

Quantum homogenization and state randomization in semi-quantal spin systems

Mátyás Koniorczyk*

*Institute of Physics, University of Pécs, H-7624 Pécs, Ifjúság útja 6, Hungary and
Research Institute for Solid State Physics and Optics,
Hungarian Academy of Sciences, H-1525 Budapest, P.O. Box 49., Hungary*

Árpád Varga

Institute of Physics, University of Pécs, H-7624 Pécs, Ifjúság útja 6, Hungary

Peter Rapčan and Vladimír Bužek

*Research Center for Quantum Information, Institute of Physics,
Slovak Academy of Sciences 84511 Dúbravská cesta 9, Bratislava Slovakia*

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We investigate dynamics of semi-quantal spin systems in which quantum bits are attached to classically and possibly stochastically moving classical particles. The interaction between the quantum bits takes place when the respective classical particles get close to each other in space. We find that with Heisenberg XX couplings quantum homogenization takes place after a time long enough, regardless of the details of the underlying classical dynamics. This is accompanied by the development of a stationary bipartite entanglement. If the information on the details of the motion of a stochastic classical system is disregarded, the stationary state of the whole quantum subsystem is found to be a complete mixture in the studied cases, though the transients depend on the properties of the classical motion.

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I. INTRODUCTION

Quantum mechanics provides the best known description of microscopic physical systems. In most of the cases, however, the system to be described is a part of a larger system, its environment, and these interact. As a result there are correlations, usually both of classical and quantum nature, developing between them. From the system's point of view this results in a modification of the behavior which usually leads to the loss of the very quantum mechanical features. This phenomenon is referred to as a decoherence.

The study of decoherence leads to the understanding of mesoscopic and macroscopic systems, that is, the emergence of classical behavior. In addition, decoherence constitutes the main obstacle of exploiting the quantum mechanical nature of system in applications such as quantum information processing. The mathematical description of decoherence aims at the derivation of the system's dynamics without the detailed description of its environment. In many of the cases this is done based on the full description of the system and its environment, which are realistic physical systems, by applying several approximations. In quantum optics, for instance, the decoherence of a single mode of the electromagnetic field is described by taking into account the interaction of the infinite set of other field modes by perturbation theory and the Born and Markov approximations. The result is

a Lindblad master equation for the density operator of the system.

Besides of the study of "real" physical systems, the introduction of simplified models is of some use. These are the microscopic models of decoherence. In this case the aim is to give a detailed understanding of the process of decoherence via a more detailed description of the system and the environment, which is feasible either analytically or numerically. Some of such models address the decoherence of a one or a few two-level systems (quantum bits) in the presence of additional quantum bits, that is, a spin-bath environment. Besides its simplicity, this choice is motivated by the relevance in quantum computing and the relation to real-world scenarios such as two-level atoms or solid-state systems. Such models bear experimental relevance in some cases and many of them can be implemented with cold bosons on optical lattices, as pointed out by Rossini et al. [1] recently.

There are several analytically solvable and physically realistic models of spin baths, such as e.g. the Tessieri-Wilkie model [2], in which the role of the interactions amongst the quantum bits of the environment is apparent. The issue of the interaction between the environment spins is studied in detail by Dawson et al. [3], who point out the role of the monogamy of entanglement as described by the Coffman-Kundu-Wootters inequalities [4, 5] in the decoherence features. In a very recent publication Camalet and Chitra [6] have studied the decoherence of a qubit due to the presence of random interactions with a spin bath. They point out the non-Markovian behavior at low temperatures as well as the role of the intra-environment interactions. Piñeda

*Electronic address: kmatyas@gamma.ttk.pte.hu

et al. [7] have studied the decoherence of two non-interacting qubits in the framework of random matrix theory in very detail.

An important class of microscopic models is that of quantum homogenization [8, 9]. An environment consisting of many qubits is considered, each in the same single-particle state ϱ (Though the global state of the environment may be a pure one assuming the presence of initial correlations in the environment.). The system quantum bit is initially in an arbitrary state σ . The interaction between the quantum bits is defined by a two-qubit quantum gate, the partial swap gate generated by a Heisenberg-XXX (that is, the isotropic Heisenberg) interaction. In its simplest formulation quantum homogenization is a collision-type model: the system qubit interacts with the reservoir qubits one-by-one. (Note that a collision-based model with a qubit as a system and a classical field as a reservoir was used by Diósi et al. [10] recently to reveal the relation between thermodynamical and information theoretic entropy.) The aim of the model is to demonstrate that in this framework the state of the system quantum bit and each reservoir quantum bit will evolve infinitesimally close to each other and also to the initial state ϱ of the reservoir. This process is rather similar to the thermalization in classical thermodynamics: the temperature of a small system will become close to the original temperature of a large environment, while this latter does not change significantly. For qubits the single-qubit density operator plays the role of the "temperature", and qualitatively the same situation arises. The evolution of the system qubit is Markovian described by a discrete semigroup which is a stroboscopic image of a continuous-time Markovian evolution. It is also shown that a distributed pairwise entanglement arises in the system which tends to a stationary value and saturates the Coffman-Kundu-Wootters inequalities [11]. The process is basis independent (or covariant) due to the nature of the partial swap operation, thus it works for arbitrary initial system and reservoir states. The partial swap is the only gate with this property.

The model of quantum homogenization was further generalized to include pairwise interactions within the reservoir. Assuming that at any time step a randomly chosen pair of qubits (system or reservoir) can interact, one obtains a Markovian evolution of the whole system again. However due to the intra-environment interactions, the evolution of the single qubits is not Markovian anymore. It was found that for an actual sequence of interactions, the state of each qubit will fluctuate around the one obtained by the collision-based homogenization. The fluctuations appear to decrease with the growing size of environment. If one simulates many actual evolutions and constructs an effective density operator for each time step, this will show "genuine" homogenization. The same holds for time-averaged states. The model with a very small (i.e. two-qubit) reservoir was recently studied by Benenti and Palma [12] who have found that irreversibility can emerge even in this case, after time-averaging.

