# Entanglement swapping of noisy states: A kind of superadditivity in nonclassicality

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We address the question as to whether an entangled state that satisfies local realism will give a violation of the same after entanglement swapping in a suitable scenario. We consider such a possibility as a kind of superadditivity in nonclassicality. Importantly, it will indicate that checking for violation of local realism, in the state obtained after entanglement swapping, can be a *method* for detecting entanglement in the input state of the swapping procedure. We investigate various entanglement swapping schemes, which involve mixed initial states. The strength of violation of local realism by the state obtained after entanglement swapping is compared with the one for the input states. We obtain a kind of superadditivity of violation of local realism for Werner states, consequent upon entanglement swapping involving Greenberger-Horne-Zeilinger-state measurements. We also discuss whether entanglement swapping of specific states may be used in quantum repeaters with a substantially reduced need to perform the entanglement distillation step.

## I. INTRODUCTION

Quantum nonseparability, in its operational sense, is the existence of states which cannot be prepared by distant observers acting locally and without any supplementary quantum channel. So it may seem that particles which do not share a common past (i.e., which have not been acted on by an interaction Hamiltonian) cannot be nonseparable (or entangled). Surprisingly, however, two particles *can* get entangled even if they do not share a common past. This is achieved in entanglement swapping [1–4]. The phenomenon was experimentally confirmed in [5].

Let us first describe very briefly the phenomenon of entanglement swapping. Suppose Alice and Bob share an entangled state. Similarly Claire and Danny also share some entangled state. See Fig. 1. Now the question is as follows: Can it be possible that Alice's and Danny's particles become entangled without an interaction between their particles?

In Refs. [1–4], the authors have shown that the answer is yes. If their partners Bob and Claire (whose particles are entangled with the particles of Alice and Danny, respectively) come together and make a measurement in a suitable basis and communicate their measurement results clasically (say, by phone call), then Alice's and Danny's particles may become entangled.

A simple example of this phenomenon can be seen if one has two singlets $(1/\sqrt{2})(|01\rangle - |10\rangle)$, one of which is shared by Alice and Bob and the other by Claire and Danny. ($|0\rangle$ and $|1\rangle$ are mutually orthonormal.) Now Bob and Claire make jointly a projection measurement (on their parts of the two singlets) in the Bell basis, which is given by (with the + sign applying to states with odd indices)

$$|B_{1,2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$

$$|B_{3,4}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{1}$$

It is easy to check that if Bob and Claire communicate (over a classical channel) the result of their measurement to Alice and Danny, they will know that they share one of the Bell states given by Eq. (1). Note that, depending on the measurement results, unitary operations $\sigma_x$, $\sigma_y$, $\sigma_z$, or $I$ may be performed by Alice (or Danny) on her (his) qubit to obtain just a singlet. ($I$ is the identity operator on the qubit Hilbert space, and $\sigma_x$, $\sigma_y$, and $\sigma_z$ are the Pauli matrices.) The particles of Alice and Danny are completely independent, and nevertheless they share entanglement after Bob and Claire's Bell measurement (and sending its outcome to them). Note that entanglement swapping can be seen as a specific case of teleportation [6]. In the entanglement swapping process, Bob and Claire make a measurement on their systems and send (teleport) the qubit (say, Bob's subsystem) through a channel to Danny. And after communication to Danny, Alice and Danny share an entangled state. Actually, if all the parties agree on the desired output state of the swapping procedure, Bob and Claire can communicate their results only to Danny, and Alice does not need to know the content of the communication.
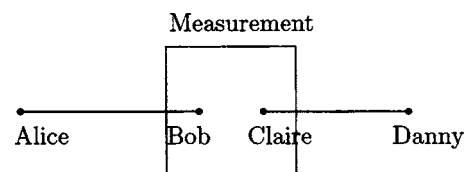


FIG. 1. Entanglement swapping between two states.

Entanglement in shared multiparty states is a fundamental resource in several quantum communication processes. However, it is usually a hard problem to detect whether a given state is entangled (see, e.g., [7]). One way to detect entanglement is to check for violation of local realism. However, there seems to exist entangled states that do not violate local realism. In this paper, we address the following question.

*Question.* Consider a state that is entangled and yet does not violate local realism. Is it possible to find some entanglement swapping process, after which the swapped state will violate local realism?

We provide a partial answer to this question. To that end, we investigate various entanglement swapping schemes which involve mixed entangled states as initial states. These will be, for simplicity, modeled as partially depolarized states [8]. We will be particularly interested to investigate the extent to which the states resulting out of the entanglement swapping process violate local realism. We address the case where the swapping itself (i.e., the measurement required for swapping) is perfect. A study of the complementary situation in which the (multiple) swapping is nonperfect, for the *realistic* case of the parametric down-conversion process, is given in [9].

The parent states considered here are the "Werner mixtures" of certain pure states (say, $|\psi\rangle$, shared between $n$ partners) and the white noise [8]:

$$\varrho = p|\psi\rangle\langle\psi| + (1-p)\varrho_{noise}. \tag{2}$$

The parameter $p$ will be called here visibility. Clearly it shows to what extent the processes that can be described by $|\psi\rangle$ are operationally visible despite the presence of noise. It can be associated with the notion of visibility in multipartite interference experiments. We shall study the relation of the visibility parameter for the initial states and the states after swapping. This will be done in various configurations.

(i) Chain configuration: a chain of entanglement swappings involving initially a sequence of pairs (sharing the parent states) [10,11]. Bell measurements—i.e., measurements projecting onto the Bell states given by Eq. (1)—are performed upon two particles of all adjacent pairs (see Fig. 1 for the case of one entanglement swapping with two pairs). This is described in Sec. II.

