INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6051–6064 doi:10.1088/0305-4470/38/26/014

Bounds on action of local quantum channels

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Received 17 February 2005, in final form 27 April 2005
Published 15 June 2005
Online at stacks.iop.org/JPhysA/38/6051

Abstract

We derive an upper bound on the action of a direct product of two quantum maps (channels) acting on bi-partite quantum states. We assume that the individual channels Λ_j affect single-particle states so that for an arbitrary input ρ_j, the distance $D_j(\Lambda_j[\rho_j], \rho_j)$ between the input ρ_j and the output $\Lambda_j[\rho_j]$ of the channel is less than ϵ. Given this assumption we show that for an arbitrary *separable* two-partite state ρ_{12}, the distance between the input ρ_{12} and the output $\Lambda_1 \otimes \Lambda_2[\rho_{12}]$ fulfils the bound $D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant \sqrt{2+2\sqrt{(1-1/d_1)(1-1/d_2)}}\,\epsilon$ where d_1 and d_2 are the dimensions of the first and second quantum system respectively. In contrast, entangled states are transformed in such a way that the bound on the action of the local channels is $D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant 2\sqrt{2-1/d}\,\epsilon$, where d is the dimension of the smaller of the two quantum systems passing through the channels. Our results show that the fundamental distinction between the set of separable and the set of entangled states results in two different bounds which in turn can be exploited for discrimination between the two sets of states. We generalize our results to multi-partite channels.

PACS numbers: 03.67.−a, 03.65.Ud, 03.65.Yz

1. Introduction

Investigation of properties of communication channels is more than ever today a central issue of information science. It is generally accepted that quantum systems have the capacity to carry information efficiently, and any transformation of these systems can be considered as an action of a quantum channel (see, e.g., [1–5]).

Some general questions arising from the transmission of quantum entaglement through quantum channels have been analysed by Schumacher in [6]. He has considered a pure entangled state of a pair of two systems R and Q, and the system Q has been subjected to a dynamical evolution (quantum channel). Schumacher has shown that the two quantities of interest, the entanglement fidelity F_e and the entropy exchange S_e, can be related to various other fidelities and entropies and are connected by an inequality reminiscent of the Fano inequality of classical information theory.

In this paper we address the question how do two local channels (each acting independently) affect a bi-partite quantum state. This scenario is rather general and can be applied to a number of situations, e.g. quantum computation with quantum computer imperfectly isolated from the environment or analysis of quantum error correcting codes. In the context of quantum error correction, this problem has been addressed by Knill and Laflamme in [7] for a particular case of two qubits and for a particular choice of a distance (fidelity) characterizing the change of the bi-partite state. In [8] Aharonov *et al* have analysed errors for a general model of quantum computation with mixed states and non-unitary operations. There however a different measure was introduced. The rationale being that measurable distinguishability of gates (super-operators) should not increase if we consider additional quantum systems which do not evolve. Here in contrast we are not interested in the distinguishability of superoperators but rather in the actions of the channels on a given state and how to relate these local actions to the change of the global state.

Specifically, consider a pair of quantum channels characterized by maps Λ_1 and Λ_2, respectively. It means that after sending a quantum system over, for instance, the first channel, the final state of the quantum system (or equivalently the output of the channel) is $\Lambda_1[\rho_1]$ where ρ_1 is the corresponding input. Moreover, let the two channels fulfil the following condition,

$$D_j(\Lambda_j[\rho_j], \rho_j) \leqslant \epsilon, \qquad \forall \rho_j \in \mathcal{S}(\mathcal{H}_j), \quad j = 1, 2, \tag{1}$$

where $D_j(.,.)$ for $j = 1, 2$ are some distance functions (metric) defined on the set of all density operators $\mathcal{S}(\mathcal{H}_1)$ and $\mathcal{S}(\mathcal{H}_2)$ representing the set of all physically realizable states of quantum systems passing through channels 1 and 2, respectively. These conditions restrict the action of each of the two channels independently of the action of the other channel. Specifically, the state of a quantum system affected by one of the two channels has to be in a small (epsilon) neighbourhood of the state describing the quantum system before the system was sent through the channel.

The parameter ϵ quantifies the action of the two quantum channels. For $\epsilon = 0$ the two channels are 'perfect' (i.e., the information transmitted via channels is not disturbed) as the output equals the input while for ϵ large the output can be significantly different from the corresponding input[4].

The question we would like to address is, how big is the change induced by the two local channels when the inputs are correlated. That is, let us prepare an arbitrary initial state ρ_{12} of two quantum systems. The first part of the jointly prepared system is sent over the first channel while the second part is sent over the second channel. Both channels individually fulfil condition (1), where, e.g., $\rho_1 = \mathrm{Tr}_2\, \rho_{12}$. In what follows we will show that the two-partite action of the channel $\Lambda_1 \otimes \Lambda_2$ for all possible physical states $\rho_{12} \in \mathcal{S}(\mathcal{H}_{12})$ fulfils a bound on its action that is determined by single-partite conditions given by equation (1).

Let us note that the problem can be transformed into the estimation of the map $\Omega = \Lambda_1 \otimes \Lambda_2 - \mathbb{1}_{12}$ where the map $\mathbb{1}_{12}$ is the identity acting on the joint system. If the distance $D_{12}(.,.)$ as well as distances $D_1(.,.)$ and $D_2(.,.)$ are defined via a norm then our task is to estimate the norm $\|\Omega(\rho_{12})\|$. Similar expressions for a general class of the so-called p-norms has been studied extensively for Ω being a physical map (more specifically the product of two physical maps) in [9, 10]. However, in our case the map Ω is neither a positive map nor

[4] Let us note that there is no relation between the parameter ϵ and the capacity of the channel in general.

a direct product of two maps. Due to the fact that the map Ω is not positive and subsequently not physical our situation is not applicable to [9, 10] and similar studies.

The paper is organized as follows. In section 2 we introduce necessary definitions and discuss a particular case of separable states, i.e. the initial state of the joint system (the system composed of two quantum systems that are sent over the two quantum channels) is separable. As a next step we drop any assumptions on the initial state and analyse the most general case of an arbitrary initial state in section 3. The results obtained are discussed in subsection 3.1. In section 4 we extend our analysis to the case of more than two quantum channels and illustrate the nature of changes on a simple example. Finally, in section 5 we summarize our results and outline possible extensions.