The stochastic homogenization model is a special case of a more general scenario termed as a semi-quantal spin gas model. In this case a set of classical particles (e.g. classical atoms) is considered to move according to a certain model, e.g. an ideal Boltzmann gas. Each particle has an internal degree of freedom which is considered to be quantum mechanical. This can be a half spin or two possible hyperfine states, termed as a quantum bit in what follows. The quantum mechanical part of the system is thus multipartite, consisting of the quantum bits attached to the respective classical particles. This essentially models any system in which quantum bits are attached to addressable and distinguishable entities which are allowed to move in space for some reason. The state of the quantum system has no effect on the classical motion whatsoever. On the other hand, if some of the particles fulfill a certain *collision condition*, e.g. they collide in the case of the Boltzmann gas model, a prescribed quantum gate acts on the respective qubits.

Semi-quantal spin gases were studied by Briegel et al. [13]. Besides the Boltzmann gas as an underlying classical model, they have considered a lattice gas with on-site exclusion, which shows a spatially correlated behavior. In this case the particles are located on a discrete lattice and they can hop to the neighboring empty sites stochastically. The collision condition is that the two particles should be at neighboring sites. In Ref. [13] an Ising coupling is considered between the respective quantum bits. For a particular evolution of the microstate of the classical system one can calculate the evolution of the quantum mechanical one, given its initial state. In the case of the Boltzmann gas, however, only the macrostate of the classical system is known, while the lattice gas moves stochastically by nature. Thus given a pure quantum state as initial condition, one obtains a pure state depending on time which has random parameters. Due to the advantageous properties of the pairwise Ising couplings (e.g. they commute), for certain initial states it is possible to calculate the density operator of an arbitrary subsystem of the rather big quantum mechanical system. In Ref. [13] the authors study various aspect of the arising non-Markovian evolution including decoherence and entanglement behavior.

Motivated by the above summarized results we intend to consider further microscopic models of decoherence and address additional issues. In particular we will consider the following classical models:

- completely random pairwise interactions,
- classical particles moving in a three-dimensional box, colliding elastically with each other and the wall (the "billiard ball" model), the qubits interact if the respective particles collide,
- discrete sites along a line, on which particles can hop to the neighboring empty sites (one-dimensional lattice gas), the qubits interact if the respective particles are next to each other.

Our one-dimensional lattice gas can be imagined as a spin chain with vacancies where the classical particles carrying the spins can jump to a vacancy. Since a similar dynamics is present in any solid for thermodynamical stability reasons, we believe that this model might bear some experimental relevance. From the point of view of decoherence models, we are able to describe a very strongly self-interacting reservoir. We will consider the Heisenberg-XX and the Ising Hamiltonians as well. We perform computer simulations of the system to deduce our results.

First we will consider a given particular evolution of the classical system and Heisenberg-XX type couplings. This choice of the Hamiltonian is motivated by the previous studies of quantum homogenization. In the case of the billiard ball model the classical trajectory is uniquely determined by the initial conditions for the classical system while for stochastic dynamics it is generated by random deviates. We seek for quantum homogenization and find that it indeed appears for all the classical dynamics after a time long enough. The main conclusion will be that the characteristics of the classical evolution do not play a relevant role in the homogenization, that is, in the state of a single-qubit subsystem. We also show that there is a distributed bipartite entanglement in the system.

Another possible attitude in the case of the stochastic evolutions is that one ignores the information on the actual classical motion. This loss of information results in the increasing entropy of the state of the quantum mechanical subsystem. We will study the so-arising decoherence in detail, both for Ising and Heisenberg-XX couplings. We show that it leads to a completely mixed state in the accessible part of the Hilbert space rather quickly, for all the considered models.

This paper is organized as follows: in Section II we consider individual evolutions and study the behavior of quantum homogenization. In Section III we consider the dynamics of the effective density matrix of the whole quantum subsystem, built up as the convex combination of the system's state in case of different particular evolutions. In Section IV the results are summarized and the conclusions are drawn.

II. INDIVIDUAL TRAJECTORIES: FROM SPIN-CHAINS TO HOMOGENIZATION MODELS

In this Section we seek for generalizations of the quantum homogenization models. In the already studied versions of these, a set of qubits is considered, one of which plays the role of the system while the others play the role of the environment. The environment qubits are initially in a state ϱ , while the system qubit is in another arbitrary state σ . The interactions are assumed to be bipartite: in each time step a chosen pair of two quantum bits interacts via a partial swap quantum gate which can be written, up to an irrelevant phase factor, as

$$U_{\text{pswap}}(\eta) = \exp\left(-i\eta\hat{H}^{(\text{XXX})}\right), \quad (1)$$

where

$$H^{(\text{XXX})} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z \quad (2)$$

is the Heisenberg XXX interaction. Depending on the model the reservoir qubit interacts with the reservoir particles one-by-one once, or randomly chosen pairs will interact. Finally, all the single-qubit density operators, describing either the state of the system qubit or any of the reservoir qubits will be approximately equal: the state of the system qubit will be “diluted” in its environment.

Instead of $H^{(\text{XXX})}$ let us consider a pairwise interaction described by the Heisenberg XX Hamiltonian

$$H^{(\text{XX})} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y. \quad (3)$$

The motivation for this choice is twofold. First, these kind of couplings play an important role in quantum state transfer in spin systems [14]. Secondly, the evolution generated by this pairwise Hamiltonian between any configuration of pairs has an invariant subspace spanned by the vectors

$$|\underline{k}\rangle = |0_1 0_2 \dots 0_{k-1} 1_k 0_{k+1} \dots 0_N\rangle, \quad (4)$$

thus with the initial state in Eq. (6), that is, $|\underline{1}\rangle$, the state of the system at time t will be a superposition of the vectors in Eq. (4) with different values of k . (Throughout this paper we will denote the elements of the computational basis on the Hilbert-space of a qubit by $|0\rangle$ and $|1\rangle$, with the convention $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$.) In this basis the Hamiltonian in Eq. (3) has a rather intuitive form. If the k -th and l -th qubit is coupled, the only nonzero matrix elements are $\langle \underline{k} | H^{(\text{XX})} | \underline{l} \rangle = \langle \underline{l} | H^{(\text{XX})} | \underline{k} \rangle = 2$. We assume for the moment that in each time step a randomly chosen pair of quantum bits interact via this coupling, thus we apply the gate

$$U_{XX}(\eta) = \exp\left(-i\eta\hat{H}^{(\text{XX})}\right). \quad (5)$$

As an initial state let us consider

$$|\Psi(t=0)\rangle = |\underline{1}\rangle. \quad (6)$$

where the first qubit is considered to model the system, while the others constitute the reservoir.

