Process POVM: A mathematical framework for the description of process tomography experiments

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In this paper we shall introduce the mathematical framework for the description of measurements of quantum processes. Using this framework the process estimation problems can be treated in the similar way as the state estimation problems, only replacing the concept of positive operator valued measure (POVM) by the concept of process POVM (PPOVM). In particular, we will show that any measurement of qudit channels can be described by a collection of effects (positive operators) defined on two-qudit system. However, the effects forming a PPOVM are not normalized in the usual sense. We will demonstrate the usage of this formalism in discrimination problems by showing that perfect channel discrimination is equivalent to a specific unambiguous state discrimination.

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Introduction. Born's trace rule predicting the quantum probabilities is the central object of quantum physics relating quantum theory with quantum experiments [1, 2]. According to this probability rule any quantum experiment is composed of two parts: preparation and measurement. In quantum theory the preparation part is described by the concept of quantum states represented by density operators ϱ, i.e. positive operators ($\varrho \geq 0$) normalized to unity ($\mathrm{Tr}\varrho = 1$). The measurement devices giving rise to outcomes x_j are described by collections of quantum effects F_j, that is, positive operators smaller than identity ($0 \leq F_j \leq I$), and summing up to identity ($\sum_j F_j = I$). We say that these operators form a positive operator valued measure (POVM). For a system prepared in a state ϱ the probability of measuring the outcome associated with an effect F is defined by Born's rule as $p(F|\varrho) = \mathrm{Tr}\varrho F$.

Quantum process is an independent part of quantum experiment that can be placed (in time) between all possible preparations and measurements. It is described by a completely positive tracepreserving linear map [3], the so-called quantum channel. Implementation of specific quantum processes is one of the main goals of the area of quantum information processing [4] aiming to run useful quantum algorithms and simulations on quantum computers.

All the problems (such as state estimation [5, 6, 7], state discrimination [7, 8, 9], state comparison [10], etc.) related to the identification of quantum states can be mathematically formulated in the language of POVMs. This concept is sufficient for our purposes despites the precise description of a particular experimental setup is not specified. The main aim of this paper is to introduce a resembling mathematical framework for the description of all possible measurements of quantum channels.

Measurements of channel parameters. Consider an unknown quantum channel $\mathcal{E}$ acting on a d-dimensional quantum system (qudit). The most general *process/channel measurement* $\mathcal{M}$ consists of the following three steps:

1. Preparation of a (test) state $\varrho_j \in \mathcal{S}(\mathcal{H}_{\rm anc} \otimes \mathcal{H})$ of $D_j \times d$- dimensional system, thus, initially the testing system is composed of a qudit and an ancillary system of dimension D_j. The ancillary system can be of different size for a different test state ϱ_j.

2. Application of an unknown process $\mathcal{E}$ on the qudit and some known channel $\mathcal{T}_{j,\rm anc}$ on the ancillary quantum system.

3. A measurement M_j (given as a collection of positive operators, $F_{jk} \geq 0$, summing up to identity operator, $\sum_k F_{jk} = I$ for all j) of the output state $\varrho_j' = (\mathcal{T}_{j,\rm anc} \otimes \mathcal{E})[\varrho_j]$ results in an outcome k with a probability $p_{jk}(\mathcal{E}) = \mathrm{Tr}[F_{jk}\varrho_j']$.

