Realization of positive-operator-valued measures using measurement-assisted programmable quantum processors

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We study possible realizations of generalized quantum measurements on measurement-assisted programmable quantum processors. We focus our attention on the realization of von Neumann measurements and informationally complete positive-operator-valued measures. Nielsen and Chuang [Phys. Rev. Lett. **79**, 321 (1997)] have shown that two unitary transformations implementable by the same programmable processor require mutually *orthogonal* states. We show that two different von Neumann measurements can be encoded into nonorthogonal program states. Nevertheless, given the dimension of a Hilbert space of the program register the number of implementable von Neumann measurements is still limited. As an example of a programmable processor we use the so-called quantum-information distributor.

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I. INTRODUCTION

General quantum measurements are formalized as *positive-operator-valued measures* (POVMs), i.e., sets of positive operators $\{F_k\}$ that fulfill the resolution to the identity, $\Sigma_k F_k = I$ (see, for instance, Refs. [1–4]). From the general structure of quantum theory [1] it follows that each collection of such operators corresponds to a specific quantum measurement. However, the theory does not directly specify a particular physical realization of a given POVM. The aim of this paper is to exploit the so-called *measurement-assisted quantum processors* to perform POVMs.

The *Stinespring-Kraus theorem* [5] relates quantum operations (*linear completely positive trace-preserving maps*) with unitary transformations. In particular, any quantum operation $\mathcal{E}$ realized on the system A corresponds to a unitary transformation U performed on a larger system $A+B$, i.e.,

$$\mathcal{E}[\varrho] = \mathrm{Tr}_B[G\varrho \otimes \xi G^\dagger], \tag{1.1}$$

where ξ is a suitably chosen state of the ancillary system B and Tr_B denotes a *partial trace* over the ancilla B. The assignment $\mathcal{E} \mapsto (G,\xi)$ is one to many, because the dilation of the Hilbert space of a system A can be performed in many different ways. However, if we fix the transformation G, the states ξ of the ancillary system B control and determine quantum operations that are going to be performed on system A. In this way one obtains a concept of a *programmable quantum processor*, i.e., a "piece of hardware" that takes as an input a data register (system A) and a program register (system B). Here the state of the program register ξ encodes the operation $\varrho \to \varrho' = \mathcal{E}_\xi[\varrho]$ that is going to be performed on the data register.

In a similar way, any quantum generalized measurement (POVM), that is represented by a set of positive operators $\{F_j\}$, can be understood as a *von Neumann measurement* performed on the larger system [4]. The von Neumann measurements are those for which $F_j \equiv E_j$ are mutually orthogonal projectors, i.e., $E_j E_k = \delta_{jk} E_k$. The *Neumark theorem* (see, e.g., Ref. [6]) states that for each POVM $\{F_j\}$ there exists a von Neumann measurement $\{E_j\}$ on a larger Hilbert space $\mathcal{H}_{AB}$ and $\mathrm{Tr}\varrho F_j = \mathrm{Tr}[(\varrho \otimes \xi)E_j]$ for all ϱ, where ξ is some state of the system B. Moreover, it is always possible to choose a von Neumann measurement such that $E_j = G^\dagger(I \otimes Q_j)G$ where G is a unitary transformation and Q_j are projectors defined on system B. Using the cyclic property of a trace operation, i.e., $\mathrm{Tr}[(\varrho \otimes \xi)G^\dagger(I \otimes Q_j)G] = \mathrm{Tr}[G(\varrho \otimes \xi)G^\dagger(I \otimes Q_j)]$, we see that the von Neumann measurement can be understood as a unitary transformation G followed by a von Neumann measurement $M \leftrightarrow \{Q_j\}$ performed on the ancillary system only (see Fig. 1).

As a result we obtain the couple (G,M) that determines a programmable quantum processor assisted by a measurement of the program register, i.e., *measurement-assisted programmable quantum processor*. Such device can be used to perform both generalized measurements as well as quantum operations.

Programmable quantum processors (gate arrays of a finite extent) have been studied by Nielsen and Chuang [7] who have shown that no programmable quantum processor can

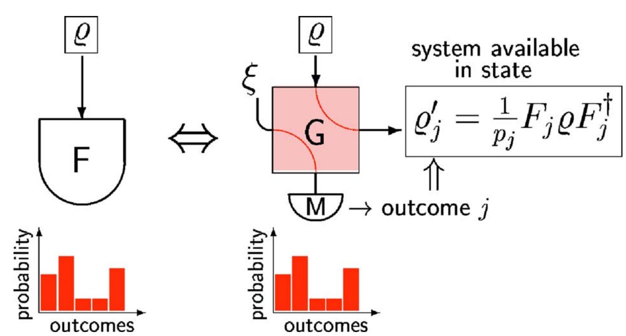

FIG. 1. (Color online) With the help of a measurement-assisted quantum processor (the right part of the figure) one can realize an arbitrary POVM F (the left part of the figure) as a nondemolition measurement. After measuring the outcome j on the program register the system (i.e., the data register) is in the state ϱ_j. The correspondence between both schemes is given by the probability rule $p_j = \mathrm{Tr}\, \varrho F_j = \mathrm{Tr}[(I \otimes Q_j)G(\varrho \otimes \xi)G^\dagger]$.

perform *all* unitary transformations of a data register. To be specific, in order to encode N unitaries into a program register one needs N mutually orthogonal program states. Consequently, the required program register has to be described by an inseparable Hilbert space, because the number of unitaries is uncountable. However, if we work with a measurement-assisted programmable quantum processor, then with a certain probability of success we can realize all unitary transformations [8–11]. The probability of success can be increased arbitrarily close to unity utilizing conditioned loops with a specific set of error correcting program states [8,12–14].

