle 2 is not defined, but this simply means that the intersection of circles 1 and 4 is inside the circle 2 for all admissible γ.

Similarly, one finds that the intersection of circles 1 and 4 is inside circle 3 for $\gamma > \max(\gamma_m, 0)$, where

$$\gamma_m := \frac{\beta}{2}\frac{(2-\alpha-\beta)^2 - \|\mathbf{a}+\mathbf{b}\|^2}{\alpha\beta - \mathbf{a}\cdot\mathbf{b} - 2\beta}\,. \tag{18}$$

Again, the denominator in Eq. (18) is zero only in if vectors **a**, and **b** are collinear and equal to α, and β, respectively, and $\alpha = 1$.

Finally, the vectors $b = \beta$ are on the boundary of the allowed region if the interval (γ_m, γ_M) is non-empty. Solving equations $b = \beta$ and $\gamma_m = \gamma_M$ we obtain the limit cases

$$b_{\|}^{\pm} \equiv b_0 \pm l := \frac{1}{a}\left((1-\alpha)(1-\beta) \pm \sqrt{D}\right), \tag{19}$$

where

$$D := (1-\alpha)^2 - \beta[(1-\alpha)^2 + 1 - a^2] + \beta^2. \tag{20}$$

From Eq. (19) we may conclude that $b_{\|}^{\pm}$ is not always meaningful with respect to our needs. First of all, it is clear that if D is negative, the solutions $b_{\|}^{\pm}$ are not real numbers. But even if D is positive, the solutions $b_{\|}^{\pm}$ may not lay inside the relevant interval $[-\beta, \beta]$.

For $\mathbf{b} = (\pm\beta, 0)$ we have $\gamma_m = \frac{1}{2}[\alpha \pm a - 2(1-\beta)]$, $\gamma_M = \frac{1}{2}(\alpha \pm a)$, and therefore these two vectors are always on the boundary. By solving equation $D = 0$ for variable β, we obtain two real solutions β_1 and β_2 telling us when for $D = 0$ the effects A and B are coexistent for all the vectors $\mathbf{b}$, $b \leq \beta$. However only the larger solution β_2 is important for us, since due to scaling property from Cor. 1, all effects B' having $\beta' \leq \beta$ will be coexistent with A as well. Together with the knowledge, that for β_2 Eq. (19) gives $b_{\|}^{+} = b_{\|}^{-} \in [-\beta, \beta]$, we may conclude, that for $\beta_1 < \beta_2$, the solutions $b_{\|}^{\pm}$ must lie outside of the interval $[-\beta, \beta]$,[3] thus giving us the boundary $\beta = b$ for all $\mathbf{b}$.

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[3]It can be seen as follows: take the expression $ab_{\|}^{-}$. It is a continuous real function of α, a, β, defined in two simply connected regions: $[0,1] \times [0,\alpha] \times [0,\beta_1]$ and $[0,1] \times [0,\alpha] \times [\beta_2, 1]$, for the three variables, respectively. That these regions are simply connected follows from inequality $0 \leq \beta_1 \leq \beta_2 \leq 1$: First, since $\mathfrak{S} \in [0,1]$, it follows that $\beta_2 \leq 1$. Second, we have chosen β_2 to be the larger from the two solution, therefore $\beta_2 \geq \beta_1$. And the last part follows from identity $\beta_1\beta_2 = (1-\alpha)^2 \geq 0$. Putting $ab_{\|}^{-}$ equal to βa we find that a necessary condition for this is

$$0 = \beta(1-\beta)(\alpha - a)(2 - \alpha + a). \tag{21}$$

In another words, $ab_{\|}^{-}$ is equal to βa only on the boundary of the definition region. It then follows, that if for one point inside the region it holds $ab_{\|}^{-} < \beta a$, then it holds $ab_{\|}^{-} \leq \beta a$ for the whole region. Analogous equation to Eq. (21) allows similar reasoning for the lower interval limit, $-\beta a$ and also for the second expression, $ab_{\|}^{+}$. From an example for the first part of the region, $\alpha = 1/4$, $a = 1/8$, $\beta = 1/4$ one finds that here both $ab_{\|}^{\pm}$ are outside $[-\beta a, \beta a]$ (both are larger or equal). On the other hand, putting $\alpha = 3/4$, $a = 1/2$, $\beta = 3/4$ shows that in this part of the region (that is for $\beta > \beta_2$), the expressions $ab_{\|}^{\pm}$ are both inside the interval $[-\beta a, \beta a]$.

In fact, this property leads us to the definition of sharpness for effects in Eq. (4) – we define the sharpness as $\mathfrak{S}(A) = 1 - \beta_2$. Moreover we may notice that as a function of a, $\mathfrak{S}(\alpha, a)$ increases if a increases. On the other hand, $\mathfrak{S}(\alpha, \alpha) = \alpha$. Therefore $\mathfrak{S}(\alpha, a) \leq \alpha$, if $a \leq \alpha$. This means that $\beta_2 \geq 1 - \alpha$. The necessary condition for $b_\parallel^\pm$ to be real and inside the interval $[-\beta, \beta]$ is then $\alpha + \beta \geq 1$.