It follows that a general experiment measuring a process $\mathcal{E}$ is associated with a collection of triples $\mathcal{M}_{jk} = \langle \varrho_j, \mathcal{T}_{j,\rm anc}, F_{jk}\rangle$ occurring with probabilities p_{jk} defined above. However, a channel $\mathcal{T}_{j,\rm anc}$ can be considered as being a part of a preparation, or a measurement process, i.e. the triples $\langle \varrho_j, \mathcal{T}_{j,\rm anc}, F_{jk}\rangle$, $\langle \mathcal{T}_{j,\rm anc} \otimes \mathcal{I}[\varrho_j], \mathcal{I}_{\rm anc}, F_{jk}\rangle$, and $\langle \varrho_j, \mathcal{I}_{\rm anc}, \mathcal{T}^*_{j,\rm anc} \otimes \mathcal{I}[F_{jk}]\rangle$, (where $\mathcal{I}$ is the identity quantum channel, and $\mathcal{T}^*_{j,\rm anc}$ is defined via the duality relation $\mathrm{Tr}\{B^\dagger \mathcal{T}_{\rm anc}[A]\} = \mathrm{Tr}\{(\mathcal{T}^*_{\rm anc}[B])^\dagger A\}$ holding for all operators A, B) define the same probabilities $p_{jk}(\mathcal{E})$. Without the loss of generality we may assume that the ancilla system evolves trivially, $\mathcal{T}_{j,\rm anc} = \mathcal{I}_{\rm anc}$ for all j, hence, the triples can be replaced by couples $\mathcal{M}_{jk} = \langle \varrho_j, F_{jk}\rangle$ occurring with probabilities $p_{jk}(\mathcal{E}) = \mathrm{Tr}\{(\mathcal{I}_{\rm anc} \otimes \mathcal{E})[\varrho_j]F_{jk}\}$.

The following lemma is a version of the so-called Choi-Jamiolkowski isomorphism [11, 12] relating qudit linear maps with linear operators on $d \times d$ system.

Lemma 1. *For arbitrary state of $D \times d$ system ($\varrho \in \mathcal{S}(\mathcal{H}_D \otimes \mathcal{H}_d)$) there exists a completely positive channel $\mathcal{R}_\varrho : \mathcal{B}(\mathcal{H}_d) \to \mathcal{B}(\mathcal{H}_D)$ such that*

$$(\mathcal{R}_\varrho \otimes \mathcal{I})[\Psi_+] = \varrho\,,$$

where $\Psi_+ = |\Psi_+\rangle\langle\Psi_+|$ and $|\Psi_+\rangle = \sum_{j=1}^d |j\rangle \otimes |j\rangle$ is an unnormalized maximally entangled quantum state on $d \times d$ system.

Proof. Consider a pure state $\varrho = |\phi\rangle\langle\phi| = \Phi$ of a $d \times D$-dimensional system and $|\phi\rangle = \sum_{j=1}^{d}\sum_{\alpha=1}^{D} \Phi_{\alpha j}|\alpha\rangle \otimes |j\rangle$. Define an operator $A_\Phi : \mathcal{H}_d \to \mathcal{H}_D$ acting as follows $A_\Phi \otimes I_d|\Psi_+\rangle = |\Phi\rangle$, i.e. $A_\phi|j\rangle = \sum_{\alpha=1}^{D} \Phi_{\alpha j}|\alpha\rangle$. We can write $\Phi = (\mathcal{R}_\Phi \otimes \mathcal{I})[\Psi_+] = (A_\Phi \otimes I_d)\Psi_+(A_\Phi^\dagger \otimes I_d)$, where $\mathcal{R}_\Phi$ is a unique linear completely positive map, because the expression of $|\Phi\rangle$ in the basis $\{|\alpha\rangle \otimes |j\rangle\}$ is unique. Generalization to an arbitrary mixed state is straightforward. For $\varrho = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ we can define a map $\mathcal{R}_\varrho = \mathcal{R}_{\sum_j \lambda_j \Phi_j} = \sum_j \lambda_j \mathcal{R}_{\Phi_j}$ that maps the maximally entangled state Ψ_+ into $\varrho = (\mathcal{R}_\varrho \otimes \mathcal{I})[\Psi_+] = \sum_j \lambda_j (\mathcal{R}_{\Phi_j} \otimes \mathcal{I})[\Psi_+] = \sum_j \lambda_j \Phi_j$. Different convex decompositions of ϱ into pure states define different Kraus decompositions of the same completely positive map $\mathcal{R}_\varrho$, hence, this map is unique. It is straightforward to see that similar result holds for a general positive operator F, that is, the transformation $\mathcal{R}_F$ defined as $\mathcal{R}_F \otimes \mathcal{I}[\Psi_+] = F$ is completely positive, too. □