So far, the properties of quantum processors with respect to realization of quantum operations has been studied by several authors [7–11,15,16]. In the present paper we will exploit measurement-assisted quantum processors to perform POVMs. The problem of the implementation of a von Neumann measurement by using programmable "quantum multimeters" for a discrimination of quantum state has been first formulated in Ref. [17] and subsequently it has been studied in Refs. [18–20]. An analogous setting of a unitary transformation followed by a measurement has been used in Ref. [21] to evaluate and measure the expectation value of any operator. The quantum network based on a controlled-SWAP gate can be used to estimate nonlinear functionals of quantum states [22] without any recourse to quantum tomography. Recently D'Ariano and co-workers [23–25] have studied how programmable quantum measurements can be efficiently realized with finite-dimensional ancillary systems. In the present paper we will study how von Neumann measurements and informationally complete POVMs can be realized via programmable quantum measurement devices. In particular, we will show that this goal can be achieved using the so-called quantum-information distributor [26,27].

II. GENERAL CONSIDERATION

Let us start our investigation with an assumption that the program register is always prepared in a pure state, i.e., $\xi=|\Xi\rangle\langle\Xi|$. In this case the action of the processor can be written in the following form:

$$G|\psi\rangle\otimes|\Xi\rangle=\sum_k A_k(\Xi)|\psi\rangle\otimes|k\rangle, \tag{2.1}$$

where $|k\rangle$ is some basis in the Hilbert space of the program register and the operator $A_k(\Xi)=\langle k|G|\Xi\rangle$ act on the data register. In particular, we can use the basis in which the measurement M is performed, i.e., $Q_a=\sum_{k\in J_a}|k\rangle\langle k|$, where J_a is a subset of indices $\{k\}$. Note that $J_a\cap J_{a'}=\emptyset$, because $\sum_a Q_a=I$.

Measuring the outcome a the data evolve according to the following rule (*the projection postulate*):

$$\begin{aligned}\varrho\to\varrho'_a&=\frac{1}{p_a}\mathrm{Tr}_p[(I\otimes Q_a)G(\varrho\otimes|\Xi\rangle\langle\Xi|)G^\dagger]\\&=\frac{1}{p_a}\sum_{k\in J_a}A_k(\Xi)\varrho A_k^\dagger(\Xi),\end{aligned} \tag{2.2}$$

with the probability $p_a=\mathrm{Tr}[(I\otimes Q_a)G(\varrho\otimes|\Xi\rangle\langle\Xi|)G^\dagger]=\mathrm{Tr}[\varrho\sum_{k\in J_a}A_k^\dagger(\Xi)A_k(\Xi)]=\mathrm{Tr}[\varrho F_a]$. Consequently for the elements of the POVM we obtain

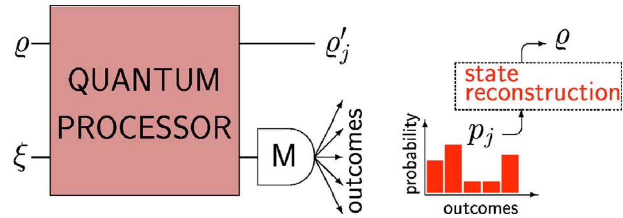

FIG. 2. (Color online) Measurement-assisted quantum processors can be exploited to perform state tomography. Based on the measured probability distribution p_j one can infer the original state ϱ.

$$F_a=\sum_{k\in J_a}A_k^\dagger(\Xi)A_k(\Xi). \tag{2.3}$$

If we consider a general program state with its spectral decomposition in the form $\xi=\sum_n\pi_n|\Xi_n\rangle\langle\Xi_n|$, then the transformation reads

$$\varrho\to\varrho'_a=\frac{1}{p_a}\sum_{n,k\in J_a}\pi_n A_{kn}\varrho A_{kn}^\dagger, \tag{2.4}$$

with $A_{kn}=\langle k|G|\Xi_n\rangle$ and $p_a=\sum_{n,k\in J_a}\pi_n\,\mathrm{Tr}[\varrho A_{kn}^\dagger A_{kn}]$. Therefore the operators

$$F_a=\sum_{n,k\in J_a}\pi_n A_{kn}^\dagger A_{kn} \tag{2.5}$$

constitute the realized POVM.

Given a processor G and some measurement M one can easily determine which POVM can be performed. Note that the same POVM can be realized in many physically different ways. Two generalized measurements M_1, M_2 are equivalent, if the resulting functionals $f_k^{(x)}(\varrho)=\mathrm{Tr}\,\varrho F_k^{(x)}$ $(x=1,2)$ coincide for all k, i.e., they result in the same probability distributions. For the purpose of the realization of POVMs, the state transformation during the process is irrelevant. However, two equivalent realizations of POVM can be distinguished by the induced state transformations (for more on quantum measurement see Ref. [4]).