(ii) Star configuration: a generalized entanglement swapping involving initially $N$ parent states (each consisting of $M$ particles). An $N$-qubit Greenberger-Horne-Zeilinger- (GHZ-) state measurement [12] is made on $N$ qubits, each belonging to a different state. This is discussed in Sec. III. GHZ-state measurements project onto the GHZ basis. The three-qubit GHZ basis, for example, consists of the states

$$G_{1,2} = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle),$$

$$G_{3,4} = \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle),$$

$$G_{5,6} = \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle),$$

$$G_{7,8} = \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \tag{3}$$

where again the $+$ sign applies to states with odd indices. Similarly one may define an $N$-qubit GHZ basis by considering the binary decompositions of $2^N - 1$.

We shall be interested whether the resulting states after different forms of entanglement swapping are nonclassical. As our benchmark of nonclassicality, we shall use the threshold value of visibility, allowing for violation of suitable Bell inequalities [13,14]. That is, we compare the critical visibility for violation of the local realism of the state obtained after entanglement swapping, with the critical visibility for violation in the input state (parent state) itself.

We obtain a kind of superadditivity in violation of local realism for the case of Werner states in the Hilbert space of dimension $2 \otimes 2$, consequent to entanglement swapping in a specific scenario (Sec. III F). In the concluding section (Sec. V), we discuss the possibility of the existence of such superadditivity for other states in suitably chosen configurations of entanglement swapping. There we will come back to the general question that we have asked in the beginning. We indicate that checking for violation of local realism in the state obtained after entanglement swapping in suitably chosen configurations can be an efficient entanglement witness for the input state. We also discuss the possibility of using entanglement swapping with specific states in the so-called "quantum repeater" [10,11], where the distillation step may not be required or its requirement may be substantially reduced.

## II. CHAIN CONFIGURATION OF ENTANGLEMENT SWAPPING

In this section, we will compare the visibilities of the input state to that of the swapped state, in the case of entanglement swapping between pairs of states in a chain configuration. See Fig. 1 for a chain of two pairs.

The chain configuration of entanglement swapping and the exponential increase of noise in the swapped state have been studied in Refs. [10,11]. We give this brief discussion here to compare with the ring configuration to be considered later.

### A. Entanglement swapping between two pairs

Consider the $(2 \otimes 2)$-dimensional Werner state [15]

$$\rho = p|B_1\rangle\langle B_1| + (1-p)\rho_{noise}^{(2)}. \tag{4}$$

(In this paper, we denote the completely depolarized state of $n$ qubits, $I_n/2^n$, as $\rho_{noise}^{(n)}$, where $I_n$ is the identity operator of the Hilbert space of $n$ qubits.) The state $\rho$ is entangled when $p > \frac{1}{3}$, but the state violates local realism only for $p > 1/\sqrt{2}$. Let Alice (A) and Bob (B) share the state $\rho$, and let Claire (C) and Danny (D) also share such a state [16]. Bob and Claire (who are together) make a measurement on their part of the two states in the Bell basis $\{B_i\}$ given by Eq. (1) (see Fig. 1). We are interested in whether the state of the qubits of Alice and Danny after the swap violates local realism. After

the Bell measurement, if the measurement result is $B_1$, the state shared by Alice and Danny is a Werner state of the form

$$\xi_{AD}^{(2)} = p^2 |B_1\rangle\langle B_1| + (1-p^2)\rho_{noise}^{(2)}. \qquad (5)$$

Since $\xi_{AD}^{(2)}$ is a Werner state, it is entangled for $p > 1/\sqrt{3}$, but violates local realism when $p > (\frac{1}{2})^{1/4}$. Of course, the same condition is obtained for the other Bell measurement outcomes. Therefore the region in which the final state $\xi_{AD}^{(2)}$ violates Bell inequalities is strictly contained in the region in which the initial state $\rho_{AB}$ has the same property. We see that there is a region of $p$—namely, $p \in (1/\sqrt{2}, (\frac{1}{2})^{1/4})$, for which the output state will not be able to show any violation of local realism (but it is still entangled), whereas the input states do violate in that region. Therefore we have a "loss in the region of violation of local realism" after entanglement swapping.

### B. Chain of $N$ states: "Loss" increases with $N$

This phenomenon of "loss" becomes more and more pronounced as the number of swappings is increased. Starting with $N$ initial Werner states in Eq. (4) shared between $A_k$ and $B_k$ ($k = 1, 2, \ldots, N$), the swapped state between $A_1$ and $B_N$ (after Bell measurements performed by $B_1A_2, B_2A_3, \ldots, B_{N-1}A_N$) is again the Werner state

$$p^N |B_1\rangle\langle B_1| + (1-p^N)\rho_{noise}^{(2)}. \qquad (6)$$

Hence the swapped state violates local realism for

$$p > \left(\frac{1}{2}\right)^{1/N}. \qquad (7)$$

Therefore in the case of a series of a large number of entanglement swappings, the swapped state can violate local realism only when initial state is almost pure.

Note that if we consider a chain of $N$ Werner states with different visibilities—i.e., if

$$p_k |B_1\rangle\langle B_1| + (1-p_k)\rho_{noise}^{(2)} \qquad (8)$$

is shared between $A_k$ and $B_k$ ($k = 1, 2, \ldots, N$)—then the swapped state between $A_1$ and $B_N$ is the Werner state

$$p_1 p_2 \cdots p_N |B_1\rangle\langle B_1| + (1 - p_1 p_2 \cdots p_N)\rho_{noise}^{(2)}. \qquad (9)$$

Therefore, again we have that the region of violation of local realism of the swapped state is strictly smaller than the region of violation of the parent states in the $(p_1, p_2, \ldots, p_N)$ space. The former is vanishing for sufficiently large $N$.