Using this lemma and the definition of the dual map $\mathcal{R}_\varrho^*$ the probability for a couple $\langle \varrho, F\rangle$ of the test state ϱ and a measurement resulting in an outcome associated with an effect F can be expressed as follows

$$\begin{aligned} p(\mathcal{E}) &= \mathrm{Tr}\{(\mathcal{I}_{\mathrm{anc}} \otimes \mathcal{E})[\varrho]F\} \\ &= \mathrm{Tr}\{(\mathcal{R}_\varrho \otimes \mathcal{E})[\Psi_+]F\} \\ &= \mathrm{Tr}\{(\mathcal{I}_{\mathrm{anc}} \otimes \mathcal{E})[\Psi_+](\mathcal{R}_\varrho^* \otimes \mathcal{I})[F]]\} \\ &= \mathrm{Tr}\{(\mathcal{I}_{\mathrm{anc}} \otimes \mathcal{E})[\Psi_+]\mathcal{M}\}\,. \end{aligned}$$

Based on this calculation we see that an operator $\mathcal{M} = (\mathcal{R}_\varrho^* \otimes \mathcal{I})[F]$ completely describes the considered process measurement outcome associated with $\langle \varrho, F\rangle$. Since both the operations $\mathcal{R}_\varrho, \mathcal{R}_\varrho^*$ are completely positive, but not necessarily trace-preserving, it follows that an operator $\mathcal{M} = (\mathcal{R}_\varrho^* \otimes \mathcal{I})[F]$ is positive and $\mathcal{M} \le I_{d\times d}$, that is, $\mathcal{M}$ is an effect defined on a $d \times d$-dimensional system that we shall call a *process/channel effect*. The most general *process/channel measurement* is defined as a collection of process effects $\mathcal{M}_{jk} = p_j(\mathcal{R}_{\varrho_j}^* \otimes \mathcal{I})[F_{jk}]$ associated with couples $\langle p_j\varrho_j, F_{jk}\rangle$ with $\sum_k F_{jk} = I_{D\times d}$ for all j labeling potentially different test states ϱ_j chosen with a prior distribution p_j.

Let us assume that the process is probed only by a single test state ϱ, i.e. $\mathcal{M}_k \leftrightarrow \langle \varrho, F_k\rangle$. In such case $\sum_k \mathcal{M}_k = (\mathcal{R}_\varrho^* \otimes \mathcal{I})[\sum_k F_k] = (\mathcal{R}_\varrho^* \otimes \mathcal{I})[I_{D\times d}]$. Since $\mathcal{R}_\varrho[X] = \sum_j \lambda_j A_{\Phi_j} X A_{\Phi_j}^\dagger$ the action of the dual map can be expressed as $\mathcal{R}_\varrho^*[X] = \sum_j \lambda_j A_{\Phi_j}^\dagger X A_{\Phi_j}$. Consequently, we get that the following normalization condition holds

$$\sum_k \mathcal{M}_k = \sum_j \lambda_j A_{\Phi_j}^\dagger A_{\Phi_j} \otimes I_d = (\mathrm{Tr}_{\mathrm{anc}}\varrho)^T \otimes I_d\,.$$

Thus, the process effects $\mathcal{M}_k$ form a positive operator valued measure not necessarily normalized in the usual sense, because $\sum_k \mathcal{M}_k \le I_{d\times d}$.

