Open Sys. & Information Dyn. (2005) **12**: 81–91

Description of Quantum Dynamics of Open Systems Based on Collision-Like Models†

Mário Ziman[1,2], Peter Štelmachovič[1], and Vladimír Bužek[1,2]

[1] *Research Center for Quantum Information*
Slovak Academy of Sciences
Dúbravská cesta 9, 845 11 Bratislava, Slovakia

[2] *Faculty of Informatics, Masaryk University*
Botanická 68a, 602 00 Brno, Czech Republic

(Received: July 19, 2004)

Abstract. Master equations in the Lindblad form describe evolution of open quantum systems that are completely positive and simultaneously have a semigroup property. We analyze the possibility to derive this type of master equations from an intrinsically discrete dynamics that is modelled as a sequence of collisions between a given quantum system (a qubit) with particles that form the environment. In order to illustrate our approach we analyze in detail how the process of an exponential decay and the process of decoherence can be derived from a collision-like model in which particular collisions are described by SWAP and controlled-NOT interactions, respectively.

1. Introduction

The central issue in the study of evolution of open systems is the observed *irreversibility* of both classical as well as quantum dynamics. Open systems interacting with an environment gradually loose their information content and decohere, which means that after some time their states are to some extent determined by the initial parameters of the environment. Such behaviour cannot be described by unitary transformations and has led to an introduction of a phenomenological dynamical postulate for open systems [1 − 3] — the *semigroup property* of the time evolution $\mathcal{E}_t$

$$\mathcal{E}_{t+s} \;=\; \mathcal{E}_t\mathcal{E}_s \quad \forall\, t,s \geq 0 \qquad \text{and} \qquad \lim_{t\to 0^+} \mathcal{E}_t \;=\; \mathcal{I}\,, \tag{1}$$

where $\mathcal{E}_t : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is a transformation acting on the set of all possible states $\mathcal{S}(\mathcal{H})$ of a given quantum system corresponding to the Hilbert space $\mathcal{H}$. In this case it is possible that pure quantum states evolve into mixtures, and vice versa.

The irreversibility is related to the non-existence of the inverse evolution $\mathcal{E}_t^{-1}$. In particular, the inverse map can exist in the mathematical sense, but the resulting map does not describe a valid quantum evolution. The physical maps $\mathcal{E}_t$ must

† Presented at the 36th Symposium on Mathematical Physics, *"Open Systems & Quantum Information"*, Toruń, Poland, June 9–12, 2004.

satisfy several constraints [4]: They have to be linear, trace-preserving and they have to be completely positive for all t.

The state of the system ϱ_t is obtained by the application of the map on the initial state ϱ, i.e. $\varrho_t = \mathcal{E}_t[\varrho]$. Equations of motion that describe this type of dynamics has the so-called *Lindblad form* [1]:

$$\dot{\varrho}_t \;=\; \mathcal{G}[\varrho_t] \;=\; -i[H,\varrho_t] + \frac{1}{2}\sum_{\alpha,\beta} c_{\alpha\beta}\left([\Lambda_\alpha, \varrho_t\Lambda_\beta] + [\Lambda_\alpha\varrho_t, \Lambda_\beta]\right), \tag{2}$$

where $\Lambda_\alpha = \Lambda_\alpha^\dagger$, $\mathrm{Tr}\,\Lambda_\alpha = 0$, $\mathrm{Tr}\,\Lambda_\alpha\Lambda_\beta = \delta_{\alpha\beta}$ for $\alpha,\beta = 1,\ldots,d^2-1$ and the coefficients $c_{\alpha\beta}$ form a positive matrix. The Hamiltonian H is traceless, i.e. $H = \sum_\alpha h_\alpha\Lambda_\alpha$. This differential equation is supplemented by the initial condition $\varrho_{t=0} = \varrho$.