2. Separable inputs

In the formulation of the problem we encounter three different metric (distance) functions: $D_1(.,.)$, $D_2(.,.)$ and $D_{12}(.,.)$ acting on different sets and thus measuring distances between different types of objects. In order to make our discussion explicit we will consider a specific choice of the distances offered by the norm of the Hilbert–Schmidt spaces corresponding to systems 1, 2 and the joint system 12, respectively[5]

$$D_a(\rho_a, \sigma_a) \equiv \|\rho_a - \sigma_a\|_a \\ = \sqrt{\mathrm{Tr}_a[(\rho_a - \sigma_a)(\rho_a - \sigma_a)^\dagger]}. \tag{2}$$

The label a denotes systems 1, 2 or the joint system 12, and $\rho_a, \sigma_a \in \mathcal{S}(\mathcal{H}_a)$ are the density operators representing possible physical states of the system labelled a. The norms that we have used to define distances $D_1(.,.)$, $D_2(.,.)$ and $D_{12}(.,.)$ are called 2-norms and are only a particular case of the so-called p-norms. However, due to the fact that we will use only basic properties of the distances $D_1(.,.)$, $D_2(.,.)$ and $D_{12}(.,.)$, we will keep our derivation as general as possible so that it can be repeated with a broad class of different distances. Only in the end will we use the specific choice of distances to derive a tight bound.

Our task is to estimate the distance

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}), \tag{3}$$

for all physically reasonable initial states $\rho_{12} \in \mathcal{S}(\mathcal{H}_{12})$ provided the two maps Λ_1 and Λ_2 fulfil condition (1). First, note that for any distance (this follows from the triangle property of a distance) holds

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant D_{12}(\Lambda_1 \otimes \mathbb{1}[\rho_{12}], \rho_{12}) \\ + D_{12}(\mathbb{1} \otimes \Lambda_2[\Lambda_1 \otimes \mathbb{1}[\rho_{12}]], \Lambda_1 \otimes \mathbb{1}[\rho_{12}]). \tag{4}$$

It means that instead of considering the case with two local channels, it is sufficient to consider only an action of a single local channel acting on one of the two subsystems and estimate the distance $D_{12}(\Lambda_1 \otimes \mathbb{1}[\rho_{12}], \rho_{12})$.

We start with the simplest case—the case of factorizable states of the form $\rho_{12} = \rho_1 \otimes \rho_2$. This corresponds to the situation as if the two channels were considered separately so that the two quantum systems that are sent through the channels are prepared individually. In this case we exploit the following property,

$$D_{12}(\rho_1' \otimes \rho_2, \rho_1 \otimes \rho_2) \leqslant D_1(\rho_1', \rho_1), \tag{5}$$

[5] The set of all density operators representing the set of physical states of a quantum system is a subset of a vector space. In such a case it is natural to define the metric (distance function) with the help of a norm so that the linear structure of the vector space is respected. There are several ways to introduce a norm on a vector space. However, the set of all density operators is also a subset of the Hilbert–Schmidt space which is a Hilbert space and we can use the norm induced with the scalar product of the Hilbert space.

of the distances $D_1(.,.)$, $D_2(.,.)$ and D_{12} where the operators ρ_1, ρ_1' and ρ_2 are the density operators representing states of the first and the second system, respectively. Let us note that this relation holds even if the distances are defined with any p-norm or even fidelity. Using equation (5) we have that $D_{12}(\Lambda_1 \otimes \mathbb{1}[\rho_1 \otimes \rho_2], \rho_1 \otimes \rho_2) \leqslant D_1(\Lambda_1[\rho_1], \rho_1)$ and consequently, for the initial state of the form $\rho_1 \otimes \rho_2$, the distance (3) is always less than or at most equal to 2ϵ

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_1 \otimes \rho_2], \rho_1 \otimes \rho_2) \leqslant 2\epsilon, \tag{6}$$

due to equations (4) and (1).

The same holds for the initial state ρ_{12} of the form $\rho_{12} = \sum_i \alpha_i \rho_1^i \otimes \rho_2^i$, where $\alpha_i \geqslant 0$, $\sum_i \alpha_i = 1$ and ρ_1^i and ρ_2^i denote the density operators of systems 1 and 2 respectively, that follows from the linearity of the map $\Lambda_1 \otimes \Lambda_2$

$$D_{12}\left(\Lambda_1 \otimes \Lambda_2\left[\sum_i \alpha_i \rho_1^i \otimes \rho_2^i\right], \sum_i \alpha_i \rho_1^i \otimes \rho_2^i\right) \\ = D_{12}\left(\sum_i \alpha_i \Lambda_1 \otimes \Lambda_2\left[\rho_1^i \otimes \rho_2^i\right], \sum_i \alpha_i \rho_1^i \otimes \rho_2^i\right),$$

and the fact that the distance $D_{12}(.,.)$ is *jointly convex*, that is

$$D_{12}\left(\sum_j \alpha_j \Lambda_1 \otimes \Lambda_2\left[\rho_1^j \otimes \rho_2^j\right], \sum_j \alpha_j \rho_1^j \otimes \rho_2^j\right) \\ \leqslant \sum_j \alpha_j D_{12}\left(\Lambda_1 \otimes \Lambda_2\left[\rho_1^j \otimes \rho_2^j\right], \rho_1^j \otimes \rho_2^j\right).$$

The last expression is a sum of terms where each term is bounded by 2ϵ, and the sum of the coefficients α_i is equal to unity. In consequence we obtain the bound

$$D_{12}\left(\Lambda_1 \otimes \Lambda_2\left[\sum_i \alpha_i \rho_1^i \otimes \rho_2^i\right], \sum_i \alpha_i \rho_1^i \otimes \rho_2^i\right) \leqslant 2\epsilon, \tag{7}$$

for an arbitrary separable state.

2.1. Hilbert–Schmidt distance

The bound on the action of a product of two quantum channels on separable states (7) is valid for any triple of distances $D_1(.,.)$, $D_2(.,.)$ and $D_{12}(.,.)$ that satisfy relation (5) (the distance $D_2(.,.)$ has to fulfil relation (5) with swapped labels 1 and 2) and in addition the distance $D_{12}(.,.)$ has to be *jointly convex*. That is, the bound is valid if $D_1(.,.)$, $D_2(.,.)$ and $D_{12}(.,.)$ are trace distances[6] or, more generally, the distances defined with p-norms or even fidelity. The question is whether it is possible to derive a better (tighter) bound or, in other words, whether the bound is optimal. For the trace distances the bound is optimal indeed, and it can be shown that there is a pair of maps such that the bound is saturated. In what follows we will show that for the distances introduced in equation (2) the bound can be further optimized.