### III. STAR CONFIGURATION ENTANGLEMENT SWAPPING

Let us consider entanglement swapping in a different configuration than that was considered in Sec. II. Assume a multiparty situation in which initially disjoint subsets of parties share entangled states. In the next stage, single representatives of each subset of parties meet together and perform a GHZ-state measurement. The result of the measurement is sent to the remaining parties. This procedure results in an entangled state shared by them. We shall call this type of
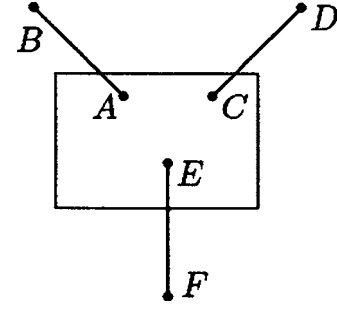


FIG. 2. A star configuration swapping. A GHZ basis measurement is performed on A, C, and E. This is represented by a box.

entanglement swapping as entanglement swapping in a "star configuration."

### A. Star swapping between three states

Suppose that pairs AB, CD, and EF each share the same Werner state $\rho = p |B_1\rangle\langle B_1| + (1-p)\rho_{noise}^{(2)}$ (see Fig. 2). A, C, and E come together and perform a measurement in their $(2 \otimes 2 \otimes 2)$-dimensional Hilbert space in the GHZ basis as given in Eq. (3). After the measurement, if $G_1$ clicks, then B, D, and F share the state

$$\xi_{BDF}^{(3)} = p^3 |G_1\rangle\langle G_1| + (1-p^2)\rho_{noise}^{(3)}$$
$$+ \frac{1}{2}p^2(1-p)(|000\rangle\langle 000| + |111\rangle\langle 111|). \qquad (10)$$

Other measurement results give the same state up to local unitary transformations. This indicates that the amount of violation of local realism, considered below for the state $\xi_{BDF}^{(3)}$, is also attained for the swapped states for other measurement results. Thus, although the partners B, D, and F need to know the result of the measurement performed by A, C, and E, they can keep the output state in every run of the measurement. Note that the swapped state now is not a mixture of white noise and $|G_1\rangle\langle G_1|$ only, and this is so whenever $p \neq 1$.

We will now use the Mermin-Klyshko (MK) inequalities to study the violation of local realism by the swapped state (see the Appendix). Let us first calculate $\text{tr}(B_3 \xi_{BDF}^{(3)})$. [See Eq. (A1).] Suppose that the observables are chosen from the $x$-$y$ plane [17]. That is, we choose

$$\sigma_{a_j} = |+, \phi_j\rangle\langle +, \phi_j| - |-, \phi_j\rangle\langle -, \phi_j|,$$

$$\sigma_{a'_j} = |+, \phi'_j\rangle\langle +, \phi'_j| - |-, \phi'_j\rangle\langle -, \phi'_j|, \qquad (11)$$

where

$$|\pm, \phi_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\phi_j}|1\rangle),$$

$$|\pm, \phi'_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\phi'_j}|1\rangle). \qquad (12)$$

The only term in the state given by Eq. (10) that will contribute to the expression $\mathrm{tr}(B_3 \xi_{BDF}^{(3)})$ is the first one, and more precisely, its part given by

$$\frac{p^3}{2}(|000\rangle\langle 111| + |111\rangle\langle 000|). \qquad (13)$$

(This observation would help us in the more general cases that we consider in the succeeding subsections.)

For the GHZ state $|G_1\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$, one has

$$\max \mathrm{tr}(B_3 |G_1\rangle\langle G_1|) = 2, \qquad (14)$$

and this maximal violation of local realism by the GHZ state is reached in the $x$-$y$ plane. Therefore the maximal value reached by $\mathrm{tr}(B_3 \xi_{BDF}^{(3)})$, for any choice of $\phi_j$ and $\phi'_j$ by the parties, is given by

$$\max \mathrm{tr}(B_3 \xi_{BDF}^{(3)}) = 2p^3. \qquad (15)$$

Consequently, the state $\xi_{BDF}^{(3)}$ violates a MK inequality for $\max \mathrm{tr}(B_3 \xi_{BDF}^{(3)}) > 1$ for

$$p > \left(\frac{1}{2}\right)^{1/3} \simeq 0.7937. \qquad (16)$$

Our initial Werner state $\rho$ violates Bell inequalities when

$$p > \frac{1}{\sqrt{2}} \simeq 0.7071. \qquad (17)$$

One should compare this with the case of entanglement swapping between *two* Werner states, where the swapped state gives violation for

$$p > \left(\frac{1}{2}\right)^{1/4} \simeq 0.8409. \qquad (18)$$

While considering violation of local realism by the state $\xi_{BDF}^{(3)}$, we have used only the Mermin-Klyshko inequalities. However, in this case one can also consider the Weinfurter-Werner-Wolf-Żukowski-Brukner (WWWZB) inequalities [18–20], which are a necessary and sufficient condition for the violation of local realism by the $N$-qubit correlations of an arbitrary state of $N$ qubits, when there are two settings at each site.

Let us first define the correlation tensor for $N$-qubit states. An $N$-qubit state $\rho$ can always be written down as

$$\frac{1}{2^N} \sum_{x_1,\ldots,x_N = 0, x, y, z} T_{x_1,\ldots,x_N} \sigma_{x_1}^{(1)} \otimes \cdots \otimes \sigma_{x_N}^{(N)}, \qquad (19)$$

where $\sigma_0^{(k)}$ is the identity operator and the $\sigma_{x_i}^{(k)}$'s $(x_i = x, y, z)$ are the Pauli operators of the $k$th qubit. The coefficients

$$T_{x_1,\ldots,x_N} = \mathrm{tr}(\rho \sigma_{x_1}^{(1)} \otimes \cdots \otimes \sigma_{x_N}^{(N)}) \quad (x_i = x, y, z) \qquad (20)$$

are elements of the $N$-qubit correlation tensor $\hat{T}$ and they fully define the $N$-qubit correlation functions of the state $\rho$.