Differentiating the time evolution $\varrho_t = \mathcal{E}_t[\varrho]$ we derive a formal expression for the generator of the dynamics $\mathcal{G}$

$$\dot{\varrho}_t \;=\; \dot{\mathcal{E}}_t[\varrho] \;=\; \dot{\mathcal{E}}_t\mathcal{E}_t^{-1}[\varrho_t] \quad\Longrightarrow\quad \mathcal{G} \;=\; \dot{\mathcal{E}}_t\mathcal{E}_t^{-1}\,. \tag{3}$$

This expression in a strict mathematical sense has a meaning only when inverse transformations $\mathcal{E}_t^{-1}$ do exist for any time t. For general one-parameter set of completely positive maps $\mathcal{E}_t$ the generator can be time-dependent, however, if these maps possess the semigroup property, then the generator is independent of time.

2. Discrete Dynamical Semigroup

The derivation of the particular master equation, which drives the evolution of an open system, is usually based on the idea that the open system is a part of a larger closed system with the underlying unitary dynamics described by Schrödinger equation. However, after tracing out the environment, the induced set of maps $\mathcal{E}_t$ essentially never fulfills the semigroup property. In order to obtain this feature of dynamics various approximations have to be applied [3, 5], for instance the Born-Markov approximation, the weak-coupling limit, the mean-field limit, etc. Master equations of the Lindblad form are valid only under specific physical conditions, but still provide us with a very reasonable approximative picture of the exact dynamics of open quantum systems.

In this paper we will present a novel method how to derive master equations. We will consider that the interaction between the system and its environment consists of bipartite collisions. For this we assume that the environment consists of N particles (quantum systems) of the same physical origin as the system under consideration. In our model we assume that the environment is initially "prepared" in a factorized state $\xi^{\otimes N}$ and the system interacts with each particle from the environment at most once. Each collision is described by a unitary transformation U, which induces a map $\mathcal{E}$ (see Fig. 1). As a result this *collision-like* model determines a discrete evolution described by powers of the map $\mathcal{E}$, i.e. $\mathcal{E}^n$ with

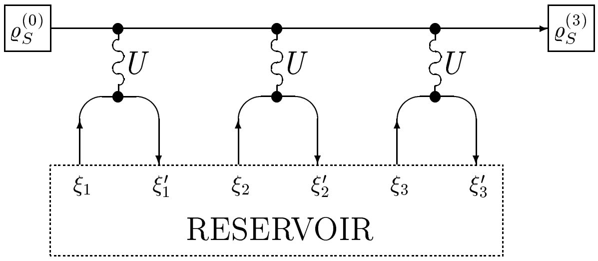

Fig. 1: A schematic visualization of a sequence of collisions between the system and particles from the environment. This interactions lead to a dynamics of the system that is described by a specific sequence of maps $\mathcal{E}_n$ that in the continuous limit can be described by a specific master equation.

$n = 1, 2, \ldots$ corresponding to the n-th collision. The set of maps $\mathcal{E}_n := \mathcal{E}^n$ with $\mathcal{E}_0 = \mathcal{I}$ obviously fulfills the semigroup property, i.e.

$$\mathcal{E}_n \mathcal{E}_m \;=\; \mathcal{E}_{n+m} \quad \text{for all} \quad n, m = 0, 1, 2, \ldots \tag{4}$$

In other words the maps $\mathcal{E}_n$ form a *one-parameter discrete semigroup.*

The question is whether we may replace the discrete dynamics $\mathcal{E}_n$ with a continuous one $\mathcal{E}_t$ such that $\mathcal{E}_{t_n} = \mathcal{E}_n$ for times $t_n = n\tau$ (τ is some fixed time interval between subsequent collisions), and the maps $\mathcal{E}_t$ satisfy the semigroup property. In other words the task is to interpolate a discrete set of points with a line in the abstract space of quantum maps. Obviously, on the time scale less than τ the continuous evolution is different from the discrete one. However, our interest is to describe the overall evolution rather than a sequence of isolated collision. In this sense, the continuous dynamics is a good approximation of the discrete one. Our aim is to describe the intrinsically discrete collision model in the language of quantum dynamical semigroups, i.e. continuous master equations.

We derive the master equation in the following way. Firstly, we will express the powers of the map $\mathcal{E}$ in a specific form. The discrete parameter n numbering the order of interaction of the system with environment particles will be replaced by a continuous parameter $t = n\tau$, i.e. $n \to t/\tau$. Correspondingly, we have a continuous set of maps $\mathcal{E}_t$. The question is, whether these maps are completely positive, and whether they form a semigroup. If, moreover, the inverse maps $\mathcal{E}_t^{-1}$ exist, then we can derive the master equation by using the expression for the generator in (3).