Let $\rho_1 = 1/d_1 \mathbb{1} + \bar{c}\cdot\bar{\sigma}$ be an input of channel 1. We have expressed the state of the system labelled as '1' using the identity operator $\mathbb{1}$ and $d_1^2 - 1$ generators $\bar{\sigma} = \{\sigma_1, \sigma_2, \ldots\}$ of the group $SU(d_1)$ multiplied with the complex unity where d_1 is the dimension of the Hilbert space

[6] The trace distance is defined with the help of the 1-norm, and $D(\rho, \sigma)$ is equal to the sum of eigenvalues of the positive operator $|\rho - \sigma|$ where $|\rho - \sigma| \equiv \sqrt{(\rho - \sigma)^\dagger(\rho - \sigma)}$ and $\rho, \sigma \in \mathcal{B}(\mathcal{H})$.

of system 1 and the vector $\bar{c} = \{c_1, \ldots\}$ is a real vector with $d_1^2 - 1$ elements. In addition we require that the set of operators $\{\sigma_\alpha\}$ satisfy the ortho-normalization condition $\mathrm{Tr}\,\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta}$. After the quantum system has been sent through the quantum channel Λ_1, the state of the system (the output) can be expressed using the same notation $\Lambda_1[\rho_1] = 1/d_1 \mathbb{1} + \bar{c}' \cdot \bar{\sigma}$ with new coefficients $\bar{c}'$ where the prime indicates the fact that the system has been sent through the quantum channel. Equivalently, $\rho_2 = 1/d_2 \mathbb{1} + \bar{d} \cdot \bar{\tau}$ is the most general state of system 2 where $\bar{\tau}$ are generators of $SU(d_2)$ multiplied with complex unity, d_2 is the dimension of the Hilbert space of system 2 and the operators $\{\tau_\beta\}$ satisfy the relation $\mathrm{Tr}\,\tau_\beta\tau_\omega = \delta_{\beta\omega}$.

We estimate the distance (3) for an arbitrary separable state and the particular choice of distances (2). Due to the *joint convexity* of the distance $D_{12}(.\,,.)$ and the linearity of the map $\Lambda_1 \otimes \Lambda_2$ it is sufficient to consider the case where the state ρ_{12} is a pure state (for more details see the end of the previous section)

$$\rho_{12} = (1/d_1 \mathbb{1} + \bar{c} \cdot \bar{\sigma}) \otimes (1/d_2 \mathbb{1} + \bar{b} \cdot \bar{\tau}) \tag{8}$$

where $\bar{c} \cdot \bar{c} = (1 - 1/d_1)$ and $\bar{b} \cdot \bar{b} = (1 - 1/d_2)$.

In this case we do not use relation (4) which means that the two channels are not considered separately and the output of the product of the two channels Λ_1 and Λ_2 is

$$\Lambda_1 \otimes \Lambda_2[\rho_{12}] = (1/d_1 \mathbb{1} + \bar{c}' \cdot \bar{\sigma}) \otimes (1/d_2 \mathbb{1} + \bar{b}' \cdot \bar{\tau}). \tag{9}$$

Inserting the two expressions, input (8) and output (9), into the definition of the distance (2) we obtain that

$$\begin{aligned} D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) = {} & \|(\bar{c}' - \bar{c}) \cdot \bar{\sigma} \otimes 1/d_2 \mathbb{1} + 1/d_1 \mathbb{1} \otimes (\bar{b}' - \bar{b}) \cdot \bar{\tau} \\ & + \bar{c}' \cdot \bar{\sigma} \otimes \bar{b}' \cdot \bar{\tau} - \bar{c} \cdot \bar{\sigma} \otimes \bar{b} \cdot \bar{\tau}\|_{12}. \end{aligned} \tag{10}$$

Last expression squared can be bounded from above by a sum of three terms

$$\begin{aligned} & \|(\bar{c}' - \bar{c}) \cdot \bar{\sigma} \otimes 1/d_2 \mathbb{1}\|_{12}^2 + \|1/d_1 \mathbb{1} \otimes (\bar{b}' - \bar{b}) \cdot \bar{\tau}\|_{12}^2 \\ & \quad + [\|(\bar{c}' - \bar{c}) \cdot \bar{\sigma} \otimes \bar{b}' \cdot \bar{\tau}\|_{12} + \|\bar{c} \cdot \bar{\sigma} \otimes (\bar{b}' - \bar{b}) \cdot \bar{\tau}\|_{12}]^2. \end{aligned}$$

Observing that $\|(\bar{c} - \bar{c}') \cdot \bar{\sigma} \otimes 1/d_2 \mathbb{1}\|_{12}^2 = 1/d_2 D_1^2(\Lambda_1[\rho_1], \rho_1)$, and equivalently $\|1/d_1 \mathbb{1} \otimes (\bar{b} - \bar{b}') \cdot \bar{\tau}\|_{12}^2 = 1/d_1 D_2(\Lambda_2[\rho_2], \rho_2)$ and $\bar{b}' \cdot \bar{b}' \leqslant (1 - 1/d_2)$ we can bound the distance squared with the expression $1/d_2\epsilon^2 + 1/d_1\epsilon^2 + (\sqrt{1 - 1/d_1} + \sqrt{1 - 1/d_2})^2\epsilon^2$. Finally, the distance between the input and the corresponding output of the product of the two channels fulfils the bound

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant \sqrt{2 + 2\sqrt{(1 - 1/d_1)(1 - 1/d_2)}}\,\epsilon, \tag{11}$$

where d_1 and d_2 are the dimensions of the Hilbert spaces corresponding to the quantum systems sent through channels 1 and 2, respectively. Even though we have proved the bound for pure separable states, we note that the result is valid for an arbitrary separable state due to the linearity of the map $\Lambda_1 \otimes \Lambda_2$ and the *joint convexity* of the distance $D_{12}(.\,,.)$. Bound (11) is undoubtedly better than bound (7) as it has been derived for a specific choice of distances. In addition, it can be shown that the bound is optimal in the sense that there is a pair of maps Λ_1 and Λ_2 and a separable state ρ_{12} such that bound (11) is saturated (optimality is discussed in more detail in section 5).

3. Entangled states

We have seen that if the initial state of the joint system 12 is factorizable or even separable then the action of the two channels is bounded by the expression $\sqrt{2 + 2\sqrt{(1 - 1/d_1)(1 - 1/d_2)}}\,\epsilon$. It may be tempting to say that the same holds for an arbitrary state. However, as the next example illustrates, if the joint state of the two systems 1 and 2 is entangled then for certain maps the separable bound can be broken.