Consider now the state $\xi_{BDF}^{(3)}$, obtained via entanglement swapping, as given in Eq. (10). One can check that the three-qubit correlation tensor $\hat{T}$ of this state contains only those terms which are also present for the GHZ state $G_1$. Precisely, the correlation tensor $\hat{T}$ of $\xi_{BDF}^{(3)}$ is given by

$$\hat{T}_{\xi_{BDF}^{(3)}} = p^3 \hat{T}_{G_1}, \qquad (21)$$

where $\hat{T}_{G_1}$ is the correlation tensor of the GHZ state $G_1$ given by

$$\begin{aligned} \hat{T}_{G_1} = &\vec{x}_1 \otimes \vec{x}_1 \otimes \vec{x}_1 - \vec{x}_1 \otimes \vec{x}_2 \otimes \vec{x}_2 \\ &- \vec{x}_2 \otimes \vec{x}_1 \otimes \vec{x}_2 - \vec{x}_2 \otimes \vec{x}_2 \otimes \vec{x}_1, \end{aligned} \qquad (22)$$

with $\vec{x}_1 = \vec{x}$ and $\vec{x}_2 = \vec{y}$. Hence, when the quantum correlation function is computed by inserting $\hat{T}_{\xi_{BDF}^{(3)}}$ into the generalized Bell inequality of WWWZB, one gets the value $2p^3$. This is because for the GHZ state, the value is 2. This maximal value $(2p^3)$ is attained for the measurement in the $x$-$y$ plane and was already obtained [in Eq. (15)] for the state $\xi_{BDF}^{(3)}$ when we considered the MK inequalities. Therefore the state $\xi_{BDF}^{(3)}$ violates local realism for $p > (1/2)^{1/3}$. Moreover, from our considerations of the WWWZB inequalities, we have that for lower values of the parameter $p$, the three-qubit correlations of $\xi_{BDF}^{(3)}$ have a local realistic model for two measurement settings at each site.

### B. Other forms of the star configuration of swapping

In the preceding subsection, we have shown that the "star configuration" leads to stronger resistance to noise admixture than with Bell measurements in the "chain configuration" (discussed in Sec. II). The parent states that we considered (in the preceding subsection) were bipartite states. Let us now consider the case of entanglement swapping with measurements in a GHZ basis when the parent states are multipartite states.

Consider therefore the state

$$\rho_3 = F|G_1\rangle\langle G_1| + (1 - F)\rho_{noise}^{(3)}, \qquad (23)$$

where $|G_1\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$. This state violates local realism for

$$F > \frac{1}{2}. \qquad (24)$$

Let two such states be shared between A, B, C and D, E, F, with A and D placed together (Fig. 3).

A and D make a Bell measurement on their parts of the two states. After the measurement, the resulting state violates the MK inequalities in the $x$-$y$ plane for

$$F > (2^{3/2})^{-1/2} \simeq 0.5946. \qquad (25)$$

Note here that we do not need the explicit form of the state, just like the case for three states. The terms in the state that contribute to the violation of MK inequality in the $x$-$y$ plane are $|0\cdots0\rangle\langle 1\cdots1|$ and $|1\cdots1\rangle\langle 0\cdots0|$.

With three $\rho_3$'s and a swapping in the three-qubit GHZ basis [given in Eq. (3)] on the three qubits (one from each of
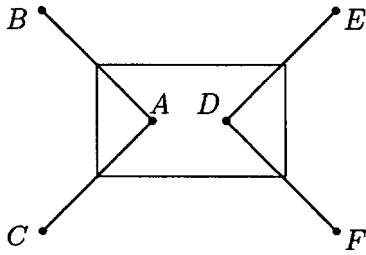
FIG. 3. A star configuration swapping. A Bell measurement is performed on A and D which is denoted by a box.

the $\rho_3$'s) (see Fig. 4), the MK inequality is violated in the $x$-$y$ plane for

$$F > (2^{5/2})^{-1/3} \simeq 0.5612. \qquad (26)$$

*Thus the following picture is emerging*: Entanglement swapping involving GHZ measurements is less fragile (to violation of local realism) than Bell measurements, with respect to the noise admixtures in the initial states.

### C. General star configuration entanglement swapping

We will now generalize the entanglement swapping process in the star configuration. Consider the following $M$-qubit state:

$$\rho_M = V|\text{GHZ}_M\rangle\langle\text{GHZ}_M| + (1 - V)\rho_{noise}^{(M)}, \qquad (27)$$

where $|\text{GHZ}_M\rangle = (1/\sqrt{2})(|0\rangle^{\otimes M} + |1\rangle^{\otimes M})$. Take $N$ copies of $\rho_M$. The $i$th copy $(i=1,2,\ldots,N)$ is shared between $A_i$ and $B_{i1}, B_{i2}, \ldots, B_{i(M-1)}$. We suppose that all $A_i$'s are at the same location of the observer called Alice. (The schematic diagram in Fig. 4 is drawn when both $N$ and $M$ are 3.) She makes a measurement in the $N$-qubit GHZ basis. [See Eq. (3) for the three qubit GHZ basis.] As in the previous cases, we take the observables in the $x$-$y$ plane—i.e., the ones given by Eq. (11)—in the MK inequality. Here also we do not need the explicit form of the state. The terms that contribute to the violation of MK inequality in the $x$-$y$ plane are $|0\cdots0\rangle\langle1\cdots1|$ and $|1\cdots1\rangle\langle0\cdots0|$. Therefore we obtain that the resulting $N(M-1)$-qubit state violates this inequality for
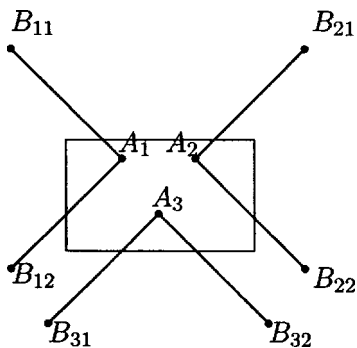