In what follows we will present two specific examples, for which the derived master equations correspond to an exponential decay, and to a decoherence process. It is important to understand which bipartite interactions underly these type of processes. We will be interested only in two-dimensional systems, i.e. qubits.

3. Qubit Formalism

In what follows we will use the so-called *left-right formalism*, or real matrix representation of quantum maps, and real vector representation of quantum states of a qubit. The state space of a qubit is a subset of four-dimensional linear space of hermitian operators. It follows that any density operator can be written as $\varrho = \frac{1}{2}I + \vec{r}\cdot\vec{\sigma}$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the *Pauli operators*. The positivity of ϱ restricts the choices of vector $\vec{r}$ such that $|\vec{r}| \leq 1/2$, i.e. the states of qubit form a three-dimensional sphere — the *Bloch sphere*.

The quantum evolution maps $\mathcal{E}$ as well as the generators $\mathcal{G}$ of quantum dynamics $\mathcal{E}$ then correspond to 4x4 real matrices acting on the vectors $(1/2, \vec{r})$. In particular, the evolution $\mathcal{E}$ has the affine form

$$\mathcal{E} \;=\; \begin{pmatrix} 1 & \vec{0} \\ \vec{t} & T \end{pmatrix},$$

where T is some 3x3 matrix and $\vec{t}$ is the translation vector. Under the action of $\mathcal{E}$ the Bloch sphere vectors $\vec{r}$ transform in the following way $\vec{r} \to \vec{r}' = T\vec{r} + \vec{t}$. The generator $\mathcal{G}$ of the dynamics has very similar form, only the first row of the matrix vanishes, i.e.

$$\mathcal{G} \;=\; \begin{pmatrix} 0 & \vec{0} \\ \vec{g} & G \end{pmatrix}.$$

The matrix elements are given by the relation $[\mathcal{G}]_{jk} = \frac{1}{2}\mathrm{Tr}(\sigma_j \mathcal{G}[\sigma_k])$ for $j,k = 0,1,2,3$ and $\sigma_0 = I$.

For our purposes it will be useful to know how to rewrite this matrix form of the generator into the operator form

$$\dot{\varrho}_t \;=\; -i[H,\varrho_t] + \frac{1}{2}\sum_{j,k=1}^{3} c_{jk}\left([\sigma_j, \varrho_t\sigma_k] + [\sigma_j\varrho_t, \sigma_k]\right), \qquad (5)$$

i.e. how to rewrite the coefficients $[\mathcal{G}]_{jk}$ into the parameters h_j and c_{jk}. The hermitian matrix c_{jk} can be rewritten as $c_{jk} = d_{jk} - ie_{jk}$, where $d_{jk} = \frac{1}{2}(c_{jk} + c_{kj})$ is the real symmetric matrix and $e_{jk} = i\frac{1}{2}(c_{jk} - c_{kj})$ is the real antisymmetric matrix. Using the operator expression of the generator (5), one can easily find the matrix

$$\mathcal{G} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4e_{23} & -2d_{22} - 2d_{33} & 2d_{12} - 2h_3 & 2d_{13} + 2h_2 \\ 4e_{31} & 2d_{12} + 2h_3 & -2d_{11} - 2d_{33} & 2d_{23} - 2h_1 \\ 4e_{12} & 2d_{31} - 2h_2 & 2d_{32} + 2h_1 & -2d_{11} - 2d_{22} \end{pmatrix}. \qquad (6)$$