For a general process measurement (described by process effects $\mathcal{M}_{jk} = p_j(\mathcal{R}_{\varrho_j}^* \otimes \mathcal{I})[F_{jk}]$) it follows that $\sum_{jk} \mathcal{M}_{jk} = \sum_j (p_j \mathrm{Tr}_{\mathrm{anc}}\varrho_j)^T \otimes I_d = (\mathrm{Tr}_{\mathrm{anc}}\overline{\varrho})^T \otimes I_d$, where the operator $\overline{\varrho} = \sum_j p_j \varrho_j$ is the average test state. Even if the test states are using different ancillas, it is always possible to consider them as joint states of the qudit and the largest of the ancilla systems. It follows that the process measurement consisting of process effects $\mathcal{M}_{jk} \leftrightarrow \langle p_j\varrho_j, F_{jk}\rangle$ can be understood as a process measurement composed of $\mathcal{M}_{jk} \leftrightarrow \langle \Xi, |j\rangle\langle j| \otimes F_{jk}\rangle$ with a single test state $\Xi = \sum_j p_j |j\rangle\langle j| \otimes \varrho_j$.

We have shown that each qudit channel measurement can be associated with a *process positive operator valued measure* (PPOVM), i.e. by a collection of effects $\mathcal{M}_\alpha$ of $d \times d$-dimensional system summing up to $\varrho^T \otimes I_d$, where ϱ is a qudit quantum state. In the following we will prove that the converse of this statement also holds.

Theorem 1. *Each PPOVM can be implemented as a process measurement.*

Proof. Consider a PPOVM $\{\mathcal{M}_\alpha\}$ with $\sum_\alpha \mathcal{M}_\alpha = \varrho^T \otimes I_d$. Our aim is to show that this PPOVM really corresponds to a process measurement. As it was argued before we can restrict ourselves to a process measurement using only a single test state Ξ such that $\mathrm{Tr}_{\mathrm{anc}}\Xi = \varrho$. Moreover, assuming that the test state is a pure state, the question is whether $\mathcal{M}_\alpha = (A_\Xi^\dagger \otimes I_d)F_\alpha(A_\Xi \otimes I_d)$ implies that

$$F_\alpha = ([A_\Xi^\dagger]^{-1} \otimes I_d)\mathcal{M}_\alpha(A_\Xi^{-1} \otimes I_d)\,,$$

hence, whether the operators $A_\Xi, A_\Xi^\dagger$ are invertible. Let $r = \mathrm{rank}\varrho \le d$ and assume that Ξ is a pure state of a qudit and an ancilla of the dimension r, hence, $\mathcal{R}_\Xi^*[X] = A_\Xi^\dagger X A_\Xi$ and $A_\Xi^\dagger A_\Xi = (\mathrm{Tr}_{\mathrm{anc}}\Xi)^T = \varrho^T$. The support of each operator $\mathcal{M}_\alpha$ is a subset of the support of $\varrho^T \otimes I$, i.e. both are defined on $(r \times d)$-dimensional system. Since $\mathrm{rank}(A_\Xi^\dagger A_\Xi) = \mathrm{rank}(A_\Xi A_\Xi^\dagger) = \mathrm{rank}A_\Xi = \mathrm{rank}A_\Xi^\dagger$ it follows that operators $A_\Xi, A_\Xi^\dagger, \varrho^T, \varrho$ have the same rank (equal to r). Because the operators $A_\Xi, A_\Xi^\dagger$ act on r-dimensional ancilla system (they have full rank) it follows they are invertible. Consequently, the above equation defines positive operators F_α forming a POVM, because $\sum_\alpha F_\alpha = ([A_\Xi^\dagger]^{-1}\varrho^T A_\Xi^{-1}) \otimes I_d = I_r \otimes I_d$. □