Let us consider, for instance, the trivial POVM, which consists of operators $F_k=c_kI$ $(c_k\geqslant 0,\sum_k c_k=1)$. In this case the observed probability distribution is data independent and some quantum operation is realized. In all other cases, the state transformation depends on the initial state of the data register, and is not linear [12]. In these cases the resulting distribution is nontrivial and contains some information about the state ϱ. In the specific case when the state ϱ can be determined (reconstructed) perfectly, the measurement is *informationally complete*. In this case we can perform the *complete state reconstruction* (see Fig. 2). Any collection of d^2 linearly independent positive operators F_k determine such informationally complete POVM. In particular, they form an operator basis, i.e., any state ϱ can be written as a linear combination $\varrho=\sum_j\varrho_jF_j$. Using this expression the probabilities read

$$p_j = \mathrm{Tr}[\varrho F_k] = \sum_k \varrho_k \mathrm{Tr}[F_j F_k] = \sum_k \varrho_k L_{jk}, \tag{2.6}$$

where the coefficients $L_{jk}=\mathrm{Tr}[F_jF_k]$ define a matrix L. In this setting the (inverse) problem of the state reconstruction reduces to a solution of a system of linear equations $p_j = \Sigma_k L_{jk}\varrho_k$, where ϱ_k are unknown. The solution exists only if the matrix L is invertible and then $\varrho_k = \Sigma_j L^{-1}_{kj} p_j$.

The purpose of any measurement is to provide us with an information about the state of the physical system based on results of a measurement. Our scheme of the measurement-assisted quantum processor represents a general model of a physical realization of any POVM.

III. QID: COMPLETE STATE TOMOGRAPHY

In this section we will present a specific example of a quantum processor, the so-called *quantum-information distributor* (QID) [26]. This device uses as an input a two-qubit program register and a single-qubit data register. The processor consists of four controlled-NOT (CNOT) gates. Its name reflects the property [26] that in special cases of program states the QID acts as an optimal cloner and the optimal universal NOT gate, i.e., it optimally distributes quantum information according to a specific prescription. Moreover, it can be used to perform an arbitrary qubit rotation with the probability $p=1/4$ [10]. The action of the QID can be written in the form [12]

$$G_{\mathrm{QID}}|\psi\rangle \otimes |\Xi\rangle = \sum_k \sigma_k A(\Xi)\sigma_k|\psi\rangle \otimes |k\rangle, \tag{3.1}$$

where σ_k are Pauli sigma matrices, the operator $A(\Xi) = \langle k|G_{\mathrm{QID}}|\Xi\rangle$ acts on the data register, and $|k\rangle \in \{|0+\rangle, |1+\rangle, |0-\rangle, |1+\rangle\}$ is a two-qubit program-register basis in which the measurement M is performed $[|\pm\rangle = (1/\sqrt{2}) \times (|0\rangle \pm |1\rangle)]$.

In what follows we shall extend the list of applications of the QID processor and show how to realize a complete POVM, i.e., a complete state reconstruction. For a general program state $|\Xi\rangle = \Sigma_k \alpha_k |\Xi_k\rangle$ with $|\Xi_k\rangle = (\sigma_k \otimes I)|\Xi_0\rangle$ [here $|\Xi_0\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$] the POVM consists of the following four operators:

$$F_k = \sigma_k F_{0+} \sigma_k = \sigma_k A(\Xi)^\dagger A(\Xi)\sigma_k, \tag{3.2}$$

with $F_{0+} = \frac{1}{4}I + \frac{1}{4}[\alpha_0 \vec{\alpha}^* + \alpha_0^* \vec{\alpha} + i\vec{\alpha}^* \times \vec{\alpha}] \cdot \vec{\sigma}$ and $\vec{\alpha}^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*)$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

Note that for the initial program state $|\Xi\rangle$ with $\alpha_0 = \cos\mu$, $\vec{\alpha} = i \sin\mu (\vec{\mu}/\mu)$ $(\mu = \|\vec{\mu}\|)$ the probabilities $p_{0+} = \mathrm{Tr}\, F_{0+}\varrho = 1/4$ are ϱ independent, and a unitary operation $U_\mu = \exp(i\vec{\mu}\cdot\vec{\sigma})$ is realized [10]. The question of interest is whether an informationally complete POVM can be encoded into a program state. In fact, the problem reduces to the question of a linear independence of operators F_k for some $|\Xi\rangle$. Using the vector representation of operators, $F_k = 1/4(I + \vec{r}_k \cdot \vec{\sigma})$, one can show that the operators F_k are linearly independent only if none of the coefficients of $\vec{r}_{0+} = \alpha_0 \vec{\alpha}^* + \alpha_0^* \vec{\alpha} + i\vec{\alpha}^* \times \vec{\alpha}$ vanishes.