Let us consider the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ corresponding to systems 1 and 2 to be two-dimensional spaces. This is the simplest possible case though the physical representations of such systems are numerous. As an example we can mention spin one-half particles, polarized photons or particular internal degrees of freedom of an ion. Let us note that in quantum information theory such systems are denoted as qubits since they represent the quantum analogue of a classical bit of information.

Then, any physical state of system 1 (or equivalently of system 2) can be written as $\rho_1 = \frac{1}{2}(\mathbb{1} + \vec{\alpha}\cdot\vec{\sigma})$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a vector in a three-dimensional real vector space and the three matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the well-known Pauli operators. For the matrix ρ_1 to represent a physical state the norm of the real vector $\vec{\alpha}$ has to be less than or equal to 1. It follows that the set of all physically realizable states of system 1 corresponds to a unit ball (Bloch sphere) in the three-dimensional vector space $\mathbb{R}^3$.

The map Λ_1, we will consider in this particular example, is a simple contraction of the ball representing the set of states such that

$$\Lambda_1 : \rho_1 \to \tfrac{1}{2}(\mathbb{1} + (1-k)\vec{\alpha}\cdot\vec{\sigma}), \tag{12}$$

where $(1-k)$ is a parameter of the contraction. Physically, the map Λ_1 describes a channel with uncoloured ('white') noise since each input state is mixed with the absolute mixture $1/2\,\mathbb{1}$ which is the fixed point of the Λ_1. In order to preserve condition (1) the parameter k has to fulfil the relation $k \leqslant \sqrt{2}\epsilon$. In what follows we assume $k = \sqrt{2}\epsilon$.

In the same way the most general state of system 2 is $\rho_2 = \frac{1}{2}(\mathbb{1} + \vec{\beta}\cdot\vec{\sigma})$, where $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ is a real vector and $|\beta| \leqslant 1$. The map Λ_2 has been chosen to be the same as the map Λ_1

$$\Lambda_2 : \rho_2 \to \tfrac{1}{2}(\mathbb{1} + (1-k')\vec{\beta}\cdot\vec{\sigma}), \tag{13}$$

with the same contraction parameter $k' = k = \sqrt{2}\epsilon$ so that condition (1) is fulfilled in this case too.

To show that the separable bound can be broken we have to consider an entangled state. However, we will not consider an arbitrary state but a very specific one—a maximally entangled state known as the Bell state of the form $\rho_{12} = 1/2(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$, where 0 and 1 denote two basis vectors of $\mathcal{H}_1$ (or $\mathcal{H}_2$). For subsequent calculations, it is useful to rewrite the state using the Pauli operators $\rho_{12} = 1/4(\mathbb{1}\otimes\mathbb{1} - \sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2 - \sigma_3\otimes\sigma_3)$. Inserting ρ_{12} into equation (3) and using the linearity of the transformation $\Lambda_1\otimes\Lambda_2$ as well as equation (2) the distance in equation (3) reads

$$D_{12}(\Lambda_1\otimes\Lambda_2[\rho_{12}], \rho_{12}) = \left\| \frac{-k-k'+kk'}{4}\{\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3\} \right\|_{12}.$$

Both constants, k as well as k', are equal to $\sqrt{2}\epsilon$. Neglecting terms of the order ϵ^2 and evaluating the norm using the scalar product we find

$$D_{12}(\Lambda_1\otimes\Lambda_2[\rho_{12}], \rho_{12}) \approx \sqrt{6}\epsilon. \tag{14}$$

This result clearly shows that even though the two maps Λ_1 and Λ_2 fulfil relations (1) the map $\Lambda_1\otimes\Lambda_2$ constructed as a *direct* product of the two maps can affect the states it acts on in a much stronger way. How much the joint (and particularly entangled) states can be changed by two arbitrary maps Λ_1 and Λ_2 is addressed in the next paragraph.

Let ρ_{12} be an arbitrary mixed state. The deviation of the output of the channel $\Lambda_1\otimes\Lambda_2$ from the input ρ_{12} is characterized by the distance (3). In order to estimate the distance we exploit (as in the case of separable states) the bound given by equation (4)

$$D_{12}(\Lambda_1\otimes\Lambda_2[\rho_{12}], \rho_{12}] \leqslant D_{12}(\Lambda_1\otimes\mathbb{1}[\rho_{12}], \rho_{12}) + D_{12}(\mathbb{1}\otimes\Lambda_2[\tilde{\rho}_{12}], \tilde{\rho}_{12}), \tag{15}$$

where $\tilde{\rho}_{12} = \Lambda_1 \otimes \mathbb{1}[\rho_{12}]$. As we do not make any assumptions, neither about the maps Λ_1 and Λ_2 nor the initial state ρ_{12}, the two states $\tilde{\rho}_{12}$ and ρ_{12} can be arbitrary physical states of the joint quantum system, i.e. arbitrary density operators. It means that taking, for instance, the first term on the right-hand side of equation (15) we need to estimate this term for all possible maps Λ_1 and all possible states ρ_{12}. This fact allows us to rewrite the bound for (3) in a different way

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant 2 \sup_{\{\Lambda_1, \rho_{12}\}} \|\Lambda_1 \otimes \mathbb{1}[\rho_{12}] - \rho_{12}\|_{12},$$

where the factor 2 appears because we have two terms in equation (15) and the supremum runs over all possible maps Λ_1 and all initial states ρ_{12}.[7]

A mixed state ρ_{12} can be decomposed into a mixture of pure states $\rho_{12} = \sum_k \alpha_k |\psi_k\rangle\langle\psi_k|$. Using a basic property of the norm (or *joint convexity* of the distance) and the normalization condition $\sum_k \alpha_k = 1$ we can simplify the last expression and instead of searching for the supremum over all possible states ρ_{12} of the joint system 12, it is sufficient to consider pure states only. It means that

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant 2 \sup_{\{\Lambda_1, |\psi\rangle\langle\psi|\}} \|\Lambda_1 \otimes \mathbb{1}[|\psi\rangle\langle\psi|] - |\psi\rangle\langle\psi|\|_{12}, \tag{16}$$

where the supremum runs over all possible maps Λ_1 and all possible pure states $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H}_{12})$ of the joint system 12. Since we have used only a basic property of the norm the last relation is valid for any distance defined with the help of a norm (or more generally any distance that is *jointly convex*). However, in what follows we will use specific properties of the Hilbert–Schmidt norm and further results are valid for that particular choice of the norm only.