To summarize, we have shown that arbitrary collection of process effects $\mathcal{M}_\alpha$ forming PPOVM can be implemented by using a pure state $|\Xi\rangle \in \mathcal{H}_r \otimes \mathcal{H}_d$ such that $\mathrm{Tr}_{\mathrm{anc}}\Xi = \varrho$ and performing a POVM given by positive operators $F_\alpha = ([A_\Xi^\dagger]^{-1} \otimes I_d)\mathcal{M}_\alpha(A_\Xi^{-1} \otimes I_d)$ with $A_\Xi = \sqrt{\varrho^T}$. This result allows us to abstract particular experimental realizations of process measurements and employ the framework of PPOVM directly. In this framework the qudit quantum channels are represented by positive two-qudit operators $\omega_\mathcal{E} = \mathcal{I} \otimes \mathcal{E}[\Psi_+]$ satisfying $\mathrm{Tr}\omega_\mathcal{E} = d$ and $\mathrm{Tr}_2\omega_\mathcal{E} = I$. Let us denote the set of all processes (process state space) by $\mathcal{S}_{\mathrm{proc}} = \{\omega \in \mathcal{B}_+(\mathcal{H} \otimes \mathcal{H}), \mathrm{Tr}_2\omega = I, \mathrm{Tr}\omega = d\}$. This set is convex and compact subset of the set of positive operators of trace d denoted as $\mathcal{B}_{+d}(\mathcal{H}\otimes\mathcal{H})$, which is isomorphic to $\mathcal{B}_{+1}(\mathcal{H}\otimes\mathcal{H})=\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$ (set of density matrices).

Maximally entangled probe. Consider that an unknown qudit channel is probed by a (normalized) maximally entangled state $|\psi_+\rangle=\frac{1}{\sqrt{d}}\sum_j|j\rangle\otimes|j\rangle$. In this case the mapping $\mathcal{R}_{\psi_+}=\frac{1}{d}\mathcal{I}$, i.e. $|\psi_+\rangle\langle\psi_+|=\frac{1}{d}\Psi_+$. That is, $\mathcal{M}=(\mathcal{R}^*_{\psi_+}\otimes\mathcal{I})[F]=\frac{1}{d}F$, where F is a two-qudit effect. Considering a POVM consisting of effects $F_1,\dots,F_n$ the corresponding PPOVM is composed of positive operators $\mathcal{M}_j=\frac{1}{d}F_j$.

Ancilla-free test states. In this case the qudit test state ϱ can be understood as being a factorized state of an ancilla and a qudit, i.e. $\Omega=\xi\otimes\varrho$. The POVM effects have the form $I_{\rm anc}\otimes F_j$ and the corresponding process effects are $\mathcal{M}_j=(\mathcal{R}^*_\Omega\otimes\mathcal{I})[I_{\rm anc}\otimes F_j]=\varrho^T\otimes F_j$. It follows that if we want to perform an equivalent (defining the same PPOVM) process measurement with the maximally entangled probe, the POVM consists of effects $X_j=d\varrho^T\otimes F_j$.

Informationally complete process tomography. A process measurement $\{\mathcal{M}_\alpha\}$ is called informationally complete if for each quantum process $\mathcal{E}$ the probability distribution $p_\alpha(\mathcal{E})={\rm Tr}[\omega_\mathcal{E}\mathcal{M}_\alpha]$ is different. Thus the process can be uniquely identified from the observed probability distribution. This happens if and only if a linear span of operators $\mathcal{M}_\alpha$ contains the whole set of process states $\mathcal{S}_{\rm proc}$.

Consider a qubit channel probed by a maximally entangled state $|\psi_+\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$. Performing the measurements of sharp observables $\sigma_\mu\otimes\sigma_\nu$ (each with the probability 1/9), where $\mu,\nu=x,y,z$. That is, the POVM is composed of effects $F_{a,b}=\frac{1}{9}|a\rangle\langle a|\otimes|b\rangle\langle b|=2\mathcal{M}_{a,b}$, where $a,b=\pm x,\pm y,\pm z$. The states $|\pm x\rangle,|\pm y\rangle,|\pm z\rangle$ are the eigenvectors of $\sigma_x,\sigma_y,\sigma_z$ associated with eigenvalues ±1, respectively. The set of operators $\mathcal{M}_{a,b}$ is overcomplete and its span contains the whole set of process states, i.e. it is an informationally complete PPOVM. Calculating the sum we find that $\sum_{a,b}\mathcal{M}_{a,b}=\frac{1}{2}I_2\otimes I_2$.