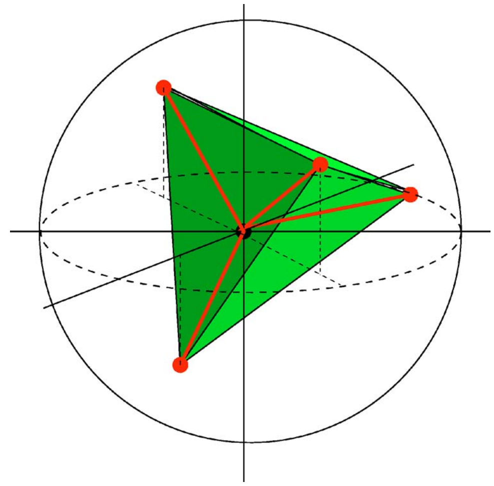

FIG. 3. (Color online) The Bloch sphere can be used to illustrate any POVM that can be realized on the QID processor. Each POVM is given by four operators that determine four points in the Bloch sphere. Using this picture one can see the structure and some properties of the realized POVM. The vertices of a tetrahedron correspond to POVM elements of the symmetric informationally complete POVM associated with the program state $|\Xi\rangle = (1/\sqrt{2})|\Xi_0\rangle + (1/\sqrt{6})(|\Xi_1\rangle + |\Xi_2\rangle + |\Xi_3\rangle)$.

The elements of a POVM can be represented in the Bloch-sphere picture. This is due to the fact that operators $F_k = \frac{1}{2}\varrho_k$ and ϱ_k represent quantum states. Choosing the program state

$$|\Xi_{POVM}\rangle = \frac{1}{\sqrt{2}}|\Xi_0\rangle + \frac{1}{\sqrt{6}}(|\Xi_1\rangle + |\Xi_2\rangle + |\Xi_3\rangle) \tag{3.3}$$

we obtain the informationally complete POVM with a very symmetric structure. In particular, the operators F_k are proportional to pure states associated with vertices of a tetrahedron drawn inside the Bloch sphere (see Fig. 3). These operators read

$$F_{0+} = \frac{1}{4}\left(I + \frac{1}{\sqrt{3}}[\sigma_x + \sigma_y + \sigma_z]\right); \tag{3.4}$$

$$F_{0-} = \frac{1}{4}\left(I + \frac{1}{\sqrt{3}}[-\sigma_x - \sigma_y + \sigma_z]\right); \tag{3.5}$$

$$F_{1+} = \frac{1}{4}\left(I + \frac{1}{\sqrt{3}}[\sigma_x - \sigma_y - \sigma_z]\right); \tag{3.6}$$

$$F_{1-} = \frac{1}{4}\left(I + \frac{1}{\sqrt{3}}[-\sigma_x - \sigma_y + \sigma_z]\right). \tag{3.7}$$

It is obvious that these operators are not mutually orthogonal, but $\mathrm{Tr}\, F_j^\dagger F_k = \frac{1}{12}\delta_{jk} + \frac{1}{4}(1-\delta_{jk})$. Using this identity one can easily compute the relation (2.6) between the observed probability distribution and the initial data state ϱ,

$$\varrho = \sum_k \left(-\frac{21}{5} p_k + \frac{9}{5} \sum_{j \neq k} p_j \right) |Q_k\rangle\langle Q_k|, \tag{3.8}$$

where we used the notation $F_k = \frac{1}{2}|Q_k\rangle\langle Q_k|$. The last equation completes the task of the state reconstruction.

Because of the identity $\mathrm{Tr}\, F_j F_k = \mathrm{const}$ for $j \neq k$ the realized POVM $\{F_k\}$ is of a special form. It belongs to a family of the so-called symmetric informationally complete measurements (SIC POVM) [28]. These measurements are of interest in several tasks of quantum information processing and possess many interesting properties. It is known (see, e.g., Ref. [28]) that for qubits there essentially exist only two (up to unitaries) such measurements. Above we have shown how one of them can be performed using the QID processor.

IV. VON NEUMANN MEASUREMENTS

An important class of measurements is described by the *projector valued measures* (PVM), which under specific circumstances enable us to distinguish between orthogonal states in a single shot, i.e., no measurement statistics is required. A set of operators $\{E_k\}$ form a PVM, if $E_j = E_j^\dagger$ and $E_j E_k = E_j \delta_{jk}$, i.e., it contains mutually orthogonal projectors. The total number of (nonzero) operators $\{E_k\}$ cannot be larger than the dimension of the Hilbert space d.

Usually the *von Neumann measurements* are understood as those that are compatible with the *projection postulate*, i.e., the result j associated with the operator $E_j = |e_j\rangle\langle e_j|$ induces the state transformation

$$\varrho \to \varrho_j' = \frac{E_j \varrho E_j}{\mathrm{Tr} \varrho E_j} = \frac{|e_j\rangle\langle e_j|\varrho|e_j\rangle\langle e_j|}{\langle e_j|\varrho|e_j\rangle} = |e_j\rangle\langle e_j| = E_j. \tag{4.1}$$

That is, the state after the measurement is described by the corresponding projector E_j.