We then conclude that

- if $\beta > 1 - \mathfrak{S}(A)$, the boundary is given by vectors $\mathbf{b}$ such that $b = \beta$, if $b_\parallel \notin (b_\parallel^-, b_\parallel^+)$, given by Eq. (19)
- if $\beta \leq 1 - \mathfrak{S}(A)$, the whole boundary is formed by vectors for which $b = \beta$ – in another words, in this case the allowed region is a circle with the diameter β and center at $\mathbf{0}$

3.3.2. *Boundary of the allowed region in the 3CI case.* The conclusion of this paragraph is that the 3CI do not make the boundary - the necessary condition for a 3CI implies that one of these circles contains another one and therefore can be disregarded, forcing the case to be a 2CI case.

Let us assume that a 3CI case defining a boundary is formed by the intersection of circles 1, 2, and 4, first – this means 1, 2 and 4 intersect in a single point which is inside circle 3. Looking at Fig. 2, points common to circles 1 and 2 are $\mathbf{p_1}$ and $\mathbf{p_2}$. The first is farther away from circle 4 than the second (since we have restricted to $b_\perp \geq 0$). Therefore, if circles 1, 2, and 4 have a single common point, it must be $\mathbf{p_2}$ (for the case $b_\perp = 0$, $\mathbf{p_1}$ and $\mathbf{p_2}$ are in the same distance from circle 4, but in this case the boundary is $b = \beta$ – we know this from the previous and can disregard case $b_\perp = 0$ here).

Take the following quantity comparing the distance of $\mathbf{p_2}$ from the center of circle 4 and its radius,

$$d(\gamma) = \|\mathbf{b} - \mathbf{p_2}(\gamma)\|^2 - (\beta - \gamma)^2, \tag{22}$$

which is a function of γ. If the point $\mathbf{p_2}$ lies on circle 4 for some γ_0, then[4] $d(\gamma_0) = 0$. If $\partial_\gamma d(\gamma)|_{\gamma_0} < 0$, then there exists an interval (γ_0, γ_1) where $d(\gamma) < 0$. Since we have assumed that the common point of circle 1, 2 and 4 is inside (that is a positive distance from the boundary of) circle 3, there exist $\gamma \in (\gamma_0, \gamma_1)$ such that the four circles intersect in a nonzero area. This is in contradiction with the fact that $\mathbf{b}$ is on the boundary. Therefore a necessary condition for a 3CI case are equations $d(\gamma_0) = 0$ and $\partial_\gamma d(\gamma_0) = 0$.

[4]Moreover, if the point $\mathbf{p_2}$ lies inside [outside] the circle 4 for some γ_0, then $d(\gamma_0) < 0$ $[d(\gamma_0) > 0]$.

TABLE 1. Necessary conditions for all four possible three circle intersections. The three circles intersecting in a single point are given in the first column. The necessary condition and its geometrical meaning for a 3CI definng the boundary is in the second column.

3CI	necessary condition and its geometrical meaning
1, 2, and 3	$0 = [b^2 - (2-\beta)^2][\|\|\mathbf{a}+\mathbf{b}\|\|^2 - (2-\alpha-\beta)^2]$ never, or 1 and 3 are one inside other
1, 2, and 4	$0 = [b^2 - \beta^2][\|\|\mathbf{a}-\mathbf{b}\|\|^2 - (\alpha-\beta)^2]$ 1 and 4 are touching, or 2 and 4 are one inside other
1, 3, and 4	$0 = [b^2 - \beta^2][\|\|\mathbf{a}+\mathbf{b}\|\|^2 - (2-\alpha-\beta)^2]$ 1 and 4 are touching, or 1 and 3 are one inside other
2, 3, and 4	$0 = [b^2 - (2-\beta)^2][\|\|\mathbf{a}-\mathbf{b}\|\|^2 - (\alpha-\beta)^2]$ never, or 2 and 4 are one inside other

Inserting the appropriate expression for $\mathbf{p_2}$ and making substitution $\gamma \to (\alpha - ag)/2$, one can express $d(\gamma)$ in the following form

$$d = c_1(c_2 - \sqrt{1-g^2} + c_3 g), \tag{23}$$

where $c_1 = b_\perp\sqrt{\alpha^2 - a^2}$, $c_2c_1 = b^2 - \mathbf{a}.\mathbf{b} + \alpha\beta - \beta^2$, $c_3c_1 = b_\parallel\alpha - a\beta$ do not depend on γ. Equation $\partial_\gamma d(\gamma) = 0$ leads to a single solution $g = -c_3/\sqrt{1+c_3^2}$. Putting this into equation $d(g) = 0$ we get equation $c_2^2 - c_3^2 - 1 = 0$. Substituting back the definitions for c_1, c_2 and c_3 we finally obtain that a necessary condition for this particular 3CI is

$$(b^2 - \beta^2)[\|\mathbf{a}-\mathbf{b}\|^2 - (\alpha-\beta)^2] = 0. \tag{24}$$