Alternatively, one of the simplest experimental implementations of an informationally complete process measurement consists of the preparation of six test states $|\pm x\rangle,|\pm y\rangle,|\pm z\rangle$ distributed with the same probability 1/6. The measurement of the output states is the complete qubit state tomography measuring all three Pauli operators $\sigma_x,\sigma_y,\sigma_z$, hence, it consists of effects $F_{\pm a}=\frac{1}{3}|\pm a\rangle\langle\pm a|$ $(\mu=x,y,z)$. Consequently, the whole setup is described by PPOVM composed of operators $\mathcal{M}_{\nu,\mu}=\frac{1}{18}|\nu\rangle\langle\nu|^T\otimes|\mu\rangle\langle\mu|$ with $\sum_{\nu,\mu}\mathcal{M}_{\nu,\mu}=\frac{1}{2}I_2\otimes I_2$, where $\nu,\mu=\pm x,\pm y,\pm z$. Let us note that PPOVMs $\{\mathcal{M}_{\mu,\nu}\}$ and $\{\mathcal{M}_{a,b}\}$ (described in the previous paragraph) coincide, because $(|\pm x\rangle\langle\pm x|)^T=|\pm x\rangle\langle\pm x|$, $(|\pm y\rangle\langle\pm y|)^T=|\mp y\rangle\langle\mp y|$, $(|\pm z\rangle\langle\pm z|)^T=|\pm z\rangle\langle\pm z|$, where the transposition is performed with respect to basis $|\pm z\rangle$.

Perfect discrimination. Two processes $\mathcal{E}_1,\mathcal{E}_2$ are perfectly distinguishable if there exists an experimental setup such that in its single run the outcomes uniquely identify the process. It corresponds to an existence of a two-outcome PPOVM, $\mathcal{M}_1+\mathcal{M}_2=\varrho^T\otimes I$, such that $p_1(\mathcal{E}_1)p_1(\mathcal{E}_2)=p_2(\mathcal{E}_1)p_2(\mathcal{E}_2)=0$. That is, the process effect $\mathcal{M}_1$ is associated with the conclusion that the process is $\mathcal{E}_1$, and the process effect $\mathcal{M}_2$ corresponds to the conclusion $\mathcal{E}_2$. The conditions ${\rm Tr}\mathcal{M}_1\omega_{\mathcal{E}_2}=0$ and ${\rm Tr}\mathcal{M}_2\omega_{\mathcal{E}_1}=0$ imply that ${\rm supp}[\mathcal{M}_1]\perp{\rm supp}[\omega_{\mathcal{E}_2}]$ and ${\rm supp}[\mathcal{M}_2]\perp{\rm supp}[\omega_{\mathcal{E}_1}]$, where $\omega_{\mathcal{E}_1}$ and $\omega_{\mathcal{E}_2}$ are the corresponding process states. Without any doubts the process and state discrimination tasks are closely related and it seems they are almost the same in the sense that process discrimination problems are reducible to state discrimination problems. It is so indeed, but there is still one important difference: PPOVMs are not normalized to identity. As the consequence of this fact we cannot make a conclusion that orthogonality of supports of $\omega_{\mathcal{E}_1}$ and $\omega_{\mathcal{E}_2}$ is the necessary condition for perfect discrimination of processes $\mathcal{E}_1$ and $\mathcal{E}_2$. In fact, there are process states with non-orthogonal supports that can be perfectly discriminated by means of PPOVM.