However, each PVM can be realized in many different ways and a particular von Neumann measurement is only a specific case. In our settings the realized POVM $\{F_k\}$ is related to the state transformation via the identity $F_k = A_k^\dagger A_k$, where $\varrho \to \varrho_k' = A_k \varrho A_k^\dagger$. The set of operators $A_k = U_k E_k$, where E_k are projectors and U_k are unitary transformations, define the same PVM given by $\{E_k\}$. In particular, $A_k^\dagger A_k = E_k U_k^\dagger U_k E_k = E_k E_k = E_k$, but the state transformation results in

$$\varrho \to \varrho_k' = U_k E_k U_k^\dagger \neq E_k. \tag{4.2}$$

Thus the final state is described by a projector, but not in accordance with the projection postulate. We refer to the PVMs that are compatible with the projection postulate as the *von Neumann measurements*. Moreover, for simplicity we shall assume that the projectors are always one dimensional, i.e., the PVM is associated with nondegenerate Hermitian operators.

The action of the processor G implementing two von Neumann measurements $\{E_j\}$ and $\{G_j\}$ can be written as

$$G|\psi\rangle \otimes |\Xi_E\rangle = \sum_j E_j|\psi\rangle \otimes |j\rangle; \tag{4.3}$$

$$G|\psi\rangle \otimes |\Xi_G\rangle = \sum_j G_j|\psi\rangle \otimes |j\rangle. \tag{4.4}$$

It is well known [11] that when two sets of Kraus operators are realizable by the same processor G, then the following necessary relation holds $\sum_j E_j G_j = \langle \Xi_E | \Xi_G \rangle I$. Using this relation for the projections $E_j = |e_j\rangle\langle e_j|$, $G_j = |g_j\rangle\langle g_j|$ we obtain the identity

$$\sum_j u_{jj} |e_j\rangle\langle g_j| = kI, \tag{4.5}$$

where $u_{jj} = \langle g_j | e_j \rangle$. For general measurements, the operator on the left-hand side of the previous equation contains off-diagonal elements. In this case the corresponding program states must be orthogonal, i.e., $k=0$. This result is similar to the one obtained by Nielsen and Chuang [7] who have studied the possibility of the realization of unitary transformations via programmable gate arrays. Nielsen and Chuang have shown that in order to perform (with certainty) two unitary transformations on a given quantum processor one needs two orthogonal program states. However, in our case we cannot be sure that given the same resources the measurement-assisted processor realizing two von Neumann measurements does exist. Moreover, we also have to consider an option that the condition holds also for nonorthogonal program states (see the case study below). From the above it follows that the realization of von Neumann measurements on programmable processors is different from implementation unitary operations on programmable processors. The reason is that for implementation of von Neumann measurements program states might not satisfy the criterion in Eq. (4.5).

A. Orthogonal program states

In order to realize a measurement described by PVM (either a von Neumann measurement or a general PVM measurement) on a d-dimensional data register the program space must be at least d dimensional. Let us start with the assumption that the Hilbert space of the program register is d dimensional and the program states are orthogonal. Our task is to analyze the possibility to perform d different (nondegenerate) von Neumann measurements M_α determined by a set of operators $E_k^\alpha = |\alpha_k\rangle\langle\alpha_k|$ ($E_k^\alpha E_j^\alpha = \delta_{kj} E_k^\alpha$ and $\sum_k E_k^\alpha = I$ for all α). Let $|\alpha\rangle$ denote the associated program states and $\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$. It is easy to see that for general measurements the resulting operator

$$G = \sum_{k,\alpha} E_k^\alpha \otimes |k\rangle\langle\alpha| \tag{4.6}$$

is not unitary. In particular, $G^\dagger G = \sum E_k^\beta E_k^\alpha \otimes |\beta\rangle\langle\alpha| \neq I$. The equality would require that the identity $\sum_k E_k^\alpha E_k^\beta = \delta_{\alpha\beta} I_d$ holds. Therefore we conclude that neither orthogonal state guarantees the existence of a programmable processor that performs the desired set of von Neumann measurements. This result makes the programming of unitaries and programming of von Neumann measurements different.

For instance, let us consider a two-dimensional program register and let us denote $E_{0,1}^0 = E_{0,1}$ and $E_{0,1}^1 = G_{0,1}$. Then the

TABLE I. The measurements $M_1, M_2, \ldots, M_N$ are realizable by a d-dimensional program register only if all vectors in the rows are mutually orthogonal. Moreover, no two columns can be related by a permutation. The orthogonality of the vectors in columns is ensured by the fact that they form a PVM. It turns out that the number of realizable measurements equals to at most $N-2$, i.e., even with qutrit one cannot encode more than a single von Neumann measurement. Moreover, the measurements that can be performed are not arbitrary.