FIG. 4. A star configuration swapping. $A_1, B_{11}, B_{12}, A_2, B_{21}, B_{22}$, and $A_3, B_{31}, B_{32}$ share noisy GHZ states. All $B$'s are at distant locations but $A$'s are in the same laboratory. A GHZ basis measurement is performed by $A_1$, $A_2$, and $A_3$, as depicted in the figure by a box.
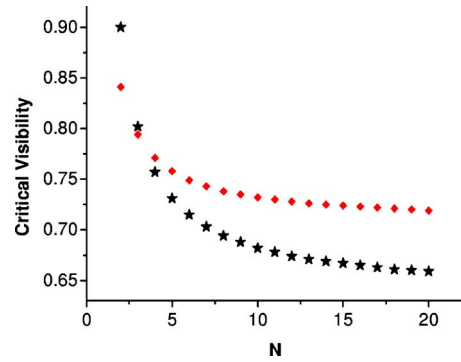


FIG. 5. (Color online) (a) Plot of the critical visibilities $V_N^{(M)}$ for $M=2$ (the stars). This is the critical visibility required (in the parent state) for violation of local realism by the swapped state in the case when one representative from each of $N$ Werner states [Eq. (4)] come together to perform an entanglement swapping in the $N$-qubit GHZ basis (see Fig. 6). The violation of local realism in the swapped state is considered by using the MK inequalities. (b) The critical visibility $V_N^f$ (the diamonds) is that for violation of local realism by considering functional Bell inequalities in the same $N$-qubit swapped state. The figure shows the relative monotonic decrease of the two visibilities with $N$.

$$V > V_N^{(M)} \equiv (2^{[N(M-1)-1]/2})^{-1/N}. \qquad (28)$$

This expression is easily obtained once we remember our observation for the derivation of Eq. (15) [21].

### D. There is no loss in the asymptotic regime

We remember that our parent state $\rho_M$, as given in Eq. (27), violates local realism for

$$V > \left(\frac{1}{\sqrt{2}}\right)^{M-1}. \qquad (29)$$

Note that $V_N^{(M)}$ [as given by Eq. (28)] is monotonically decreasing with respect to $N$. A plot of the critical visibility $V_N^{(M)}$ for $M=2$, which is the visibility obtained when the swapping in a star configuration is performed on $N$ number of copies of two-qubit Werner states, is given in Fig. 5. It clearly shows the monotonic decrease of the critical visibility in $N$.

Thus the system is surprisingly robust to noise admixture with respect to violation of local realism in the following sense. The amount of (white) noise that the parent state can afford so that the state after entanglement swapping still violates local realism increases monotonically as we consider swapping between higher number of parties in a star configuration. Moreover, one has

$$V_N^{(M)} \to \left(\frac{1}{\sqrt{2}}\right)^{M-1} \quad \text{as } N \to \infty. \qquad (30)$$

This shows that the amount of noise that the parent state can afford, so that the state obtained after entanglement swapping violates the MK inequality, in the asymptotic limit of arbitrarily large number of subsystems (in the way considered above—that is, in the star situation), *coincides* with the

amount of noise that can be afforded by the parent state itself to violate local realism. The loss in the region of violation of local realism is more and more recovered as we consider entanglement swapping between higher and higher number of parties, and ultimately in the asymptotic limit, there is *no* loss in the region of violation of local realism.

In this general situation, the state obtained after performing the entanglement swapping is an incoherent mixture of some product states and a "weakened" GHZ state (i.e., a GHZ state admixed with white noise). The product states contribute only to the $T_{z\cdots z}$ component of the correlation tensor $\hat{T}$ [cf. Eq. (20)] of the state obtained after entanglement swapping. Here we have considered violation of the MK inequalities (by the state obtained after swapping) only in the *x-y* plane. This is because the gradual reduction of loss of the region of violation of local realism after entanglement swapping and the disappearance of this loss asymptotically are already obtained in this plane. However, we do not rule out a faster reduction of loss if all the WWWZB inequalities are considered.

From the perspective of the recent works indicating that Bell inequality violation is a signature of "useful entanglement" [22–25], our result here can be also viewed as showing (in a particular case) that in an entanglement swapping process, this useful entanglement is lost, but this loss may be asymptotically vanishing. Below in Secs. III E and III F, we will show that useful entanglement can even be "gained" in an entanglement swapping process, and this gain can be possible even without going into the asymptotic regime. Here, by "gain," we mean a situation in which the swapped state violates local realism, even when the parent state does not violate. That is, there exists values of the visibility *V*, for which the parent states do not, while the swapped state does violate Bell inequalities after performing the swap. We will perform the swapping in a star configuration.

### E. Star entanglement swapping in the light of functional Bell inequality

The Bell inequalities we have considered up to now are the ones in which there is only a finite number of (in fact, 2) settings per local site. However, there are Bell inequalities in which one may consider even a continuous range of settings of the local apparatus, as described in the Appendix, subsection A 2.

Let us consider violation of local realism by the swapped state as revealed by a functional Bell inequality. For simplicity, let us consider the parent states to be a two-qubit state, although all our considerations can be generalized to a parent state of higher number of qubits. Suppose therefore that the Werner state, given by Eq. (4), is shared between two parties A and B. Numerical calculations have indicated that the Werner state violates local realism for $p > 1/\sqrt{2}$ even for a high number of settings per observer [26,27]. We use as a working hypothesis that $p = 1/\sqrt{2}$ is indeed the threshold value below which there exist an explicit local realistic model which returns the quantum predictions for the continuous range of settings.

Consider now the "star" configuration described before, where $A_1B_1, A_2B_2, \ldots, A_NB_N$ share *N* Werner states, each
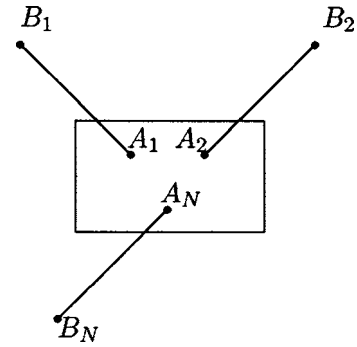


FIG. 6. *N* Werner states are distributed between $A_i$ and $B_i$ (for $i = 1, 2, \ldots, N$) and a GHZ basis measurement is performed at $A_1, A_2, \ldots, A_N$.

given by Eq. (4) (see Fig. 6). The $A_i$'s come together and perform a measurement in the *N*-qubit GHZ basis [see Eq. (3)] as has been discussed previously. We will now consider violation of local realism of the swapped state by using the functional Bell inequalities.