If the state of the system is within the subspace in argument, described by the density matrix $\varrho_{\underline{k}, \underline{k}'}$ expressed in the basis in Eq. (4), the density operator of the k -th particle is diagonal in the computational basis and it reads

$$\varrho^{(k)} = \text{diag}(1 - p_k, p_k), \quad \text{where } p_k = \varrho_{\underline{k}, \underline{k}}. \quad (7)$$

Here p_k is the probability of finding the k -th qubit in the state $|1\rangle$ in a projective measurement in the computational basis. Another quantity to consider is the concurrence, which is used prevalently to quantify the entanglement of two quantum bits:

$$C_{\underline{k},\underline{k}'} = \min(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_3), \quad \lambda_i = \text{eig}\left(\sqrt{\sqrt{\rho^{(k,k')}}} \tilde{\rho}^{(k,k')} \sqrt{\rho^{(k,k')}}\right), \quad (8)$$

where $\tilde{\rho}^{(k,k')} = \sigma_y \otimes \sigma_y \rho^{(k,k')*} \sigma_y \otimes \sigma_y$ is the Wootters-tilde and $\rho^{(k,k')*}$ is the transpose of $\rho^{(k,k')}$, the density matrix of qubits k and k' , in the computational basis. In the subspace spanned by the vectors in Eq. (4) this is simply twice the modulus of the coherences (off-diagonal matrix elements) of the system's density matrix,

$$C_{\underline{k},\underline{k}'} = 2|\rho_{\underline{k},\underline{k}'}|, \quad (9)$$

as it can be verified by calculating the bipartite density matrix and substituting it to Eq. (8).

Let us return to the evolution $|\Psi(t)\rangle$ with the initial condition $|\Psi(t=0)\rangle = |\underline{1}\rangle$, under the application of the quantum gate in Eq. (5) on a randomly chosen pair of qubits in each step. As a degree of inhomogeneity of the system we consider the square of the standard deviation of the p_k -s in Eq. (7),

$$\sigma^2(t) = \langle p_k^2 \rangle_k - \langle p_k \rangle_k^2, \quad (10)$$

where $\langle \dots \rangle_k$ stands for averaging over the particles. The smaller it is, the more homogeneous the system will become from the point of view of single particles. This is plotted in Fig. 1 (Upper figure, curve A). Though the evolution shows some stochastic features due to the nature of the interaction sequence, after a time long enough the system becomes homogeneous. The mechanism of the homogenization is similar to the already studied models of stochastic homogenization.

It is also interesting to consider the quantum correlations generated during the evolution. In Refs. [4, 15] it was shown that the states which the initial state $|\underline{1}\rangle$ evolve into during the dynamics generated by Heisenberg XX couplings have bipartite entanglement only. Let us quantify this in terms of the sum of the concurrences

$$C_{\text{tot}}(t) = \sum_{k < k'} C_{\underline{k},\underline{k}'}(t), \quad (11)$$

which therefore provides full information on the net amount of quantum correlations. As it can be seen in Fig. 1 (Lower figure, curve A), this grows and then fluctuates around a stationary value. This is how the decoherence of single qubits arises in this system.

Spin chains with vacancies

Let us consider spins aligned along a chain with periodic boundary conditions. Let the interaction be defined at the Hamiltonian level as the sum of nearest-neighbor XX interactions, with periodic boundary conditions as