Any pure state $|\psi\rangle \in \mathcal{H}_{12}$ can be expressed using the Schmidt basis

$$|\psi\rangle = \sum_{k=1}^{n_\psi} \beta_k |k\rangle_1 \otimes |k\rangle_2, \tag{17}$$

where $\{|k\rangle_1\}$ and $\{|k\rangle_2\}$ are two sets of orthonormal vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and β_k are the real positive coefficients. The integer n_ψ denotes the number of elements in the Schmidt decomposition of the given pure state and is always less than or equal to the dimension of the smaller of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. In this particular basis the state $\rho_{12} = |\psi\rangle\langle\psi|$ has the form

$$|\psi\rangle\langle\psi| = \sum_{k,l=1}^{n_\psi} \beta_k \beta_l |k\rangle_1\langle l| \otimes |k\rangle_2\langle l|. \tag{18}$$

Let us now estimate the expression $\|\Lambda_1 \otimes \mathbb{1}[|\psi\rangle\langle\psi|] - |\psi\rangle\langle\psi|\|_{12}^2$ from equation (16). Using equation (18) for the density operator $|\psi\rangle\langle\psi|$ and tracing over the degrees of freedom belonging to the second system we have that

$$\|\Lambda_1 \otimes \mathbb{1}[|\psi\rangle\langle\psi|] - |\psi\rangle\langle\psi|\|_{12}^2 = \sum_{k,l=1}^{n_\psi} \beta_k^2 \beta_l^2 \,\mathrm{Tr}_1 V_{kl} (V_{kl})^\dagger, \tag{19}$$

where $V_{kl} = \Lambda_1[|k\rangle_1\langle l|] - |k\rangle_1\langle l|$. At this point we apply relations (A.1), (B.1) and (B.2) (proved in appendices A and B) and equation (1) that establish the following inequalities:

$$\begin{aligned} &\mathrm{Tr}_1 V_{kl}(V_{kl})^\dagger \leqslant 2\epsilon^2, && \forall\, k \neq l, \\ &\mathrm{Tr}_1 V_{kk}(V_{kk})^\dagger \leqslant \epsilon^2, && \forall\, k. \end{aligned}$$

[7] Given the fact that we have specified the dimension of neither system 1 nor system 2, the two systems can be different. Therefore we should find the supremum over all Λ_1 and all ρ_{12} of the first expression in equation (15) and all Λ_2 and all $\tilde{\rho}_{12}$ of the second expression in equation (15). However, the results are the same in both cases.

These inequalities bound each contribution (trace term) in the sum on the right-hand side of equation (19). If we replace each term with the corresponding bound and maximize over all possible β_j then we estimate the expression on the left-hand side of the last equality as

$$\|\Lambda_1 \otimes \mathbb{1}[|\psi\rangle\langle\psi|] - |\psi\rangle\langle\psi|\|_{12}^2 \leqslant (2 - 1/d)\,\epsilon^2,$$

where d is the dimension of the smaller of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in the case that the two subsystems 1 and 2 are different[8]. Since the result is independent of both the map Λ_1 and the state $|\psi\rangle\langle\psi|$ it holds for all maps Λ_1 and all density operators $|\psi\rangle\langle\psi|$ (representing pure states). Consequently, the supremum over all maps Λ_1 and all pure states ρ_{12} is less than or equal to this value and so is the distance (3)

$$D_{12}(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) \leqslant 2\sqrt{2 - 1/d}\,\epsilon. \tag{20}$$

Bound (20) is valid for entangled as well as separable states. However, for separable states we have already found a tighter bound $\sqrt{2 + 2\sqrt{(1 - 1/d_1)(1 - 1/d_2)}}\,\epsilon$ (see equation (11)) which means that the entangled states can be affected by independent channels more strongly than separable states.

3.1. Detection of entanglement

The difference in the behaviour of separable and entangled states resulted in two different bounds. The bound for entangled states is weaker and this bound is obeyed by entangled and separable states. On the other hand the bound for separable states (11) is tighter and need not be fulfilled by entangled states. Subsequently, any state that violates bound (11) is necessarily entangled and a direct product of physical channels can be exploited as a kind of 'entanglement witness'. Let us point out that the entanglement witnesses, known in the literature [11, 12], are based on a different approach. They are constructed using positive but not completely positive maps (that is non-physical maps) acting on one of the two subsystems, and the non-positivity of the final operator (output) is the indication of entanglement. In contrast, in our case, we have a product of two physical maps so that a physical (completely positive) map is acting on each of the two subsystems and the difference between an input and the corresponding output is measured. In addition there is a potential advantage in this approach. Not only the question whether a state is entangled or separable can be answered. If we relate the distance to the entanglement then we could answer the question how much entanglement is shared by two quantum systems.

Similarly as in the case of entanglement witnesses, given a pair of maps, the detection need not be (and in general is not) perfect. In other words given a pair of channels only a subset of the set of all entangled states violates bound (11), and those are the only states that are detected as entangled. Naturally, it is desirable to optimize the detection so that the whole set of entangled states is detected. There are several things we can do to optimize the detection of entangled states using quantum channels:

(i) optimal choice of the distances $D_1(.\,,.)$, $D_2(.\,,.)$ and $D_{12}(.\,,.)$;
(ii) optimal choice of the pair of channels (maps Λ_1 and Λ_2) and subsequent derivation of the bound for separable states for that particular choice.

It is obvious that both elements influence detection of entanglement. Let us point out that the choice of maps is not limited to physical channels. The problem is usually formulated

[8] If the two subsystems 1 and 2 are different then the number of elements n_ψ in the Schmidt decomposition equation (17) is always less than or equal to d—the dimension of the smaller of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Consequently, the number of coefficients β_j we maximize over is always bounded by this number, which in turn bounds the maximum.

as follows: given a density matrix of a bipartite system how strongly are the two subsystems entangled. That is we have a complete knowledge of the elements of the density matrix and we are allowed to execute arbitrary operation (function) on the matrix to calculate the entanglement. Such operation can be non-physical and even nonlinear. Construction of entanglement witnesses using a general class of non-physical but linear maps has been investigated in [13]. The authors have shown that with the help of linear maps it is possible to distinguish perfectly the set of entangled states from the set of separable states. Here we show that this approach could be useful not only for the problem of detection but also for the problem of quantifying entanglement.