Consider the local observable at the *j*th location to be

$$\sigma_{a_j}(\phi_j) = |+, \phi_j\rangle\langle +, \phi_j| - |-, \phi_j\rangle\langle -, \phi_j|, \quad (31)$$

where

$$|\pm, \phi_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\phi_j}|1\rangle). \quad (32)$$

The aggregate $\xi_j$ of local parameters, in the functional Bell inequality, at the *j*th site is just the single parameter $\phi_j$ here. One can now easily show that the swapped state violates local realism [by violating the functional Bell inequality with the observables as defined in Eq. (31)] for [28,29,52]

$$p > V_N^f \equiv \frac{2}{\pi}(2)^{1/N}. \quad (33)$$

$V_N^f$ is plotted in Fig. 5 and compared with the critical visibility $V_N^{(2)}$ for the same swapped state, but when violation of the MK inequalities are considered.

### F. Kind of superadditivity for Werner states

There is an important consequence of the relation (33). For $N \geq 7$, the critical visibility $V_N^f$ is strictly less than $1/\sqrt{2}$, which is the critical visibility for the Werner state to violate local realism (on the basis of the multisetting numerical results in Ref. [26]).

Therefore, in the process of entanglement swapping, there seems to be a kind of superadditivity with respect to violation of local realism. Suppose Alice shares seven Werner states with seven Bobs $B_1, \ldots, B_7$. Each of the states is given by Eq. (4), with a visibility $p \in (2/\pi, 1/\sqrt{2})$. Such Werner states do not violate local realism [26]. Now suppose that Alice makes a measurement in the generalized GHZ basis and communicates her result to all the Bobs over a classical channel. The state created at the Bobs violates local realism (by violating the functional Bell inequality, as discussed above) for [cf. Eq. (33)]

$$p > \frac{2}{\pi}(2)^{1/7} \simeq 0.7029, \qquad (34)$$

which is strictly less than $1/\sqrt{2} \simeq 0.7071$. Yet for such visibilities which are lower than $1/\sqrt{2}$, any *single* pair of particles shared between Alice and any one Bob will not be able to violate local realism. (Recall that we have assumed that taking more settings at each site does not help to improve the critical visibility of violation of local realism by the Werner state [26].) It is in this sense that we obtain a kind of "superadditivity" in violation of local realism.

For sufficiently large $N$,

$$V_N^f \to \frac{2}{\pi} \simeq 0.6366. \qquad (35)$$

*Let us note here a surprising coincidence.* An explicit construction of local hidden variable model for the Werner state exists (until the date) for all possible projection measurements in a plane by the two parties for just $p \leq 2/\pi$ [30,31].

It must be stressed that the kind of superadditivity obtained here is not related to a distillation protocol [32]. As distinct from a distillation protocol, we do not consider measurements depending on previous measurements. Also, in our case, the Alices are together while the Bobs can be far apart. Collective operations are required on both ends in the usual distillation protocols. Both the recurrence method and the one-way hashing method [32,33] require CNOT operations at both ends, which is not possible in our case, as the Bobs are not together. The distillation protocol (for all entangled states of two qubits) in Ref. [34] starts with a filtering operation, but must be subsequently followed by the recurrence method, which is again not allowed in our case. The situation is similar for the protocol in Ref. [35]. In the distillation protocol presented in Ref. [36], measurements on more than two copies of the input are required. The experimentally feasible protocol given in Refs. [37,38] also requires collective operations at both ends. There is a further difference of the entanglement swapping protocol considered in this paper with entanglement distillation protocols [32]. As we have noted before [just after Eq. (10)], in our entanglement swapping scheme, the receivers of the swapped state (the $B_i$'s in Fig. 6) need to know the result of the measurement performed by the $A_i$'s. However, in contrast to the distillation protocols, they do not need to discard the swapped state for some measurement results. In a distillation protocol, discarding some of the outputs is absolutely essential, as entanglement cannot increase under local actions.

In Ref. [39], two Werner states shared by $A_1B_1$ and $A_2B_2$, respectively, are shown to violate local realism, although the individual states are nonviolating. But in Ref. [39], collective tests are required at both ends. That is, both $A_1, A_2$ and $B_1, B_2$ are required to be together. In our case, although the Alices must be together, the Bobs are separated. Therefore the "superadditivity" reached in this subsection is of a different kind than the one in Ref. [39].

## IV. ENTANGLEMENT SWAPPING IN QUANTUM REPEATERS

In the preceding section, we have obtained an example in which the initial state has a local realistic model, but surprisingly, after entanglement swapping, the resulting state can violate local realism. This was obtained with the initial states as Werner states. However, these results were obtained with the star configuration, for which the entanglement swapping process swaps multifold two-particle entanglement into a multiparticle entanglement. That is the initial and final entangled states apply to a different number of qubits.

Therefore we shall now ask a different question. Consider two pairs of qubits, each pair independent of the other and both in an identical quantum state $\rho^{in}$. That is, say Alice and Bob share $\rho_{AB}^{in}$, which is formally identical with the state shared by Claire and Danny $\rho_{CD}^{in}$. Is it possible that the entanglement swapping process, involving a (two-qubit) Bell measurement jointly by Bob and Claire, may lead to a new state, $\rho_{AD}^{out}$ shared by Alice and Danny, which has the property that it violates local realism more strongly than each of the initial states? Note that now we start with two (identical) two-qubit states and end with another two-qubit state. The properties of entanglement of the initial and final states can now be compared directly.