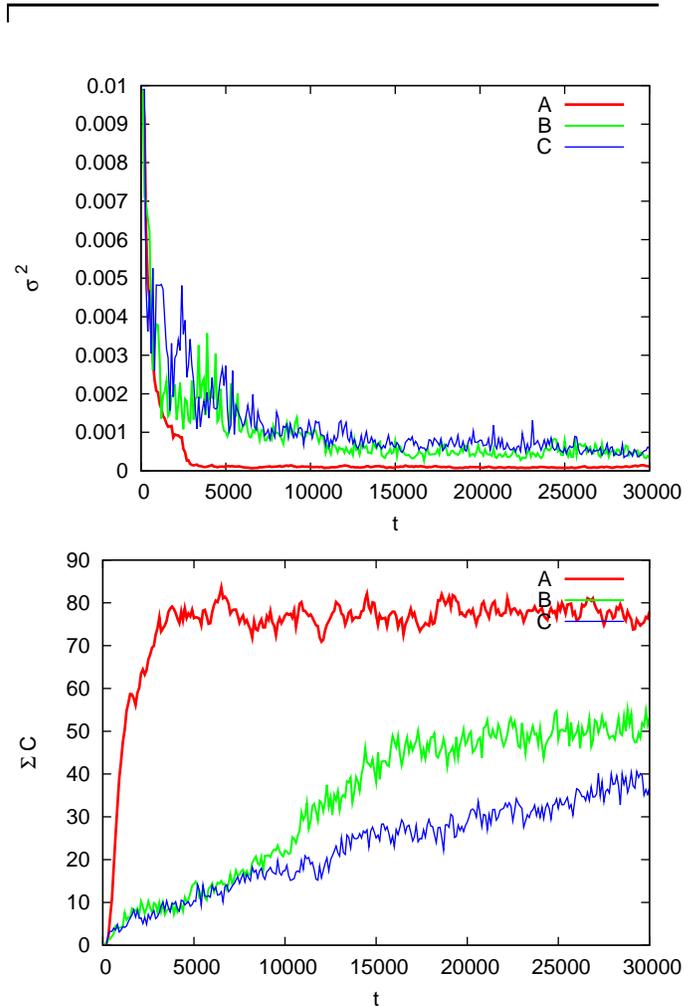


FIG. 1: (color online) Evolution of the degree of inhomogeneity in Eq. (10) (upper figure) and the sum of the pairwise concurrences in Eq. (11) (lower figure) for different scenarios. A: random pairwise interactions, B: one-dimensional diffusive gas with 150 sites, C: one-dimensional diffusive gas with 200 sites. Only the values at every 100th time step are plotted, and the data are connected with lines to guide the eye. There are $N = 100$ particles considered, the interaction strength parameter in Eq. (5) and Eq. (13) is $\eta = 0.1$. Repeated calculations of the stochastic models give figures with different details but of the same character.

usual in statistical physics:

$$H^{(\text{chain})} = \sum_k H_{k,k+1}^{(\text{XX})}. \quad (12)$$

Consider the dynamics generated by this Hamiltonian. To be comparable with the discrete-time models, we con-

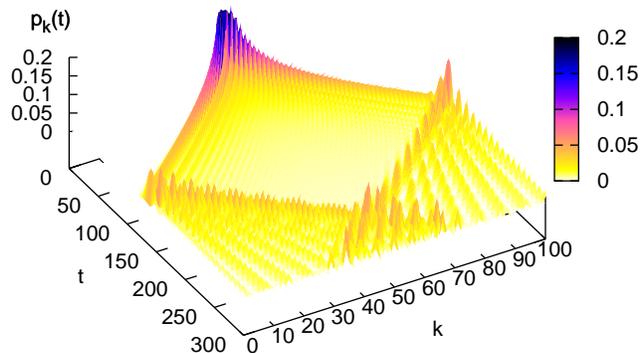


FIG. 2: (color online) “Propagation of the disturbance” in a usual Heisenberg XX chain of 100 quantum bits, with periodic boundary conditions. Initially the qubit in the middle is in the state $|1\rangle$ while all the others are in the state $|0\rangle$ (that is, initial state is $|\underline{50}\rangle$). The probability $p_k(t)$ in Eq. (7) of finding the k -th particle in the state $|1\rangle$ is plotted. The values are truncated at $p_k(t) = 0.2$ to make the detailed structure more visible. The interaction strength in Eq. (13) is $\eta = 0.1$. Notice the appearance of edge effects around the 100-th time step.

sider the repeated application of

$$U(\eta) = \exp\left(-i\eta H^{(\text{chain})}\right) \quad (13)$$

to the initial state which is again considered to be $|\underline{1}\rangle$. This results in a kind of “propagation of the disturbance”, as it can be seen in Fig. 2, where the $p_k(t)$ -s of Eq. (7) are plotted against k and t . As the qubits are aligned to a chain in this case, the value of k describes a kind of spatial position of the qubit in the chain. The propagation is a kind of broadening, but it shows interference stripes reflecting the quantum mechanical nature of the system.

The question arises if a kind of homogenization effect appears even in this very coherent scenario. Therefore we consider again the time evolution of the degree of inhomogeneity in Eq. (10), which is plotted in Fig 3. It appears that the system becomes homogeneous in a very short time from the point of view of the single-particle density operators. Though the monotonicity of the curve is slightly disturbed by the edge effects due to the finite size of the system (compare Figs. 2 and 3), even these do not significantly alter the behavior. This kind of coherent homogenization has a rather clear interpretation: since the initial “disturbance” causes a broadening and the state has to remain normalized, the disturbance is diluted along the chain like water waves generated by a single drop. Note that since there are many particles taking part in the process, the homogenization is much faster than in the case when it is due to pairwise interactions. In Fig 3 we have also plotted the time evolution

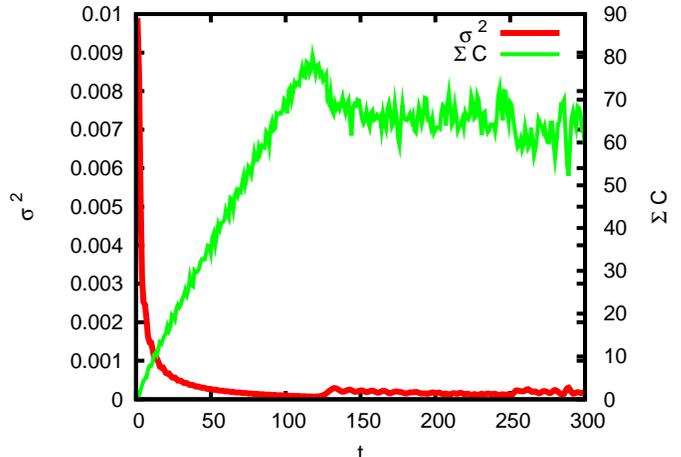


FIG. 3: (color online) Evolution of the degree of inhomogeneity in Eq. (10) and the sum of the pairwise concurrences in Eq. (11) for the same situation as in Fig. 2, for a Heisenberg- XX chain.

of the net concurrence defined in Eq. (11). Note that it grows monotonously till the edge effects appear.