In particular, consider one of the channels being the identity map ($\mathcal{E}_1=\mathcal{I}$) and second one transforming the whole state space into a fixed pure state $|0\rangle$ ($\mathcal{E}_2=\mathcal{A}_0$). The corresponding operators $\omega_\mathcal{I}=\Psi_+$, $\omega_0=I\otimes|0\rangle\langle0|$, have non-orthogonal supports, i.e. if considered as states they are not perfectly distinguishable. However, there exists a very simple experimental procedure of channel discrimination using the test state $|1\rangle$. Probing the identity the output state is $|1\rangle$, whereas probing the contraction $\mathcal{A}_0$ the output state is $|0\rangle$, i.e. which is orthogonal to $|1\rangle$. A simple measurement (described by POVM elements $|0\rangle\langle0|, I-|0\rangle\langle0|$) tells us whether the channel was $\mathcal{I}$, or $\mathcal{A}_0$. The corresponding PPOVM consists of process effects $\mathcal{M}_\mathcal{I}=|1\rangle\langle1|\otimes(I-|0\rangle\langle0|)$ and $\mathcal{M}_0=|1\rangle\langle1|\otimes|0\rangle\langle0|$, $\mathcal{M}_\mathcal{I}+\mathcal{M}_0=|1\rangle\langle1|\otimes I$. It is straightforward to verify that ${\rm Tr}\mathcal{M}_\mathcal{I}\omega_\mathcal{I}={\rm Tr}\mathcal{M}_0\omega_0=1$.

The characterization of all channels that can be perfectly discriminated is beyond the scope of this Letter. Instead we will provide qualitative arguments why the orthogonality of supports is only sufficient, but not necessary for perfect distinguishability of processes. In a sense any PPOVM can be understood as a normalized POVM (of two qudits states of trace d) if an effect $\mathcal{M}_{\rm extra}=I_d\otimes I_d-\sum_\alpha\mathcal{M}_\alpha=(I_d-\varrho^T)\otimes I_d$ is added. Because of the different normalization of states the probabilities given by the trace relation are normalized to d and we will use the term "rate" instead of "probability". In particular, the rate to get the extra outcome equals ${\rm Tr}\omega\mathcal{M}_{\rm extra}=d-1$ for all process states ω. This "extra" outcome is a fake outcome that is not really measured in the process measurement, but formally it describes the outcome of a process state measurement for which no conclusion is made. That is, the perfect discrimination of processes by means of PPOVM can be understood as the special case of unambiguous discrimination of states via POVM. The inconclusive result is associated with the "extra" outcome added to PPOVM. And in such case the orthogonality of supports is not required. In particular, the process states $\omega_0, \omega_{\mathcal{I}}$ (defined above) can be unambiguously discriminated.

Perfect discrimination of unitary channels. The discrimination of unitary processes was already investigated in several papers [13, 14, 15] and the solution is, in principle, known. In the framework of PPOVMs the unitaries are associated with maximally entangled process states $|\omega_U\rangle = U \otimes I|\Psi_+\rangle$. A pair of unitary channels can be discriminated only if there exists an unambiguous state discriminator of unnormalized maximally entangled states $\omega_U = |\omega_U\rangle\langle\omega_U|$ and $\omega_V = |\omega_V\rangle\langle\omega_V|$. Since $\mathcal{M}_{\rm extra} = (I_d - \varrho^T) \otimes I_d$ it is guaranteed that for inconclusive outcomes $\langle\omega_U|\mathcal{M}_{\rm extra}|\omega_U\rangle = \langle\omega_V|\mathcal{M}_{\rm extra}|\omega_V\rangle = d-1$, hence, the total failure rate reads $P_{\rm failure} = {\rm Tr}[\mathcal{M}_{\rm extra}(p_U\omega_U + p_V\omega_V)] = (d-1)$. If the optimal success rate of unambiguous state discrimination of two maximally entangled states is strictly less than 1 the corresponding unitary channels cannot be perfectly discriminated, because $P_{\rm failure} > d-1$ for all unambiguous discriminators. For pure states the optimal failure rate is given by the absolute value of the scalar product [9], i.e. $P_{\rm failure} = |\langle\omega_U|\omega_V\rangle|$. It follows that whenever ${\rm Tr}U^\dagger V > d-1$ the pair of unitaries cannot be perfectly discriminated.