Therefore, we consider below the case of two-qubit entanglement in the initial states, two-qubit Bell-state measurements, and two-qubit final states. As we shall see one can find specific initial states which after entanglement swapping lead to a two-qubit state which violates local realism more robustly than the initial ones.

Consider the initial state

$$\rho_\lambda = \lambda|\psi\rangle\langle\psi| + \frac{1-\lambda}{2}(|00\rangle\langle00| + |11\rangle\langle11|), \qquad (36)$$

where $|\psi\rangle = a|01\rangle - b|10\rangle$ [and $\lambda > 1/2(1-ab)$]. It is entangled whenever $\lambda > 1/(1+2ab)$ [40]. For $\lambda \leq 1/(1+a^2b^2)$, this state does not violate any Bell inequality. However, despite the fact that for $\lambda \in (1/(1+2ab), 1/(1+a^2b^2))$ the state $\rho_\lambda$ can be modeled with local hidden variable models, it was shown in Ref. [41] that after a suitable local filtering operation [42], the resulting state violates local realism.

Thus, Alice and Bob share a state $\rho_\lambda$, and so do the other two. After a Bell measurement performed by Bob and Claire, if the measurement result is $|B_1\rangle$, the final state of Alice and Danny is

$$\xi_{AD}^{(\rho_\lambda)} = \frac{1}{A}\left[ \frac{\lambda^2 a^2 b^2}{2}|B_1\rangle\langle B_1| + \frac{(1-\lambda)^2}{8}(|00\rangle\langle00| + |11\rangle\langle11|) \right.$$
$$\left. + \frac{\lambda(1-\lambda)}{2}(a^2|01\rangle\langle01| + b^2|10\rangle\langle10|) \right], \qquad (37)$$

where $A = \lambda^2 a^2 b^2 + (1-\lambda^2)/4$. This state violates local realism for

$$\lambda > \frac{1}{\sqrt{1 + 4(\sqrt{2} - 1)a^2b^2}}. \tag{38}$$

Therefore the region of violation of local realism for the swapped state $\xi_{AD}^{(\rho_\lambda)}$ is not greater than that for the parent states (that is, for $\rho_\lambda$). Despite this fact, the *amount* of violation of local realism is greater in the state $\xi_{AD}^{(\rho_\lambda)}$, as compared to that in the initial state $\rho_\lambda$, for some ranges of the parameters.

We will now indicate that it is potentially possible to use the state $\rho_\lambda$ [of Eq. (36)] in a quantum repeater, where there may be a reduced need to perform the entanglement distillation step. The entanglement of formation [32,44] of the state $\rho_\lambda$ is given by

$$E_{\rho_\lambda} = H\left( \frac{1 + \sqrt{1 - C_{\rho_\lambda}^2}}{2} \right), \tag{39}$$

$$C_{\rho_\lambda} = \max\{0, (1 + 2ab)\lambda - 1\}, \tag{40}$$

where $H$ is the binary entropy function given by $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$. The entanglement of formation of the state $\xi^{(\rho_\lambda)}$ [given in Eq. (37)], obtained after entanglement swapping with two $\rho_\lambda$, can also be similarly calculated by using the prescription given in Ref. [44]. One can then check that there exist ranges of the parameters $a$ and $\lambda$, for which the entanglement of formation of the state $\xi^{(\rho_\lambda)}$ is greater than that in the state $\rho_\lambda$. It may be possible to use this phenomenon in a quantum repeater, in which the need to perfom the distillation step is substantially reduced. We will follow this up in a later publication.

Note that the "superadditivity" reported in Sec. III F (for the Werner states) is of a different kind than in Ref. [41]. Importantly, note here that the superadditivity reported in Sec. III F is for Werner states. And at least for a single copy of a Werner state, one cannot reproduce the kind of "self-superadditivity" by using local filtering operations [43], as was done in [41]. Moreover, for the case of Werner states (Sec. III F), although we do require postselection (just as in Ref. [41]), all the postselected cases lead to the same result of increased nonclassicality (in contrast to that in Ref. [41]).

## V. DISCUSSION

To conclude, we have shown an example of entanglement swapping process, in which although the initial state has a local realistic model, after performing entanglement swapping, the final swapped state can violate local realism. This was obtained by using the initial states as Werner states. We regard this as a kind of superadditivity for Werner states. We have also considered another family of states, in which we have shown that the amount of violation of local realism, as also amount of entanglement (as quantified by entanglement of formation), is increased after entanglement swapping.

Coming back to the general question posed in the Introduction, it may be true that it is a generic feature that an entangled state which satisfies local realism will violate local realism after a suitable entanglement swapping procedure. If

this is true, then this method can be used to detect entanglement in the laboratory. Suppose Alice and Bob who are in different locations share some state. They want to find out whether their shared state is entangled or not. One way is to perform a Bell experiment and find whether their state violates local realism. If the state violates local realism, then they conclude that their state is entangled. If the state does not violate local realism, they cannot infer anything about the entanglement of the state. However, Alice and Bob can apply the method discussed in this paper. They can perform entanglement swapping on some copies of the state in a suitable configuration and then check whether the resulting state violates local realism. If yes, then they can infer that the input state was entangled. It is interesting to find out the most general class of states whose entanglement can be detected in this way.

As we have noted earlier, in general, our schemes of entanglement swapping in different configurations are not "distillation" [32]. Take, for example, the "star" configuration considered in Sec. III (see Fig. 6). There, the parties $B_1, \ldots, B_N$ do not share any entanglement before swapping. So they simply do not have entanglement before and thus cannot "distill" it. Our scheme is entanglement distribution rather than entanglement distillation. However, there is a way to see our scheme also as a distillation one.