In order to interpolate between the stochastic homogenization and the coherent one, we consider now a one-dimensional semi-quantal spin gas model with random classical dynamics. Let us consider N classical particles arranged on a chain of length L with periodic boundary conditions. A configuration of the particles is given by specifying their position. The sites are labeled by $l = 1 \dots L$. Thus the specification of a position $l(k)$ for each particle $k = 1 \dots N$ defines a configuration of the classical part of the system. Initially the system is described by a fixed configuration $(l(k))(0)$. The evolution of the system is stochastic and we consider discrete time steps $t = 0, 1, 2, \dots$. The evolution is defined in terms of a probability transition matrix $P_{(l(k))(t)|(l'(k))(t+1)}$ which gives the probability that a configuration $l(k)$ at time t evolves into $l'(k)$ at time $t + 1$. We will consider Markov chains only, thus the matrix P will be independent of time and the previous states of the system,

$$P_{(l(k))(t)|(l'(k))(t+1)} = P_{(l(k))|(l'(k))} \quad (14)$$

for all t . Actually we will consider the following evolution: at each time step a random particle and a random direction is chosen. If the target site is empty, the particle moves there. In this case the probability transition matrix in Eq. (14) is symmetric, thus it describes a doubly stochastic Markov chain. These are known to have a single stationary probability distribution which is uniform. Starting the system in an arbitrary configuration, after a time long enough all the configurations can be found with equal probability. Let us remark that though the evolution of the system is Markovian, any of its subsystems evolve in a non-Markovian way as the rest of the system acts as a memory.

The quantum mechanical part of the system is consid-

ered to be in a pure state $|\Psi_0\rangle$ at $t = 0$. In each (discrete) time step there is an interaction Hamiltonian which depends on the classical configuration. In the case of the discrete classical model we have

$$H(t) = \sum_{\langle k, k' \rangle_t} H^{(k, k')}, \quad (15)$$

where $\langle k, k' \rangle_t$ means that the particles k and k' are neighbors at time t , while $H^{(k, k')}$ is a chosen bipartite qubit interaction, in particular the Heisenberg-XX Hamiltonian in Eq (3) in this Section. Note that in this case the Hamiltonian in Eq. (15) is simply twice the adjacency matrix of the actual configuration of the particles. The evolution is given by

$$|\Psi(t+1)\rangle = \exp(-i\eta H(t))|\Psi(t)\rangle, \quad (16)$$

where η is the parameter describing the coupling strength. Though the particular value of η affects the fluctuations of the studied quantities as well as the speed of the transients, it did not affect the main features of the behaviour in our simulations. Therefore we present results where η has a fixed arbitrary value. However, special values of η for certain Hamiltonians might cause special behaviour, which would be a possible topic of further studies. The so defined evolution, which is the subject of the present study, is a sequence of unitary evolution steps where the actual unitary step is determined by the state of the underlying classical Markov chain.

Looking at the evolution of $p_k(t)$ in Eq. (7) which is plotted in Fig. 4 we find that there is again a propagation kind of phenomenon, but it reflects the randomness of the classical motion. The disturbance may propagate in the cluster but it is reflected by the vacancies. These reflections slow down the propagation and make it rather noisy. If we consider the evolution of Eq. (10), which is plotted in Fig. 1 (Upper figure, curves B and C with different particle densities), we find that even though it shows significant random oscillations, it has a clear tendency of homogenization. Also, the behavior of the net concurrence shows a growing tendency, see Fig. 1 (Lower figure, curves B and C), though it is rather oscillatory and the growth depends highly on the details of the classical evolution.

The billiard-ball model

Next we turn our attention to another similar model. Assume that each of the quantum bits is assigned to a (ball-shaped) particle of a finite diameter and mass moving classically in three dimensions. For each evolution, a classical initial condition for the positions \mathbf{r}_k and momenta \mathbf{p}_k is generated, and the classical dynamics of the system is solved. The initial positions are drawn from a uniform (within the three-dimensional region given by the containing box dimensions reduced by the perimeter of the balls on each end of the three coordinate intervals) distribution for the ball origins in three dimensions.

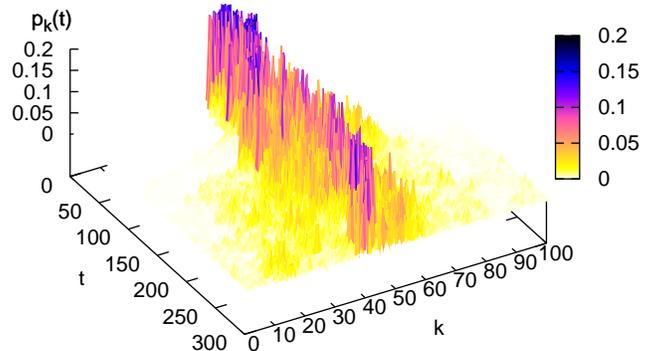


FIG. 4: (color online) “Propagation of the disturbance” in the one-dimensional lattice gas model with 100 particles at 150 sites, $\eta = 0.1$. The initial state is $|\underline{50}\rangle$, that is, all the qubits are in the state $|0\rangle$ except for the middle one, which is in the state $|1\rangle$. Note that the probability is plotted against the particle’s index k . Since the classical evolution preserves the order of the particles, this reflects spatial order of the spins even though their actual position changes.

Should the generated position of a particle be such that the ball representing the particle overlaps with any of previously generated ones, the position is discarded and a new one is generated. The distribution of momenta is Boltzmann with zero mean and a fixed standard deviation. Two particles (or a particle and the containing box) collide elastically whenever a contact of their boundaries takes place. The collisions (i.e. the appropriate changes of the colliding particles’ velocities) are instantaneous. The mass of the containing box is infinite, i.e. its position does not vary upon particle-box collisions.

The quantum system is initially in the state in Eq. (6) again. During the evolution a pair of qubits interacts via the gate in Eq. (5) iff the two respective billiard balls collide. This is a kind of semi-quantal spin gas model.

We again look at whether homogenization features are present in this model and find qualitatively the same behavior as in the case of the spin chain with vacancies model. Sequences generated by gas dynamics and random ones yield very similar results (see Fig. 5).

We can conclude that quantum homogenization at a single particle level appears in a variety of models based on the same bipartite interaction, albeit with very different underlying classical dynamics. These models range from the random bipartite interactions where each particle can interact directly to one-dimensional diffusive lattice gases or regular spin chain where there is no direct interaction of the particles.