The problem of perfect discrimination of two unitaries is equivalent to the discrimination of a single unitary channel and the identity channel, U, I. Process effects $\mathcal{M}_U, \mathcal{M}_I$ are related to POVM consisting of two projectors E_I, E_U via the relation $\mathcal{M} = (\mathcal{R}^*_\Omega \otimes \mathcal{I})[E]$, where the test state Ω can be chosen to be pure. Identity does not affect this state and therefore $E_I = |\Omega\rangle\langle\Omega| = \Omega$ is the effect identifying the identity operator. Consequently, $E_U = I - \Omega$ is associated with the unitary channel U. The no-error condition $\langle\omega_U|\mathcal{M}_I|\omega_U\rangle = 0$ and the definition of $\mathcal{M}_I = (A^\dagger_\Omega \otimes I)E_I(A_\Omega \otimes I)$ result in identity

$$0 = |\langle\Psi_+|A^\dagger_\Omega A_\Omega \otimes U|\Psi_+\rangle|^2 = |\langle\Omega|I \otimes U|\Omega\rangle|^2 ,$$

hence, the existence of perfect discrimination is guaranteed if and only if there exists a pure state Ω such that $\langle\Omega|I \otimes U|\Omega\rangle$ vanishes. As it was argued in [13, 14, 15] this is possible if and only if zero belongs to a convex hull of eigenvalues of U distributed on a unit circle in the complex plane. For a general pair of unitaries U, V the problem is reduced to the analysis of eigenvalues of $UV^\dagger$.

In the qubit case each unitary has two eigenvalues, thus the perfect discrimination of a pair I, U requires that $U = e^{i\eta}|\varphi\rangle\langle\varphi| + e^{i(\eta+\pi)}|\varphi_\perp\rangle\langle\varphi_\perp|$, i.e. ${\rm Tr}U = 0$. Consequently, qubit unitary channels U, V can be perfectly discriminated if and only if they are orthogonal. However, such statement no longer holds for qudits and as it was shown in [13] for an arbitrary pair of (qudit) unitary processes U, V there exists a finite n such that $U^{\otimes n}$ and $V^{\otimes n}$ can be perfectly discriminated, i.e. the distinguishability is not equivalent to the orthogonality.

Conclusions. The goal of this paper has been to introduce a mathematical framework for the description of measurements on quantum processes. This idea led us to the definition of the so-called process POVM (PPOVM) defined as a collection of effects $\mathcal{M}_1, \dots, \mathcal{M}_n$ ($0 \leq \mathcal{M}_j \leq I_d \otimes I_d$), such that $\sum_j \mathcal{M}_j = \varrho^T \otimes I_d$, where ϱ is an arbitrary single qudit state and ^{T} denotes its transposition. In this framework the channels are associated with positive operators of $d \times d$ system with trace equal to d. An arbitrary process measurement can be described by PPOVMs and we have shown that also each PPOVM can be implemented experimentally although the experimental realization is not unique. This ambiguity is one of important open problems left for further investigation.

The framework of PPOVMs provides us with a powerful tool for different process estimation problems [13, 14, 15, 16, 17, 18], mostly in answering the optimality questions. Moreover, the concepts originally developed for POVMs can be directly translated and applied for PPOVMs as it was demonstrated in the case of informational completeness of PPOVMs. Using the PPOVM framework we have argued that perfect discrimination problems for quantum channels are equivalent to very specific unambiguous discrimination problems of quantum states.

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