For example, in the case of a chain of two pairs of entangled particles (A-B and C-D), if Alice has particle A and Bob has particles B, C, and D, one can consider our scheme as entanglement distillation. Then Bob performs Bell-type analysis on particles B and C and projects particles A and D on a new entangled state. It is a distillation because Alice and Bob had previously shared entanglement in A-B and after swapping has entanglement in A-D (see [3,4]). To see this as a distillation scheme, we must see whether the output in A-D is more entangled than the input in A-B. If that is true in some cases, then only a subensemble of swapped pairs will be more entangled than the parent pairs. Others must be less entangled, as entanglement cannot increase under local operations. Interestingly, there exist states for which an increase of entanglement is possible after entanglement swapping (as shown in Sec. IV). This means that for those states, a "quantum repeater" [10,11] may potentially be based only on entanglement swapping. Note that for the Werner states, one needs both entanglement swapping *and* entanglement distillation (see [10,11]). In Sec. IV, we indicated a possible candidate for such a phenomenon.

Other entanglement swapping schemes considered in this paper, such as the star configuration, can also be considered as a "distillation" scheme in the following sense. Consider Fig. 6 with $N = 3$ (for example), and imagine that party $B_1$ has particle $B_1$, party $B_2$ has particle $B_2$, and party $B_3$ has particles $A_1$, $A_2$, $A_3$, and $B_3$. Then, indeed the three parties shared two-particle entanglement even before swapping, and the swapped state contains genuine three-particle entanglement [45,46]. So our scheme might not only be a kind of distillation, but also a procedure which can transform one type of entanglement to another one (two-particle to three-particle entanglement, in our example).

Finally, it is intriguing to find out whether there exists a Bell inequality for which the superadditivity of Werner states

considered in this paper can be explained in the following way. Let us consider the case of superadditivity for seven Werner states (Sec. III F). In the star configuration, there are therefore eight partners who share these states (see Fig. 6). It will be interesting if there exists a Bell inequality for eight partners such that the seven Werner states shared by them will violate the inequality for $p > (2/\pi)(2)^{1/7}$ [see Eq. (34)].

## APPENDIX: BELL INEQUALITIES

For obtaining violation of local realism by the swapped state, we have considered two different types of multipartite Bell inequalities: multiparticle Mermin-Klyshko inequalities [47–51] (subsection A 1) and the functional Bell inequality [52] (subsection A 2).

### 1. Mermin-Klyshko inequalities

A Bell operator for the so-called Mermin-Klyshko inequality for $N$ qubits (shared between observers $A_1$, $A_2, \ldots, A_N$) can be defined recursively as [53]

$$B_k = \frac{1}{2} B_{k-1} \otimes (\sigma_{a_k} + \sigma_{a'_k}) + \frac{1}{2} B'_{k-1} \otimes (\sigma_{a_k} - \sigma_{a'_k}),$$
(A1)

with $B'_k$ obtained from $B_k$ by interchanging $a_k$ and $a_k'$ and

$$B_1 = \sigma_{a_1} \quad \text{and} \quad B'_1 = \sigma_{a'_1}.$$
(A2)

The party $A_j$ is allowed to choose between the measurements $\sigma_{a_j}$ and $\sigma_{a'_j}$. Here $\vec{a}_j$ and $\vec{a'}_j$ are two three-dimensional unit vectors $(j = 1, 2, \ldots, N)$, and for example, $\sigma_{a_j} = \vec{\sigma} \cdot \vec{a}_j$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

An $N$-qubit state $\eta$ violates MK inequality if

$$\text{tr}(B_N \eta) > 1.$$
(A3)

### 2. Functional Bell inequalities

To study the violation of local realism of the swapped state, we will (along with the MK inequalities) also consider the functional Bell inequalities [52].

The functional Bell inequalities [52] essentially follow from a simple geometric observation that in any real vector space, if for two vectors $h$ and $q$ one has $\langle h | q \rangle < \|q\|^2$, then this immediately implies that $h \neq q$. In simple words, if the scalar product of two vectors has a lower value than the length of one of them, then the two vectors cannot be equal.

Let $\varrho_N$ be a state shared between $N$ separated parties. Let $O_n$ be an arbitrary observable measured at the $n$th location $(n = 1, \ldots, N)$. The quantum mechanical prediction $E_{QM}$ for the correlation in the state $\varrho_N$, when these observables are measured, is

$$E_{QM}(\xi_1, \ldots, \xi_N) = \text{tr}(O_1 \cdots O_N \varrho_N),$$
(A4)

where $\xi_n$ is the aggregate of the local parameters at the $n$th site. Our objective is to see whether this prediction can be reproduced in a local hidden variable theory. A local hidden variable correlation in the present scenario must be of the form

$$E_{LHV}(\xi_1, \ldots, \xi_N) = \int d\lambda \rho(\lambda) \Pi_{n=1}^{N} I_n(\xi_n, \lambda),$$
(A5)

where $\rho(\lambda)$ is the distribution of the local hidden variables and $I_n(\xi_n, \lambda)$ is the predetermined measurement result of the observable $O_n(\xi_n)$ corresponding to the hidden variable $\lambda$.

Consider now the scalar product

$$\langle E_{QM} | E_{LHV} \rangle = \int d\xi_1 \cdots d\xi_N E_{QM}(\xi_1, \ldots, \xi_N) E_{LHV}(\xi_1, \ldots, \xi_N)$$
(A6)

and the norm

$$\|E_{QM}\|^2 = \int d\xi_1 \cdots d\xi_N [E_{QM}(\xi_1, \ldots, \xi_N)]^2.$$
(A7)

If we can prove that a strict inequality holds—namely, for all possible $E_{LHV}$, one has

$$\langle E_{QM} | E_{LHV} \rangle \leq B,$$
(A8)

with the number $B < \|E_{QM}\|^2$—we will immediately have $E_{QM} \neq E_{LHV}$, indicating that the correlations in the state $\varrho_N$ are of a different character than in any local realistic theory. We then could say that the state $\varrho_N$ violates the "functional" Bell inequality (A8), as this Bell inequality is expressed in terms of a typical scalar product for square integrable functions. Note that the value of the product depends on a continuous range of parameters (of the measuring apparatuses) at each site.

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