Measurement	M_1	M_2	$\cdots$	M_N
Result 1	$\|\alpha_1\rangle$	$\|\beta_1\rangle$	$\cdots$	$\|\omega_1\rangle$
Result 2	$\|\alpha_2\rangle$	$\|\beta_2\rangle$	$\cdots$	$\|\omega_2\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Result d	$\|\alpha_d\rangle$	$\|\beta_d\rangle$	$\cdots$	$\|\omega_d\rangle$

above condition reads $E_0G_0 = E_1G_1 = 0$. Using the definition $E_k = |e_k\rangle\langle e_k|$ and $G_k = |g_k\rangle\langle g_k|$ we obtain the orthogonality conditions $\langle e_0|g_0\rangle = \langle e_1|g_1\rangle = 0$. Consequently, because in the two-dimensional case the orthogonal state is unique, we obtain $|g_0\rangle = |e_1\rangle$ and $|g_1\rangle = |e_0\rangle$, i.e., the measurements are the same. Similarly one can show that even for qutrit ($d_p = d = 3$) one can perform only one von Neumann measurement, too. In particular, $\langle e_0|g_0\rangle = 0$ implies $|g_0\rangle = a_0|e_1\rangle + b_0|e_2\rangle$, $|g_1\rangle = a_1|e_0\rangle + b_1|e_2\rangle$ and $|g_2\rangle = a_2|e_0\rangle + b_2|e_1\rangle$. Orthogonality of $|g_j\rangle$ results in a set of equations $a_0b_2 = 0$, $b_0b_1 = 0$, $a_1a_2 = 0$ with the solution that set of vectors $\{|g_j\rangle\}$ is just a permutation of the set $\{|e_j\rangle\}$. However, this solution does not correspond to a realization of two noncommutative measurements. To perform two such measurements one needs an extra dimension, i.e., for $d_p = d = 4$ we can realize two von Neumann measurements. An addition of new dimension enables us to perform one more noncommuting measurement [29], i.e., with d orthogonal states one can implement at most $N = d-2$ noncommuting von Neumann measurements. See Table I for the properties that the corresponding eigenvectors have to satisfy.

In order to implement a set of von Neumann measurements on a qudit (with d-dimensional Hilbert space) one has to utilize a program register with the dimension of the Hilbert space such that $\dim \mathcal{H}_p = d_p > d$. In general, in this case we work with d_p outcomes and d_p projective operators Q_k that define the realized measurement. However, each PVM consists of maximally d projectors. Therefore $d_p - d$ of the induced operators E_k should represent the *zero operator*. It means that we are realizing the von Neumann measurement such that some of the outcomes do not occur, i.e., the probability of these outcomes is equal to zero for all data states. However, there is one more option, that the set of operators $\{E_k\}$ (corresponding to the outcomes $k = 1, \ldots, d_p$) contains exactly only d different operators (projectors). This means that more outcomes specify the same projection and define a single result of the realized von Neumann measurement.

We can utilize the so-called "zero" operators to formulate a general approach to implement any set of arbitrary von Neumann measurements. Let us consider N von Neumann measurements M_α ($\alpha = 1 \cdots N$) given by nonzero operators $\{E_k^\alpha\}$ (number of k equal to d). We can define new sets of d_p operators $\{\tilde{E}_k^\alpha\}$ by adding to $\{E_k^\alpha\}$ zero operators so that the condition $\sum_k \tilde{E}_k^\alpha \tilde{E}_k^\beta = \delta_{\alpha\beta} I$ holds. Using this approach we find that any collection of N von Neumann measurements can be realized on a single quantum processor given by Eq. (4.6) with (maximally) $N \cdot d$-dimensional program space.

Let us summarize our results in the following propositions:

Proposition 1. Using $d_p = d$-dimensional program space and orthogonal program states allows us to encode maximally $N = d-2$ specific (noncommuting) von Neumann measurements on a qudit (see Table I).

Proposition 2. Let $M_1, \ldots, M_N$ be N (noncommuting) von Neumann measurements on a qudit. Then it is sufficient to use $N \cdot d$-dimensional program space to encode these measurements into orthogonal program states.

B. Case study: Projective measurements on a qubit

Let us consider two von Neumann measurements $M = \{E_0, E_1\}$ and $N = \{G_0, G_1\}$ on a qubit. First we will assume a three-dimensional Hilbert space of a program register. We define measurements $M_1 = \{E_0, E_1, 0\}$ and $M_2 = \{0, G_1, G_2\}$, respectively. It is easy to see that neither of these two sets of operators satisfy the condition $0 = \sum_k E_k G_k = 0E_0 + E_1G_1 + 0G_2 = E_1G_1$. The equality holds only if $E_1G_1 = 0$, i.e., $E_1 = |\psi\rangle\langle\psi|$ and $G_2 = |\psi_\perp\rangle\langle\psi_\perp|$, but this implies that the two measurements are the same. Consequently, the dimension of the program space has to be increased in order to encode into a program register two projective measurements on a qubit. Therefore let us consider a four-dimensional Hilbert space of the program register. In this case we have $M_1 = \{E_0, E_1, 0, 0\}$, $M_2 = \{0, 0, G_0, G_1\}$, and the condition holds for all possible measurements M_1, M_2. We conclude that in order to implement N von Neumann measurements (by encoding into orthogonal states) on a qubit a $2N$-dimensional program space is required. Let us note that for qudits this is only the sufficient condition and for specific collections of measurements we can do better.