The inverse relations then read

$$\begin{aligned} h_1 &= \frac{[\mathcal{G}]_{32} - [\mathcal{G}]_{23}}{4}, & h_2 &= \frac{[\mathcal{G}]_{13} - [\mathcal{G}]_{31}}{4}, & h_3 &= \frac{[\mathcal{G}]_{21} - [\mathcal{G}]_{12}}{4}, \\ e_{23} &= \frac{[\mathcal{G}]_{10}}{4}, & e_{31} &= \frac{[\mathcal{G}]_{20}}{4}, & e_{12} &= \frac{[\mathcal{G}]_{30}}{4}, \end{aligned} \qquad (7)$$

and

$$\begin{aligned} d_{11} &= \frac{-[\mathcal{G}]_{22} - [\mathcal{G}]_{33} + [\mathcal{G}]_{11}}{4}, & d_{12} &= \frac{[\mathcal{G}]_{12} + [\mathcal{G}]_{21}}{4}, \\ d_{22} &= \frac{-[\mathcal{G}]_{11} - [\mathcal{G}]_{33} + [\mathcal{G}]_{22}}{4}, & d_{23} &= \frac{[\mathcal{G}]_{23} + [\mathcal{G}]_{32}}{4}, \\ d_{33} &= \frac{-[\mathcal{G}]_{11} - [\mathcal{G}]_{22} + [\mathcal{G}]_{33}}{4}, & d_{13} &= \frac{[\mathcal{G}]_{13} + [\mathcal{G}]_{31}}{4}. \end{aligned} \tag{8}$$

It is important to note that these relations enable us to represent not only dynamical semigroups, but any time evolution of a qubit in the Lindbland-like form. In this general case the coefficients c_{jk} will not be constant, but will explicitly depend on time. In this way we can derive more general master equations describing dynamics beyond Markovian approximation.

4. Case Study I: Quantum Homogenization

The *quantum homogenization* [6−8] is a process motivated by the thermodynamical process of *thermalization*. It describes a system-reservoir interaction in which the initial state of the system ϱ is transformed into the state ξ determined by the state of the reservoir that is composed of N systems of the same physical origin as the system. The interaction between the system and the reservoir consists of bipartite collisions. Each collision is described by some unitary map U. In order to obtain a discrete semigroup describing the dynamics of the system, we assume that initially the reservoir is in a factorized state $\xi^{\otimes N}$ and that the system interacts with each system from the reservoir at most once (see Fig. 1).

The homogenization approximates the evolution $\varrho \otimes \xi^{\otimes N} \to \xi^{\otimes(N+1)}$, which is forbidden by the *no-cloning theorem* (see [9] and references therein). Let δ be the parameter describing the quality of the homogenization in the following sense. After the homogenization process is complete, all systems are described by states, which belong to a δ-vicinity of the state ξ. Moreover, the homogenization requires the validity of the following relations corresponding to *trivial homogenization* $\mathrm{Tr}_1(U\xi\otimes\xi U^\dagger) = \mathrm{Tr}_2(U\xi\otimes\xi U^\dagger) = \xi$. If we assume that the homogenization is independent on the initial state of the system qubit (ϱ) as well as on initial states of reservoir qubits (ξ) then for qubits U must posses the form of the *partial swap* operation (for more details see [6])

$$P_\eta = \cos\eta I + i \sin\eta S\,, \tag{9}$$

where S is the *swap operation* defined by the relation $S|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$ for all $|\psi\rangle, |\phi\rangle$. In what follows we will use the notation $c = \cos\eta$ and $s = \sin\eta$.

Let us define the Bloch-sphere vectors for density operators, i.e. $\varrho \leftrightarrow \vec{r}$ and $\xi \leftrightarrow \vec{t}$. In terms of these vectors the partial-swap induces the map $\vec{r} \to \vec{r}\,' =$

$c^2\vec{r} + s^2\vec{t} - 2cs\vec{t}\times\vec{r}$, i.e. the superoperator $\mathcal{E}_\xi$ is represented by the matrix

$$\mathcal{E}_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2s^2 t_x & c^2 & 2cst_z & -2cst_y \\ 2s^2 t_y & -2cst_z & c^2 & 2cst_x \\ 2s^2 t_z & 2cst_y & -2cst_x & c^2 \end{pmatrix}. \tag{10}$$