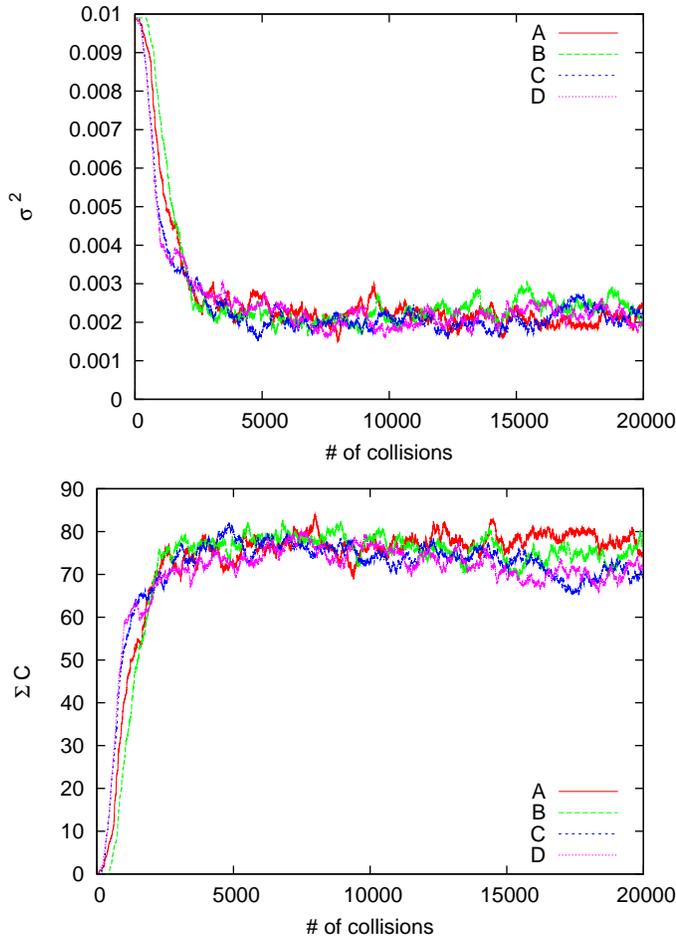


FIG. 5: (color online) Evolution of the degree of inhomogeneity in Eq. (10) (upper figure) and the sum of the pairwise concurrences in Eq. (11) (lower figure) for two billiard-ball collision sequences (A, B) and for two random collision sequences (C, D). $N = 100$ ball-shaped particles of diameter one (arbitrary units) in a cube box with dimensions $150 \times 150 \times 150$ (arbitrary units). The mass of the particles is one (arbitrary units). The initial velocities (and hence also the velocities at later times) are normally distributed with zero mean and standard deviation $\sigma = 0.32$ (arbitrary units). The interaction strength parameter in Eq. (5) and Eq. (13) is $\eta = 0.1$.

III. STATE RANDOMIZATION DUE TO STOCHASTIC CLASSICAL DYNAMICS

In this Section we consider the diffusive lattice gas model already described, and investigate the effect of the loss of information on the stochastic evolution of the classical system.

We simulate the evolution of the system on a computer with a fixed classical and quantum initial condition. Unlike in the previous Section, we run the simulation now s times independently, which yields different $|\Psi_s(t)\rangle$ evolutions, depending on the details of the evolution on the classical subsystem. We will refer to this as the s -th trajectory. Simulating N_{traj} trajectories we construct the

density operator

$$\varrho(t) = \frac{1}{N_{\text{traj}}} \sum_{s=1}^{N_{\text{traj}}} |\Psi_s(t)\rangle \langle \Psi_s(t)|. \quad (17)$$

Its interpretation is as follows: if we disregard all the information we have on the classical evolution of the system we can describe its state by a density operator

$$\tilde{\varrho}(t) = \sum_{s'} p_{s'}(t) |\Psi_{s'}(t)\rangle \langle \Psi_{s'}(t)|, \quad (18)$$

where $|\Psi_{s'}(t)\rangle$ -s are all the possible evolutions of the system and $p_{s'}(t)$ is the probability of obtaining the given evolution. We expect that for $N_{\text{traj}} \rightarrow \infty$, $\varrho(t)$ will converge to the exact $\tilde{\varrho}(t)$ stochastically. In practice we determine the required number of simulations N_{traj} empirically by increasing it till the further increase does not decrease the variance of the density operator elements. The evaluation of (17) is obviously suitable for parallel computing. Thus we have carried out our simulations on a parallel computer using GNU Octave and its MPI toolbox [16].

We consider the evolution of the von Neumann entropy of the whole system,

$$S(t) = -\text{Tr} \varrho(t) \log_2 \varrho(t) \quad (19)$$

which describes its mixedness. In addition, we will consider bipartite entanglement as measured by concurrence in Eq. (8)

First we consider the XX Hamiltonian in Eq. (3) as the bipartite interaction. As all the Hamiltonians in Eq. (15) commute with the total z component of the spin, the subspace spanned by the vectors in Eq. (4) will be invariant in this case too. Consider the state $|\underline{1}\rangle$ as an initial state. In Fig. 6 we plot the time evolution of the von Neumann entropy of the state of the whole system as defined in Eq. (17), i.e. the density operator in the absence of information on the collision history. It appears that after some transient which depends on the features of the underlying dynamics (e.g. its entropy rate), the system will be in the complete mixture of the given subspace,

$$\varrho_{\text{mix}} = \frac{1}{N} \sum_{k=1}^N |k\rangle \langle k|, \quad (20)$$

which is reflected in the maximum value of von Neumann entropy, i.e. $\log_2 N$. Note that this already appears if there is a single unoccupied site in the system, thus the effect of the classical noise is rather remarkable. Thus the situation of the microcanonical ensemble is arising in this model: the state of the system is an equal-weight mixture of all of its possible states. This implies that there is a vanishing stationary entanglement, of course. If, however, we calculate the average net concurrence for each trajectory and calculate the average afterwards,

$$\bar{C}(t) = \frac{1}{N_{\text{traj}}} \sum_{s=1}^{N_{\text{traj}}} C_{\text{tot}}(|\Psi_s(t)\rangle), \quad (21)$$