In what follows we shall show a way to realize three different von Neumann measurements on a qubit by using only four-dimensional program space. To achieve this goal we will use *nonorthogonal* program states. We will show that in special cases the condition of orthogonality [given by Eq. (4.5)] can be relaxed. The program space of the QID processor given by Eq. (3.1) consists of two qubits. Using the conclusion of the previous paragraph we see that QID allows us to perform two von Neumann measurements. It is easy to see that the operators $A_k = \sigma_k A(\Xi)\sigma_k$ with $A(\Xi) = \frac{1}{2}\sum_j \alpha_j \sigma_j$ are not projectors. Consequently, the projective measurement cannot be realized in the same way as described above. However, the QID processor can still be exploited to perform a von Neumann measurement.

Using the program state $|\Xi\rangle = (1/\sqrt{2})(|\Xi_0\rangle + |\Xi_1\rangle)$ the operator $A = (1/2\sqrt{2})[I + \sigma_x]$ (i.e., $F_0 = A^\dagger A = \frac{1}{2}P_+$, where $P_+ = \frac{1}{2}[I + \sigma_x]$) is a projection onto the vector $|+\rangle = (1/\sqrt{2})|0\rangle + |1\rangle$. It is obvious that $F_1 = \sigma_x F_0 \sigma_x = F_0$ and $F_2 = F_3 = \frac{1}{2}P_-$, where $P_- = \frac{1}{2}[I - \sigma_x]$. It turns out that we have realized the PVM described by $P_\pm$, i.e., the eigenvectors of the σ_x measurement. The state transformation reads $\varrho \to \varrho_k' = P_\pm$ (if $p_k \neq 0$), respectively. It follows that the realization of the measurement of σ_x is in accordance with the projection postulate. In the same way we can realize σ_y and σ_z measurement (in these cases different results must be paired). Basically, this corresponds to a choice of different two-valued measurements, but in reality we perform only a single four-valued measurement. As a result we find that on the QID we can realize three different von Neumann measurements. Note that we have used only two qubits as the program register. Moreover, the associated program states $|\Xi_{\sigma_j}\rangle = (1/\sqrt{2})[|\Xi_0\rangle + |\Xi_j\rangle]$ are not mutually orthogonal, but $\langle \Xi_{\sigma_j}|\Xi_{\sigma_k}\rangle = \frac{1}{2}$ (for $j \neq k$) and Eq. (4.5) holds. Namely, for the measurements of $\sigma_x \leftrightarrow \{P_\pm\}$ and $\sigma_z \leftrightarrow \{P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|\}$ the condition (4.5) reads $\frac{1}{2}[P_+P_0 + P_+P_1 + P_-P_1 + P_-P_0] = \frac{1}{2}I$.

V. PROJECTION-VALUED MEASURES

If we relax the projection postulate more PVMs can be realized on a single processor. Let us consider that the dimension of the program space equals d and $|\alpha\rangle$ is the state that encodes the PVM given by a set $\{E_k^\alpha\}$. The action of G can be written as

$$G|\psi\rangle \otimes |\alpha\rangle = \sum_k U_k^\alpha E_k^\alpha |\psi\rangle \otimes |k\rangle \tag{5.1}$$

and the condition $\Sigma_k E_k^\alpha U_k^{\alpha\dagger} U_k^\beta E_k^\beta = \delta_{\alpha\beta} I$ must hold. Let us consider two PVMs on a qubit $\{E_0 = |0\rangle\langle 0|, E_1 = |1\rangle\langle 1|\}$ and $\{G_0 = |\phi\rangle\langle\phi|, G_1 = |\phi_\perp\rangle\langle\phi_\perp|\}$. Define a unitary map U such that $|\phi\rangle \to |1\rangle$ and $|\phi_\perp\rangle \to |0\rangle$. Using this map we can define a processor by the following equations:

$$G|\psi\rangle \otimes |\Xi_E\rangle = E_0|\psi\rangle \otimes |0\rangle + E_1|\psi\rangle \otimes |1\rangle;$$

$$G|\psi\rangle \otimes |\Xi_G\rangle = \widetilde{G}_0|\psi\rangle \otimes |0\rangle + \widetilde{G}_1|\psi\rangle \otimes |1\rangle, \tag{5.2}$$

where $\widetilde{G}_0 = UG_0 = |1\rangle\langle\phi|$, $\widetilde{G}_1 = UG_1 = |0\rangle\langle\phi_\perp|$, and $\langle \Xi_E|\Xi_G\rangle = 0$. Direct calculation shows that $E_0\widetilde{G}_0 + E_1\widetilde{G}_1 = |0\rangle\langle 0|1\rangle\langle\phi| + |1\rangle\langle 1|0\rangle\langle\phi_\perp| = 0$, i.e., G is unitary. From here it follows that if one does not require the validity of the projection postulate, then any two PVMs can be performed on a processor with two-dimensional program space.

This result holds in general. Let us consider a set of d PVMs $\{E_k^\alpha\}$ on a qudit. There always exist unitary transformations U^α such that operators $\widetilde{E}_k^\alpha = U^\alpha E_k^\alpha$ satisfy the condition $\Sigma_k \widetilde{E}_k^{\alpha\dagger}\widetilde{E}_k^\beta = \delta_{\alpha\beta} I$. Without the loss of generality we can consider that the measurement M_0 is given by projectors $|0\rangle\langle 0|, \ldots, |d-1\rangle\langle d-1|$ and M_α by $|\phi_0^\alpha\rangle\langle\phi_0^\alpha|, \ldots, |\phi_{d-1}^\alpha\rangle\langle\phi_{d-1}^\alpha|$ (see Table II).