Now we turn our attention on the derivation of the master equation that corresponds to a dynamics induced by a sequence of partial-swap interactions. By a suitable (unitary) substitution of the operator basis $\{I, \sigma_x, \sigma_y, \sigma_z\} \to \{I, S_x, S_y, S_z\}$, such that $S_k = U\sigma_k U^\dagger$ ($UU^\dagger = U^\dagger U = I$), the matrix $\mathcal{E}_\xi$ can be rewritten into the form

$$\mathcal{E}_\xi = \begin{pmatrix} 1 & 0 \ \ 0 & 0 \\ 0 & & 0 \\ 0 & cA & 0 \\ 2s^2 w & 0 \ \ 0 & c^2 \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} c & 2sw \\ -2sw & c \end{pmatrix}, \tag{11}$$

where w is determined by the initial state of the reservoir, i.e. $\xi = \frac{1}{2}I + wS_z$. This change of the operator basis corresponds to a different choice of the x, y, z axes in the Bloch sphere. In what follows we will describe the process in this new basis, in which the powers of $\mathcal{E}_\xi$ can be easily find

$$\mathcal{E}_\xi^n = \begin{pmatrix} 1 & 0 \ \ 0 & 0 \\ 0 & & 0 \\ 0 & c^n A^n & 0 \\ 2w(1-c^{2n}) & 0 \ \ 0 & c^{2n} \end{pmatrix}. \tag{12}$$

The powers of the matrix A can be found as follows: Using the identity $\cos\arctan x = (1+x^2)^{-1/2}$ and defining the parameter $\omega = \arctan(2ws/c)$ we find

$$A = \begin{pmatrix} c & 2sw \\ -2sw & c \end{pmatrix} = \sqrt{c^2 + 4w^2 s^2} \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \tag{13}$$

and its powers are equal to

$$A^n = (c^2 + 4s^2 w^2)^{n/2} \begin{pmatrix} \cos(n\omega) & \sin(n\omega) \\ -\sin(n\omega) & \cos(n\omega) \end{pmatrix}. \tag{14}$$

The introduction of the continuous time is now straightforward. One has to replace n with t/τ. The dynamics of open systems is usually characterized by two parameters: the *decay rate* Γ_1 and the *decoherence rate* Γ_2. In our case we can introduce these parameters as follows

$$\begin{aligned} c^{2t/\tau} &= e^{-\Gamma_1 t} \implies \Gamma_1 = \frac{1}{T_1} = -\frac{2}{\tau}\ln c\,, \\ \left[c(c^2+4s^2w^2)^{1/2}\right]^{t/\tau} &= e^{-\Gamma_2 t} \implies \Gamma_2 = \frac{1}{T_2} = -\frac{1}{\tau}\left[\ln c\sqrt{c^2+4s^2w^2}\right], \end{aligned} \tag{15}$$

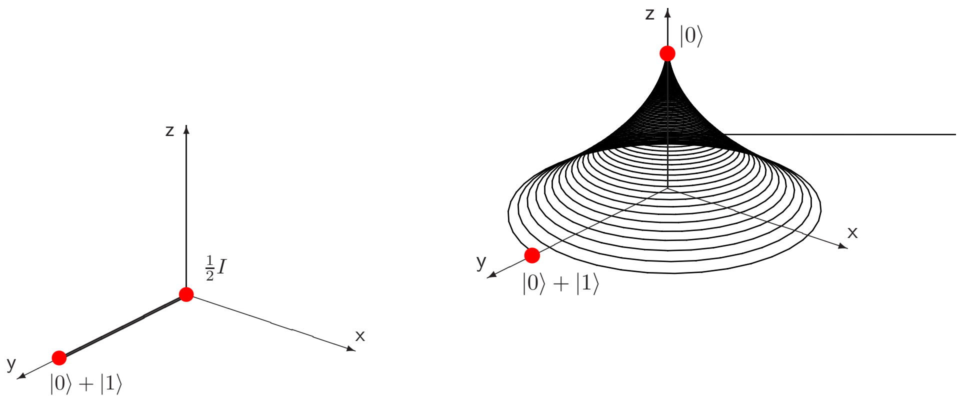

Fig. 2: Visualization of the evolution of a system qubit due to its interaction with reservoir qubit in the process of quantum homogenization. The system is supposed to be initially prepared in a pure state $(|0\rangle + |1\rangle)/\sqrt{2}$. The qubits in reservoir are prepared in the total mixture characterized by the values $w = 0$ (left figure) and and in the state $|0\rangle$ characterized by $w = 1/2$ (right figure).