Let us express the most general bipartite two-qubit state ρ_{12} using the Pauli operators σ_j, $j = 1, 2, 3$

$$\rho_{12} = \frac{1}{4}\left[\mathbb{1} + \sum_{j=1}^{3}\alpha_j\sigma_j\otimes\mathbb{1} + \sum_{k=1}^{3}\mathbb{1}\otimes\sigma_k + \sum_{j,k=1}^{3}\gamma_{jk}\sigma_j\otimes\sigma_k\right],$$

where α_j, β_k and γ_{jk} for $j, k = 1, 2, 3$ are the real parameters. Further, consider a linear map Λ_{12}

$$\Lambda_{12} : \rho_{12} \to \rho_{12} + \frac{\epsilon}{4}\left(-\mathbb{1} + \sum_{j,k=1}^{3}\gamma_{jk}\sigma_j\otimes\sigma_k\right). \tag{21}$$

With the help of the map (21) and the trace distance we define the following function:

$$\mathcal{F}(\rho_{12}) \equiv \frac{1}{\epsilon}\,\mathrm{Tr}_{12}|\Lambda_{12}[\rho_{12}] - \rho_{12}| - 1. \tag{22}$$

The factor $1/\epsilon$ is there to eliminate the dependence on the epsilon while -1 has been added for convenience only. The function $\mathcal{F}$ has the following properties:

(i) $\mathcal{F}\big(\sum_j \lambda_j\rho_{12}^j\big) \leqslant \sum_j \lambda_j\mathcal{F}\big(\rho_{12}^j\big)$, convexity.
(ii) $\mathcal{F}\big(U_1\otimes U_2\,\rho_{12}\,U_1^\dagger\otimes U_2^\dagger\big) = \mathcal{F}(\rho_{12})$, local unitary equivalence $\forall U_1$ and $\forall U_2$.
(iii) $\mathcal{F}(\rho_{12}) \geqslant 0$, $\forall\rho_{12}$, non-negativity.
(iv) $\mathcal{F}(\rho_{12}) = 0$, for all separable states.
(v) $\mathcal{F}(\rho_{12}) = C(\rho_{12})$, where ρ_{12} is a pure or Werner state (for definition of the Werner state see [14]) and $C(\rho_{12})$ is the concurrence (see [15]).

Through the extension of the proposed method to non-physical maps and a suitable choice of the map acting on the joint state ρ_{12} we have managed to construct a function that detects entanglement on all Werner states. Moreover, some of the listed properties of the function $\mathcal{F}$ are supposed to be fulfilled by a function that not only distinguishes separable and entangled states but performs a harder task—measures entanglement between two quantum systems. Though the constructed function is not a proper measure of entanglement (there are entangled states for which $\mathcal{F}$ is zero) a suitable extension might 'correct' the function so that all entangled states are detected.

4. *N* channels

In many physical situations it is less convenient to divide the system under consideration into two large subsystems than into a large number (say *N*) of smaller but equal systems. A typical example is the envisaged quantum computer composed of small micro-traps each holding a single qubit. In such a case individual qubits are spatially separated so that the interaction with the environment can be described by local maps Λ_i where the index *i* labels the qubits (or micro-traps). These maps can be derived phenomenologically or determined experimentally so their knowledge can be assumed. Obviously, we want to keep the influence of the environment as small as possible so each of the maps would satisfy a condition similar to equation (1)

$$D_i(\Lambda_i[\rho_i], \rho_i) \leqslant \epsilon, \qquad \forall\, i = 1, \ldots, N, \quad \forall\, \rho_i \in \mathcal{S}(\mathcal{H}_i), \tag{23}$$

where $D_i(.,.)$ are again metrics (distance functions) and $\mathcal{S}(\mathcal{H}_i)$ is the set of all density operators for each $i = 1, \ldots, N$. Using these maps we can find the state of a particular qubit after interaction with the environment. However, what is more important is the final state of the whole system

$$\Lambda_1 \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}], \tag{24}$$

and, in particular, how much the joint state $\rho_{1,\ldots,N}$ has changed due to the interaction with the environment. This change can be characterized by a distance between the original state $\rho_{1,\ldots,N}$ and the output of the product of the individual maps given by equation (24)

$$D_{1,\ldots,N}(\Lambda_1 \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}], \rho_{1,\ldots,N}), \tag{25}$$

where the $D_{1,\ldots,N}$ is a metric (distance function) defined on the set of all density operators $\mathcal{S}(\mathcal{H}_{1,\ldots,N})$ of the joint system $1, \ldots, N$. Here we use the same definition of the metric (distance) as before and define the functions $D_i(.,.)$ for $j = 1, \ldots, N$ and $D_{1,\ldots,N}(.,.)$ with the help of the norm of the corresponding Hilbert–Schmidt space (for more details see section 2)

$$D_i(\rho, \sigma) \equiv \|\rho - \sigma\|_i, \tag{26}$$

$$D_{1,\ldots,N}(\rho, \sigma) \equiv \|\rho - \sigma\|_{1,\ldots,N}. \tag{27}$$

Using these definitions it can be shown that the distance in equation (25) is always less than or equal to $N\sqrt{2 - 1/d}\,\epsilon$ where d is the dimension of the Hilbert space $\mathcal{H}_i$.

We note that the action of the product of local channels $\Lambda_1 \otimes \cdots \otimes \Lambda_N$ on separable states is such that $D_{1,\ldots,N}(\Lambda_1 \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}], \rho_{1,\ldots,N}) \leqslant N\epsilon$. This means that the restriction to the set of separable states leads to the decrease of the bound on $D_{1,\ldots,N}$ by the factor $\sqrt{2 - 1/d}$.[9]

To prove the statement we will use a very similar line of reasoning as in the case of two subsystems. First, taking advantage of the triangle inequality we bound the distance in equation (25) as follows:

$$\begin{aligned} D_{1,\ldots,N}&(\Lambda_1 \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}], \rho_{1,\ldots,N}) \\ &\leqslant D_{1,\ldots,N}(\Lambda_1 \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}], \mathbb{1} \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}]) \\ &\quad \vdots \\ &\quad + D_{1,\ldots,N}(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \Lambda_N[\rho_{1,\ldots,N}], \rho_{1,\ldots,N}). \end{aligned} \tag{28}$$

Each of the N terms on the right-hand side of the last equation can be rewritten as

$$D_{1,\ldots,N}\big(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \Lambda_i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}\big[\tilde{\rho}^{(i)}_{1,\ldots,N}\big], \tilde{\rho}^{(i)}_{1,\ldots,N}\big), \tag{29}$$

where

$$\tilde{\rho}^{(i)}_{1,\ldots,N} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \Lambda_{i+1} \otimes \Lambda_{i+2} \otimes \cdots \otimes \Lambda_N[\rho_{1,\ldots,N}],$$

so it is sufficient to bound expression (29). Next, we divide the whole system into two parts, an elementary system i and the rest. From this point the proof takes the same lines as in the case of two subsystems discussed in section 3. Therefore we recall result (20) obtained there