Proposition 3. A collection of arbitrary (noncommuting) N projection-valued measures can be realized on quantum processor with N-dimensional program space.

Programming unitaries vs PVMs

As an alternative to the scenario presented in the previous section one can consider the following strategy for how to realize a measurement on the programmable quantum processor. Programmable processors are designed to perform unitary operations. Since different projection-valued measures are always related by some fixed unitary transformation, it is possible to exploit the existing processor to rotate the input data state by a suitable transformation. After this transformation is implemented the *fixed* von Neumann measurement of the data register is performed. In particular, let us consider that the processor G implements the transformation $|\psi\rangle \to U|\psi\rangle$ and the fixed measurement of the data register is described by set of projectors $\{E_k\}$. Using such a processor the measured probabilities read $p_k = \langle\psi|U^\dagger E_k U|\psi\rangle = \langle\psi|F_k|\psi\rangle$, where operators $F_k = U^\dagger E_k U$ describe the realized PVM. However, the output state $|\psi_k'\rangle$ is described by the corresponding projection E_k. To obtain the state transformation that is in accordance with the projection postulate one has to apply the same unitary transformation U once more, i.e., we use the same processor twice.

TABLE II. The realization of an arbitrary collection of d PVMs $M_1, \ldots, M_d$ on a qudit. The operators $\widetilde{E}_k^\alpha$ correspond to an example of the choice of unitary transformations U^α. In particular, each U^α transforms the basis $\{|\phi_j^\alpha\rangle\}$ into some permutation of the basis $\{|j\rangle\}$. The permutation is different for each α.

$M_1 \leftrightarrow \widetilde{E}_k^1$	$M_2 \leftrightarrow \widetilde{E}_k^2$	$\cdots$	$M_d \leftrightarrow \widetilde{E}_k^d$
$\|0\rangle\langle 0\|$	$\|1\rangle\langle\phi_0^2\|$	$\cdots$	$\|d-1\rangle\langle\phi_0^d\|$
$\|1\rangle\langle 1\|$	$\|2\rangle\langle\phi_1^2\|$	$\cdots$	$\|0\rangle\langle\phi_1^d\|$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\|d-1\rangle\langle d-1\|$	$\|0\rangle\langle\phi_{d-1}^2\|$	$\cdots$	$\|d-2\rangle\langle\phi_{d-1}^d\|$

From here it follows that the implementation of a von Neumann measurement is related to a repeated usage of the processor realizing the given unitary operation. In particular, to realize N von Neumann measurements we have to use twice the processor realizing N unitary transformations, i.e., the program space is composed of two N-dimensional systems (unitary operators are encoded in orthogonal states). As a result we find that the dimension of the program space equals N^2. In the limit of large number of measurements this N^2 is larger than Nd that quantifies the number of orthogonal program states from Proposition 2. In fact, whenever the number of measurements is larger than the dimension of the object, the usage of a quantum processor realizing unitary transformations is less efficient.

Let us note that with this realization of measurements we do not have to consider the compatibility with the projection postulate, providing that the measurement is not performed in a nondemolition way. However, nondemolition measurements require additional systems and therefore the model would correspond again to some measurement-assisted quantum processor. If one does not care about a particular realization of the PVM, then the number of realizable PVMs N equals to the dimension of the number program states encoding the corresponding unitary transformations. This is exactly the content of Proposition 3.

VI. CONCLUSION

We have studied how POVMs can be physically realized using the so-called measurement-assisted quantum processors. In particular, we have analyzed how to perform a complete state reconstruction and von Neumann measurements. As a result we have found that an arbitrary collection of von Neumann measurements cannot be realized on a single programmable quantum processor of finite dimension. We have shown how to use the QID processor to perform the state reconstruction.

The number of implementable von Neumann measurements is limited by the dimension of the program register. Our main result is that with a program register containing Nd orthogonal states one can certainly find a processor which performs arbitrary N von Neumann measurements. In principle, one can do much better than this. We have shown that nonorthogonal program states can be used very efficiently. This makes the programmability of unitary transformations and von Neumann measurements different. In particular, the QID processor can be exploited to perform three von Neumann measurements by using three nonorthogonal states of only two qubits of the program register. Using $d_p=d$-dimensional program space one can encode maximally $N=d-2$ von Neumann qudit measurements into orthogonal program states (for a qubit we have $N=1$).

Relaxing the condition of compatibility with the projection postulate the processor allows us to realize any collection of N PVMs by using only $d_p=N$-dimensional program space. An open question is whether we can perform more PVMs or not. The two tasks that can be performed by programmable processors are the realization of von Neumann measurements and the application of unitary transformations on the data register. These two applications are different. According to Nielsen and Chuang [7], any collection of N unitary transformations requires N-dimensional program space. For N von Neumann measurements the upper bound reads $d_p=Nd$ and any improvement strongly depends on the specific set of these measurements. The characterization of these classes of measurements is an interesting topic that will be studied elsewhere.

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