and the continuous version of the homogenization process can be described by a one-parameter set of maps

$$\mathcal{E}_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\Gamma_2 t}\cos\Omega t & e^{-\Gamma_2 t}\sin\Omega t & 0 \\ 0 & -e^{-\Gamma_2 t}\sin\Omega t & e^{-\Gamma_2 t}\cos\Omega t & 0 \\ 2w(1-e^{-\Gamma_1 t}) & 0 & 0 & e^{-\Gamma_1 t} \end{pmatrix} \quad (16)$$

with frequency $\Omega = \omega/\tau$ describing the "rotating" part of the evolution (see Fig. 2). The general rule is that before deriving the master equation itself one has to check that the maps are valid quantum maps and that they fulfill the semigroup property. However, here we will directly derive the generator, and from its properties the character of transformation will become obvious. Using the method described in previous sections we find out that the derived generator is time independent and it takes the form

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\Gamma_2 & -\Omega & 0 \\ 0 & -\Omega & -\Gamma_2 & 0 \\ 2w\Gamma_1 & 0 & 0 & -\Gamma_1 \end{pmatrix}. \quad (17)$$

Consequently, the master equation is of the Lindblad form with *time-independent* coefficients c_{jk} and h_j. In particular, the non-vanishing parameters read $h_3 = \Omega/2$, $c_{11} = c_{22} = \Gamma_1/4$, $c_{33} = (2\Gamma_2 - \Gamma_1)/4$, and $c_{12} = -i2w\Gamma_1$ and the matrix of c_{jk} is positive whenever $\Gamma_2 \geq 1/2\Gamma_1$, and $|w| \leq 1/2$. The latter condition is trivially satisfied, because of the positivity of ξ, i.e. $|w| \leq 1/2$. The first condition is related to a more general result (for details see [10]), according to which the decay is at most two times faster than the decoherence, i.e. $\Gamma_1 \leq 2\Gamma_2$, or equivalently

$T_2 \leq 2T_1$. Its validity in the case of homogenization can be checked by direct calculation.

The positivity of the matrix c_{jk} guarantees the complete positivity of the whole evolution. The master equation of the homogenization process reads

$$\begin{aligned}\dot{\varrho}_t &= -i\frac{\Omega}{2}[S_3,\varrho_t] + \frac{\Gamma_1}{4}(S_1\varrho S_1 + S_2\varrho S_2 - 2\varrho) + \frac{2\Gamma_2-\Gamma_1}{4}(S_3\varrho S_3 - \varrho)\\ &\quad -iw\Gamma_1(S_1\varrho S_2 - S_2\varrho S_1 + i\varrho S_3 + iS_3\varrho)\,,\end{aligned} \tag{18}$$

where the operators S are given by (10). Let us note that the parameter Ω describes only the Hamiltonian part of evolution, which is in accordance with our expectations. The Hamiltonian part causes the rotation, whereas the other part of the generator induces a contraction into the fixed point ξ. Only in the case when the homogenization is completed the map cannot be inverted, but strictly speaking this happens in the limit when time goes to infinity.

Let us make the following choice of parameters: $w = -1/2$, $\Gamma_1 = 2\Gamma_2 = 2\gamma$. In this case the master equation takes a simple form

$$\dot{\varrho}_t = i\frac{\Omega}{2}[S_3,\varrho] + \gamma[2S_-\varrho S_+ - S_-S_+\varrho - \varrho S_-S_+]\,, \tag{19}$$

where $S_\pm = \frac{1}{2}(S_1 \pm S_2)$. This is the well known equation describing the process of exponential decay with the Hamiltonian $H = -\Omega/2\, S_3$ [4].