If the expression in the first bracket is zero, we obtain the condition for 2CI of circles 1 and 4. If the second bracket is zero, the circles 2 and 4 are one inside the other (if $\alpha \geq \beta$, then circle 4 is inside the circle 2, and it is the opposite if $\alpha \leq \beta$). Their intersection is then the whole smaller circle and it is important, that this fact is independent of γ. We can then disregard the larger circle completely, because the intersection does not depend on it in any respect. The 3CI is thus reduced to 2CI and can not therefore define boundary different from those found in the previous section dealing with 2CI. In the same way it can be found that the three other possible 3CI are similar and always lead to boundary defined by a 2CI intersection. The quantity $d(\gamma)$ from Eq. (22) and the resulting condition from Eq. (24) for all four possible 3CI are summarized in Tab. 1

3.3.3. *Boundary of the allowed region in the 4CI case.* Four point intersection can occur if one of the points $\mathbf{p_1}$ and $\mathbf{p_2}$ coincides with one of the points $\mathbf{p_3}$ and $\mathbf{p_4}$. Since point $\mathbf{p_2}$ is always (we have restricted to $b_\perp > 0$) closer to circle 3 than point $\mathbf{p_1}$, and point $\mathbf{p_3}$ is closer to circle 1 than $\mathbf{p_4}$, if the four circles have a single point intersection it must be case that for some γ the points $\mathbf{p_2}$ and $\mathbf{p_3}$ coincide. Putting the x coordinates to be equal we obtain the solution for γ,

$$\gamma = \frac{1}{2}\left[\alpha\beta + \mathbf{a}\cdot\mathbf{b} - 2(1-\alpha)(1-\beta)\right]. \tag{25}$$

Parameter γ given in the previous equation represents the four circle intersection if $\mathbf{p_2}$ and $\mathbf{p_3}$ are well defined. Point $\mathbf{p_2}$ exists if $\gamma \in [\frac{\alpha-a}{2}, \frac{\alpha+a}{2}]$, while point $\mathbf{p_3}$ exists if $\gamma \in [\frac{\alpha-a}{2} - (1-\beta), \frac{\alpha+a}{2} - (1-\beta)]$. Using these conditions and the solution from Eq. (25) we conclude that Eq. (25) represents a case $\mathbf{p_2}$=$\mathbf{p_3}$ if the following condition is fulfilled,

$$(1-\beta)(1-\alpha) - (\beta - 1 + a) \leq \mathbf{a}\cdot\mathbf{b} \leq (1-\beta)(1-\alpha) + (\beta - 1 + a). \tag{26}$$

If the limit expressions from Eq. (19) fulfill the above inequality, then all relevant vectors $\mathbf{b}$ do.[5] Putting equal the y coordinates for the points $\mathbf{p_2}$ and $\mathbf{p_3}$ we finally obtain the equation of the coordinate $b_\perp$ as a function of $b_\parallel$

$$\begin{aligned} b_\perp = &\frac{1}{2a}\sqrt{(\alpha^2 - a^2)\{a^2 - [(2-\alpha)(1-\beta) + ab_\parallel]^2\}} \\ &+ \frac{1}{2a}\sqrt{(2-\alpha)^2 - a^2)\{a^2 - [\alpha(1-\beta) + ab_\parallel]^2\}}, \end{aligned} \tag{27}$$

which can be rewritten into the form of Eq. (5)

Note added

A paper with identical title is being published on the arXiv simultaneously by Paul Busch and Heinz-Jürgen Schmidt. These authors solve the same problem independently with a different method. The final results have yet to be compared.

[5] That the limits fullfill the inequality can be proved analogously to the proof of Eq. (19) to be inside(outside) the allowed interval. First, the implication $\beta > 1 - \mathfrak{S}(A) \rightarrow \beta + a - 1 > 0$ holds, by which the interval in (26) is nonempty. The implication follows from inequality $\mathfrak{S}(\alpha, a) \geq a$, which follows from the fact that $\mathfrak{S}(\alpha, a) = a$ implies $a = 0$ or $a = \alpha$, and therefore can be valid only on the boundary of the definition region. By taking an explicit example from inside the region, the inequality $\mathfrak{S}(\alpha, a) \geq a$ is proved. Similarly, the equality in (26) for the limits $ab_\parallel^\pm$ implies $\alpha = a$ or $\beta = 1$, again the boundary of the definition region of $\mathbf{a}.\mathbf{b}$.

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