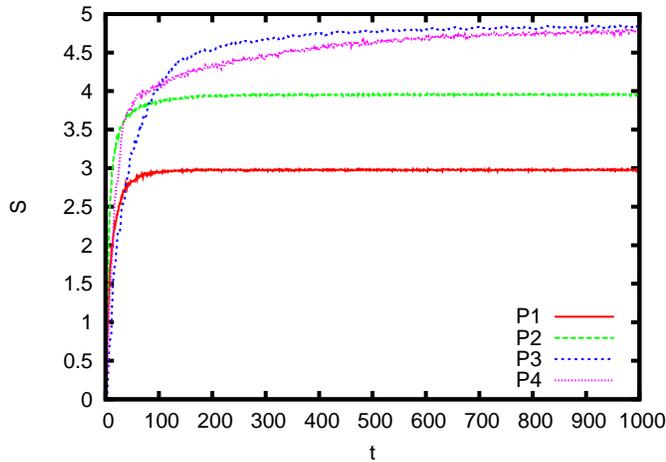


FIG. 6: (color online) Time evolution of the von Neumann entropy of the density operator in a Heisenberg-XX spin gas with various parameters. The interaction strength is $\eta = 1.0$ for all the curves, 3000 trajectories were simulated in each case. P1: 8 particles, 16 sites; P2: 16 particles, 20 sites; P3: 32 particles, 33 sites; P4: 32 particles, 40 sites.

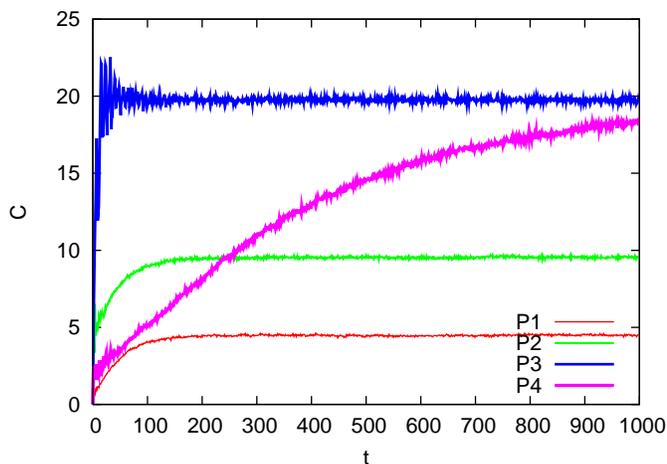


FIG. 7: (color online) The average concurrence in Eq. (21) in a one-dimensional lattice gas with Heisenberg XX interactions. The interaction strength is $\eta = 1.0$ for all the curves, 3000 trajectories were simulated in each case. P1: 8 particles, 16 sites; P2: 16 particles, 20 sites; P3: 32 particles, 33 sites; P4: 32 particles, 40 sites.

(C.f. Eq. (11)), we find that there is a nonvanishing concurrence, see Fig. 7. This is in accordance with our expectations: as we have found in the previous Section, homogenization takes place in each individual trajectory which leads to a nonvanishing concurrence. If, however, we drop the information on the classical evolution, this resource will not be accessible. Finally, it is also interesting to plot the degree of inhomogeneity in Eq. (10) of the density operator of the system calculated according to Eq. (17). This is done in Fig. 8. It appears that the homogenization becomes smooth due to the averaging over

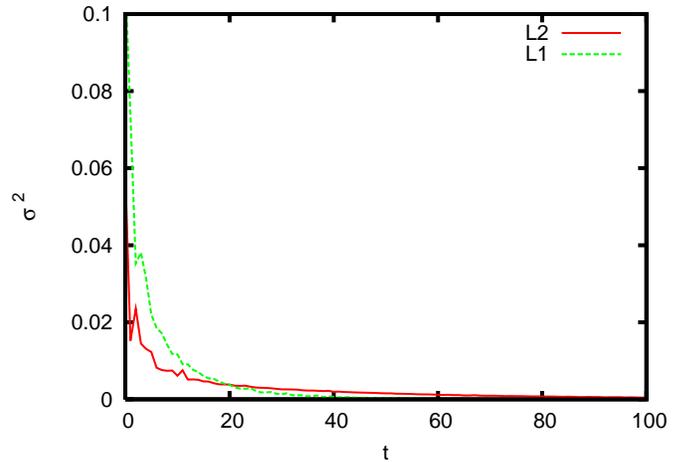


FIG. 8: (color online) The inhomogeneity in Eq. (10) of the density operator of the system calculated according to Eq. (17), in a one-dimensional lattice gas with Heisenberg-XX interactions. There were 3000 trajectories simulated to obtain the density operator. The coupling strength was $\eta = 1$. Parameters of the curves: L1: 8 particles, L2=16 particles, 40 sites in both cases.

several trajectories. It is interesting to point out here that if we consider an initial state which is a superposition of two states from different invariant subspaces, the stationary entropy will be initial state dependent. As we have verified via simulations as well, if one considers an initial state

$$|\Psi(t=0)\rangle = C_0|\underline{0}\rangle + C_1|\underline{1}\rangle, \quad (22)$$

where $|\underline{0}\rangle = |0\rangle^{\otimes N}$, the stationary entropy will be

$$S_{\text{stationary}} = H\left(|C_0|^2, \frac{|C_1|^2}{N}, \frac{|C_1|^2}{N}, \dots, \frac{|C_1|^2}{N}\right), \quad (23)$$

where $H(\cdot)$ is the Shannon entropy function, and the argument $\frac{|C_1|^2}{N}$ appears N times. This rather expectable effect is very quantum mechanical: the conservation laws produce invariant subspaces and the stationary entropy of the microcanonical distribution depends on the relation of a given initial state to this subspace structure. The situation is conceptionally similar to the phase-like transitions in the Dicke model [17].