5. Case Study II: Decoherence from Collisions

In this section, we will analyze a collision-like dynamics which models a decoherence of a qubit. A more general treatment of decoherence will be presented elsewhere [11], while here we will concentrate on the derivation of the master equation. The task will be the same as before, except that instead of the partial-swap operation we will consider a *partial CNOT* (controlled-NOT operation), i.e. $U_\eta = \cos\eta I + i\sin\eta\,\mathtt{CNOT}$. The CNOT gate performs the σ_x rotation on the *target qubit*, when the *control qubit* is in the state $|1\rangle$. If the control is in the state $|0\rangle$, then the state of the target qubit is not changed. Unlike the swap operation, the controlled NOT is asymmetric under the exchange of qubits. Therefore, we have two different evolutions determined by the role of the system qubit: it can be either the target qubit, or the control qubit.

Let us assume the system qubit acts as the target. In this case the interaction induces the map on the target qubit that reads

$$\mathcal{E}_\xi = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1-2s^2\xi_{11} & 2cs\xi_{11}\\ 0 & 0 & -2cs\xi_{11} & 1-2s^2\xi_{11}\end{pmatrix}, \tag{20}$$

where we used again the notation $c = \cos\eta, s = \sin\eta$ and ξ_{11} stands for $\langle 1|\xi|1\rangle$, i.e. it is the parameter that characterizes the initial state of reservoir qubits. For the calculation of the powers of this map we will use the same approach as before, because again we need to calculate the powers of 2×2 matrix A of the specific form

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \sqrt{a^2+b^2} \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} = \sqrt{a^2+b^2} X(\omega) . \tag{21}$$

In the last equality we exploited the identity $\cos\arctan x = (1+x^2)^{-1/2}$ and we set $\omega = \arctan(b/a)$ to obtain the form suitable for evaluating the powers. Note that $X(\omega)^n = X(n\omega)$. In our particular case $a = 1 - 2s^2\xi_{11}$ and $b = 2cs\xi_{11}$. That is,

$$\mathcal{E}_\xi^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{(1+4s^2\xi_{11}\xi_{00})^{n/2} X(n\omega)} \end{pmatrix} . \tag{22}$$

Following the same methods as in the previous section we can derive the generator of the process $\mathcal{G}$ in a very simple form

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\Gamma & \Omega \\ 0 & 0 & -\Omega & -\Gamma \end{pmatrix} \quad \Longrightarrow \quad \dot{\varrho}_t = -i\frac{\Omega}{2}[\sigma_x, \varrho_t] + \frac{\Gamma}{2}(\sigma_x \varrho_t \sigma_x - \varrho_t) . \tag{23}$$

As seen from its structure this generator describes "pure" decoherence, i.e. the process of diagonalization of the state in the basis associated with the eigenvectors of the operator σ_x. The parameters of the dynamics are defined in a similar way as before, i.e.

$$\Omega = \omega/\tau = \arctan\frac{2cs\xi_{11}}{1-2s^2\xi_{11}} \quad \text{and} \quad \Gamma = -\frac{1}{\tau}\ln\sqrt{1-4s^2\xi_{11}\xi_{00}} .$$

If we consider the system qubit to play the role of the control, then the dynamical map $\mathcal{E}_\xi$ is very similar to the previous case, i.e.

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c^2 + s^2\langle\sigma_x\rangle_\xi & cs(1-\langle\sigma_x\rangle_\xi) & 0 \\ 0 & -cs(1-\langle\sigma_x\rangle_\xi) & c^2+s^2\langle\sigma_x\rangle_\xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{24}$$

where $\langle\sigma_x\rangle_\xi = \mathrm{Tr}\xi\sigma_x$. Defining the parameters

$$\begin{aligned} \Gamma &= -\frac{1}{\tau}\ln\sqrt{c^2+s^2\langle\sigma_x\rangle_\xi} , \\ \Omega &= \frac{1}{\tau}\arctan\frac{cs(1-\langle\sigma_x\rangle_\xi)}{c^2+s^2\langle\sigma_x\rangle} \end{aligned}$$

we can directly write down the master equation

$$\dot{\varrho}_t = -i\frac{\Omega}{2}[\sigma_z, \varrho_t] + \frac{\Gamma}{2}(\sigma_z \varrho_t \sigma_z - \varrho_t) . \tag{25}$$

Like in the previous case we obtain the dynamics describing the decoherence. Nevertheless, there is a difference, if the system qubit plays the role of the control, then the decoherence is observed in the basis associated with the eigenvectors of the σ_z operator, rather then in the basis of eigenvectors σ_x that was the case when the system qubit played the role of the target. We see that irrespective of the fact whether the system qubit is the control or the target the sequence of partial CNOT collisions leads to decoherence processes. There are two differences though. The first one is the basis in which the decoherence takes place and the second is the decoherence time. In particular, the decoherence rates depend on the initial state of the reservoir ξ in different ways.