[9] Here we have used bound (7) for separable states that can easily be extended to a multi-partite case.

and refer the reader to section 3 for more details. Equation (20) states that

$$D_{1,\ldots,N}\big(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\Lambda_i\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\big[\tilde{\rho}^{(i)}_{1,\ldots,N}\big],\tilde{\rho}^{(i)}_{1,\ldots,N}\big)\leqslant\sqrt{2-1/d},$$

where d is the dimension of the ith elementary subsystem. Since we have N terms in the expression on the right-hand side of equation (28) the distance (25) is bounded by

$$D_{1,\ldots,N}(\Lambda_1\otimes\cdots\otimes\Lambda_N[\rho_{1,\ldots,N}],\rho_{1,\ldots,N})\leqslant N\sqrt{2-1/d}\,\epsilon, \tag{30}$$

where N is the number of elementary subsystems each satisfying condition (26), and d is the dimension of the Hilbert spaces $\mathcal{H}_i$ corresponding to the elementary subsystems.

4.1. Example

To illustrate the character of changes induced by the local maps on the global state of the whole system let us consider a simple model of N qubits undergoing a process of decoherence. That is the Hilbert spaces $\mathcal{H}_i$ are two-dimensional and the maps Λ_i are chosen to be

$$\Lambda_i:\tfrac{1}{2}(\mathbb{1}+\vec{\alpha}\cdot\vec{\sigma})\to\tfrac{1}{2}\{\mathbb{1}+\alpha_3\sigma_3+(1-k)[\alpha_1\sigma_1+\alpha_2\sigma_2]\}, \tag{31}$$

where k is equal to $k=\sqrt{2}\epsilon$ in order to fulfil conditions (23). The action of the map Λ_i is such that it preserves the diagonal elements in the basis formed by the eigenvectors of σ_3 while the non-diagonal elements are suppressed. Such maps describe the process of dephasing, a particular case of decoherence, since the vanishing of off-diagonal elements results in states that describe statistical mixtures.

Consider the initial state of the joint system to be the Greenberger–Horn–Zeiliner (GHZ) state

$$\rho_{1,\ldots,N}=\tfrac{1}{2}\{|0\ldots0\rangle\langle0\ldots0|+|0\ldots0\rangle\langle1\ldots1|+|1\ldots1\rangle\langle0\ldots0|+|1\ldots1\rangle\langle1\ldots1|\}. \tag{32}$$

The action of the map $\Lambda_1\otimes\cdots\otimes\Lambda_N$ on the state $\rho_{1,\ldots,N}$ described above can be evaluated straightforwardly and we obtain

$$\begin{aligned}\Lambda_1\otimes\cdots\otimes\Lambda_N[\rho_{1,\ldots,N}]&=\tfrac{1}{2}\{|0\ldots0\rangle\langle0\ldots0|+|1\ldots1\rangle\langle1\ldots1|\\&\quad+(1-k)^N(|0\ldots0\rangle\langle1\ldots1|+\text{h.c.})\}.\end{aligned} \tag{33}$$

Despite the fact that the state of each individual qubit remains unchanged (a consequence of this is that conditions (23) are trivially fulfilled) the state of the whole system changes because the off-diagonal elements are strongly suppressed. The distance (25) between the input $\rho_{1,\ldots,N}$ and the corresponding output $\Lambda_1\otimes\cdots\otimes\Lambda_N[\rho_{1,\ldots,N}]$ gives

$$D(\Lambda_1\otimes\cdots\otimes\Lambda_N[\rho_{1,\ldots,N}],\rho_{1,\ldots,N})=\sqrt{\tfrac{1}{2}[1+(1-k)^{2N}-2(1-k)^N]},$$

which for ϵ very small can be estimated as

$$D(\Lambda_1\otimes\cdots\otimes\Lambda_N[\rho_{1,\ldots,N}],\rho_{1,\ldots,N})\approx N\epsilon. \tag{34}$$

The deviation of the GHZ state under the action of the direct product of local maps Λ_i for sufficiently small ϵ scales as $N\epsilon$ which confirms our more general result (30). Though the result may seem to be optimistic (one might expect worse scaling with N) the effect of the action of local maps is to disentangle the qubits (destroy quantum correlations between the qubits). In addition, the disentanglement itself is strong since the off-diagonal elements are suppressed exponentially with the increase of the number of systems involved in the dynamics. This example nicely illustrates that though the deviation expressed with the help of the distance (25) scales as $N\epsilon$ the entanglement may be destroyed much more dramatically.

Finally note that in this example the bound for separable states $N\epsilon$, derived with the help of equation (7), is not violated in spite of the fact that we have used an entangled state. We have already pointed out that it is not necessary for any entangled state to violate the separable bound. To show that the bound can be violated indeed one can choose the map Λ_1 defined in section 3 for maps Λ_i and the initial state of the form $|\text{bell}\rangle^{\otimes N/2}$ where $|\text{bell}\rangle$ denotes one of the Bell states (see, for instance, section 3).

5. Conclusions

We have analysed the direct product of linear maps that describe local actions of a set of quantum channels. We have found a bound on the action of such a product of maps (expressed as a distance between an input and output of the product) provided the linear maps composing the product are bounded as well. We have addressed two typical scenarios. In the first, a quantum system is divided into two subsystems and the product is composed of two maps acting on the two subsystems, respectively. In the second scenario a joint system is composed of N equal subsystems and we have N linear maps acting on N subsystems of a given quantum system.

Our analysis has shown that the fundamental difference between the set of separable and entangled states yields two different bounds. For separable states the distance (3) is bounded by $\sqrt{2+2\sqrt{(1-1/d_1)(1-1/d_2)}}\,\epsilon$ while in the case of entangled states the distance can be larger and is bounded from above by $2\sqrt{2-1/d}\,\epsilon$.

Let us note that the bound for separable states (11) is optimal. That is there exists a pair of channels Λ_1 and Λ_2 such that the bound is saturated (examples are presented in appendix C). It is interesting to note that the channels that saturate the bound are the same channels that saturate the separable bound (7) for the case of the trace distance (see appendix C) or bound (20) for entangle states in the case of two-dimensional systems. Clearly, to establish the upper bound on the action of a pair of local quantum channels it is sufficient to find a pair of channels for which the action is maximal and set the bound to this maximum. The form of the channels may depend on the dimensions d_1 and d_2. However, our results suggest that the channels for which the bounds are maximal are of the same form for arbitrary d_1 and d_2 and are the channels that we have used in our examples.