Finally, let us first consider the Ising interaction

$$H_{\text{Ising}} = \sigma_x \otimes \sigma_x, \quad (24)$$

as the bipartite interaction for Eqs. (15) and (16). It would dynamically generate cluster states from the products of the eigenstates of the σ_z operators in a usual spin chain (i.e., with no empty sites or classical motion). The quantum system is the product state $|100\dots 0\rangle$ initially, which would evolve into a cluster state if the lattice was saturated. In this case the possibility of numerical simulation is limited by the exponentially growing size of the

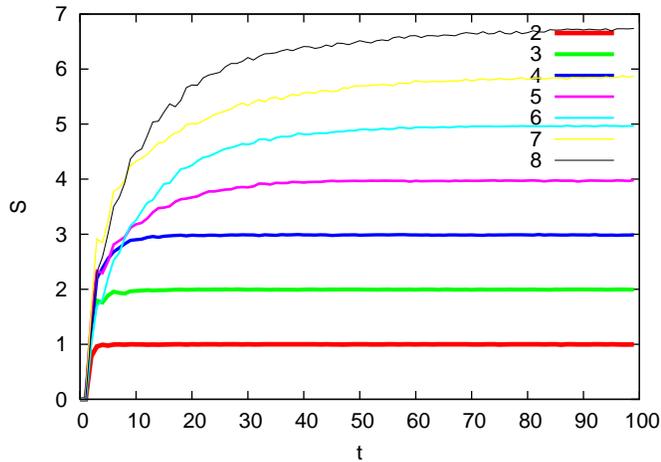


FIG. 9: (color online) Evolution of the entropy of the state of an Ising spin gas for the quantum initial state $|100\dots 0\rangle$, with a fixed random uniform classical initial condition. The interaction strength η is set to 1. The parameter of the curves is N , the number of particles, the length of the chain is chosen to be $L = 3N$ with periodic boundary conditions. 3000 trajectories were simulated to obtain the figure.

Hilbert space. However, it appears according to the simulations that the entropy tends to converge to $N - 1$ in each case. Looking at the density matrix in the computational basis it will be diagonal and it describes a complete mixture in a subspace spanned by the vectors of the computational basis with odd Hamming weights, which has a dimensionality half of the whole Hilbert space. It is easy to see that the subspaces spanned by the computational basis vectors with even or odd Hamming weights are invariant subspaces of the pairwise Ising interactions.

It is also interesting to compare the (classical Shannon) entropy of the probability distribution in the diagonal of the density matrix in the computational basis with the von Neumann entropy of the density operator. Since this latter is the infimum of the entropies of the diagonal of the density matrices taken over all the possible bases, if these two are close to each other the density matrix is almost diagonal. The evolution of these entropies is plotted in Fig. 10. Obviously since the density operator tends to a complete mixture in a subspace, the stationary values will be the same. If there are just a few vacancies in the system, the two entropies differ. If, however, there is a relevant number of unoccupied sites in the system, one finds that the density operator during the whole evolution will be “almost diagonal” in the computational basis in the sense that the entropy of the diagonal of the density matrix will be almost the same as the von Neumann entropy of the density operator. Let us remark that the same can be observed for the Heisenberg-XX interaction in the basis in Eq. (4). Since these bases are built up entirely from product states, one can conclude that the evolution for a large number of empty sites is very similar

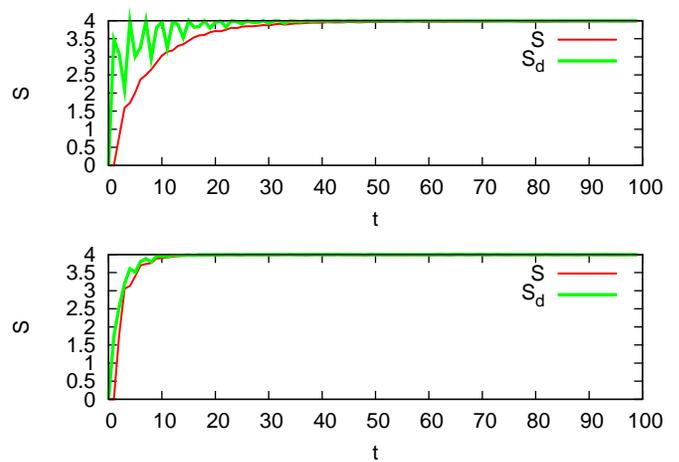


FIG. 10: (color online) Evolution of the von Neumann entropy of the state (S) and the Shannon-entropy of the diagonal of the density operator (S_d) in the computational basis of an Ising spin gas for the quantum initial state $|100\dots 0\rangle$, with a fixed random uniform classical initial condition. In the upper figure there are 5 particles on 6 sites considered, while in the lower figure the 5 particles are located at a chain of 10 sites. The interaction strength η is set to 1. 3000 trajectories were simulated to obtain the figure.

to a classical stochastic process: the arising decoherence is very fast.

All together we find that in spin gas models with random classical dynamics the details of the classical motion are required to access quantum features of the system. In the absence of these the system will attain a highly mixed stationary state, which is essentially independent of the classical motion. The transients, however, depend on these details, such as the entropy rate of the classical Markov process, rules governing the interaction, etc.

IV. SUMMARY AND CONCLUSIONS

We have investigated various semi-quantal spin systems as microscopic models of decoherence. One of the aspects was to study the presence of quantum homogenization at the single particle level, purely due to the interaction between the qubits in argument. We have found that with Heisenberg-XX couplings homogenization always appears, accompanied by a stationary net entanglement, regardless of the details of the evolution of the classical part of the system.

We have also studied the mixedness of the whole system in the absence of any information on the particular classical evolution. We have found that at least in the studied cases the system will reach a stationary state which is a microcanonical ensemble in the quantum mechanical part. This in turn implies that the system qubit will finally decohere, too. Similarly to classical non-periodic Markov chains the transients leading

to this stationary state depend on the details of the underlying quantum dynamics, while the stationary state is the same. This is true even for classical processes with a small entropy rate.

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