6. Conclusion

We have shown that using a simple collision-like model one can derive master equations for a qubit interacting with an environment. In our approach the discrete dynamics is described by a sequence of unitary transformations representing bi-partite interactions ("collisions"). As a result of the collision-like evolution the induced one-qubit dynamics is discrete, too. However, we have shown that in specific cases (partial SWAP operation and partial CNOT operation) this essentially discrete evolution can be substituted by a continuous one. The resulting time evolution fulfills the semigroup property. Using the state-space description of the dynamics of a qubit we can say that the evolution of a qubit state that undergoes a sequence of collisions can be illustrated as an ordered sequence of points in the Bloch sphere. We have shown how to connect these points with a smooth line representing the continuous time evolution driven by a Lindblad master equation.

The presented approach can be generalized to higher-dimensional quantum systems (qudits), as well as to collisions described by more general bi-partite unitary transformations U. The method essentially works for any interaction which induce an *invertible* map $\mathcal{E}$ (i.e. $\det \mathcal{E} \neq 0$). It turns out that semigroups $\mathcal{E}_t = e^{-\mathcal{G}t}$ always contain only invertible maps, because for the (usually unphysical) map $\mathcal{E}_{-t}$ the following identity holds $\mathcal{E}_t\mathcal{E}_{-t} = e^{-\mathcal{G}t}e^{+\mathcal{G}t} = e^0 = \mathcal{I}$.

Another (open) problem concerning the derivation of the master equation from a discrete collision-like dynamics is whether for all invertible mappings $\mathcal{E}$, the semigroup generated by its powers can be always interpolated with a continuous semigroup of completely positive maps. This problem leads to the question about the structure of the set of quantum semigroups. It is worth studying which bi-partite interactions in the collision-like models could stand behind the known quantum processes described approximatively by quantum master equations. We believe that this approach of derivation of the master equations provides us with a new insight into the dynamics of open systems.

Acknowledgements

This was work supported in part by the European Union projects QUPRODIS and CONQUEST.

Bibliography

[1] G. Lindblad, *On the generators of Quantum Dynamical Semigroups*, Comm. Math. Phys. **48**, 119 (1976).

[2] E. B. Davies, *Quantum Theory of Open Systems*, Academic, London, 1976.

[3] R. Alicki, K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics, Springer-Verlag, Berlin, 1987.

[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum information*, Cambridge University Press, Cambridge, 2000.

[5] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limit*, Rev. Mod. Phys. **53**, No. 3, 569 (1980).

[6] M. Ziman, P. Štelmachovič, V. Bužek, M. Hillery, V. Scarani, and N. Gisin, *Diluting quantum information: An analysis of information transfer in system-reservoir interactions*, Phys. Rev. A **65**, 042105 (2002), LANL preprint archive quant-ph/0110164.

[7] V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin, and V. Bužek, *Thermalizing quantum machines: Dissipation and Entanglement*, Phys. Rev. Lett. **88**, 097905 (2002), LANL preprint archive quant-ph/0110088.

[8] M. Ziman, P. Štelmachovič, and V. Bužek, *Saturation of Coffman-Kundu-Wootters inequality via quantum homogenization*, J. Opt. B: Quantum Semiclass **5**, 439 (2003).

[9] V. Bužek and M. Hillery, *Quantum copying: Beyond the No-Cloning Theorem*, Phys. Rev. A **54**, 1844 (1996).

[10] S. Daffer, K. Wódkiewicz, J. K. McIver, *Bloch equations and completely positive maps*, *J. Mod. Opt.* **51**, 1843 (2004), LANL preprint archive quant-ph/04011177.

[11] M. Ziman *et al.*, in preparation.