In the end let us point out that our analysis is not restricted to the case of physical maps only and can be extended to the case of linear and hermiticity preserving maps that are not physical.

Acknowledgments

This work was supported in part by the European Union projects QGATES, QUPRODIS and CONQUEST, by the Slovak Academy of Sciences via the project CE-PI, by the project APVT-99-012304 and by the Alexander von Humboldt Foundation.

Appendix A

We prove the relation

$$\begin{aligned}\|\Lambda[|k\rangle\langle l|]-|k\rangle\langle l|\|^2 &= \|\Lambda[|l\rangle\langle k|]-|l\rangle\langle k|\|^2\\ &= \tfrac{1}{4}\{\|\Lambda[(|k\rangle\langle l|+|l\rangle\langle k|)]-(|k\rangle\langle l|+|l\rangle\langle k|)\|^2\\ &\quad+\|\Lambda[(|k\rangle\langle l|-|l\rangle\langle k|)]-(|k\rangle\langle l|-|l\rangle\langle k|)\|^2\}.\end{aligned} \tag{A.1}$$

for all physical (linear, hermiticity preserving and completely positive) maps Λ with the norm defined in equation (2) and $k \neq l$.

Let us denote by V_{kl} the expression $\Lambda[|k\rangle\langle l|] - |k\rangle\langle l|$. Using the definition of the norm in equation (2) we have that for any physical map Λ

$$\begin{aligned}
&\|\Lambda[|k\rangle\langle l|] - |k\rangle\langle l|\|^2 = \mathrm{Tr}\, V_{kl} V_{kl}^\dagger,\\
&\|\Lambda[|l\rangle\langle k|] - |l\rangle\langle k|\|^2 = \mathrm{Tr}\, V_{kl}^\dagger V_{kl},\\
&\|\Lambda[(|k\rangle\langle l| + |l\rangle\langle k|)] - (|k\rangle\langle l| + |l\rangle\langle k|)\|^2 = \mathrm{Tr}\big(V_{kl} + V_{kl}^\dagger\big)\big(V_{kl}^\dagger + V_{kl}\big),\\
&\|\Lambda[(|k\rangle\langle l| - |l\rangle\langle k|)] - (|k\rangle\langle l| - |l\rangle\langle k|)\|^2 = \mathrm{Tr}\big(V_{kl} - V_{kl}^\dagger\big)\big(V_{kl}^\dagger - V_{kl}\big).
\end{aligned}$$

Equation (A.1) is a direct consequence of the last result.

Appendix B

In this appendix we prove two relations

$$\|\Lambda[(|k\rangle\langle l| + |l\rangle\langle k|)] - (|k\rangle\langle l| + |l\rangle\langle k|)\| \leqslant 2\epsilon, \tag{B.1}$$
$$\|\Lambda[(|k\rangle\langle l| - |l\rangle\langle k|)] - (|k\rangle\langle l| - |l\rangle\langle k|)\| \leqslant 2\epsilon, \tag{B.2}$$

for all physical (linear, completely positive and hermiticity preserving) maps Λ satisfying the condition given by equation (1) and $k \neq l$. The two expressions

$$\begin{aligned}
&\|\Lambda[(|k\rangle\langle l| + |l\rangle\langle k|)] - (|k\rangle\langle l| + |l\rangle\langle k|)\|,\\
&\|\Lambda[(|k\rangle\langle l| - |l\rangle\langle k|)] - (|k\rangle\langle l| - |l\rangle\langle k|)\|,
\end{aligned}$$

can be rewritten as

$$\|\Lambda[(\rho_1 - \rho_2)] - (\rho_1 - \rho_2)\|, \qquad \|\Lambda[\mathrm{i}(\rho_3 - \rho_4)] - \mathrm{i}(\rho_3 - \rho_4)\|,$$

where

$$\begin{aligned}
&\rho_1 = \tfrac{1}{2}(|k\rangle + |l\rangle)(\text{h.c.}), \qquad && \rho_2 = \tfrac{1}{2}(|k\rangle - |l\rangle)(\text{h.c.}),\\
&\rho_3 = \tfrac{1}{2}(|k\rangle + i|l\rangle)(\text{h.c.}), \qquad && \rho_4 = \tfrac{1}{2}(|k\rangle - i|l\rangle)(\text{h.c.}).
\end{aligned}$$

By using the triangle inequality

$$\begin{aligned}
&\|\Lambda[(\rho_1 - \rho_2)] - (\rho_1 - \rho_2)\| \leqslant \|\Lambda[\rho_1] - \rho_1\| + \|\Lambda[\rho_2] - \rho_2\|,\\
&\|\Lambda[\mathrm{i}(\rho_3 - \rho_4)] - \mathrm{i}(\rho_3 - \rho_4)\| \leqslant \|\Lambda[\rho_3] - \rho_3\| + \|\Lambda[\rho_4] - \rho_4\|,
\end{aligned}$$

we obtain the relations (B.1) and (B.2) owing to the conditions (1).

Appendix C

Here we present an example showing that bounds (7) and (11) are optimal. In this example we will consider a more general case of distances $D_1(.\,,.)$, $D_2(.\,,.)$ and $D_{12}(.\,,.)$ and define the distances with p-norms

$$D_a^p(\rho_a, \sigma_a) = (\mathrm{Tr}|\rho_a - \sigma_a|^p)^{1/p},$$

where a labels systems 1, 2 or 12, ρ_a and σ_a are the density operators and p is a positive integer. The map Λ_1 is chosen to be a contraction of the form

$$\Lambda_1[\rho_1] = (1 - k_1)\rho_1 + k_1 \frac{1}{d_1}\mathbb{1},$$

where d_1 is the dimension of the Hilbert space $\mathcal{H}_1$ and k_1 is the contraction parameter. In what follows we assume $k_1 = \epsilon \big/ \big[(1 - 1/d_1)^p + (d_1 - 1)\big/ d_1^p\big]^{1/p}$ so that the condition (1) is fulfilled. Similarly, the map Λ_2 is a contraction

$$\Lambda_2[\rho_2] = (1 - k_2)\rho_2 + k_2 \frac{1}{d_2}\mathbb{1},$$

where d_2 is the dimension of the Hilbert space $\mathcal{H}_2$ and $k_1 = \epsilon\big/\big[(1-1/d_2)^p + (d_2-1)\big/d_2^p\big]^{1/p}$. For the initial state we choose a pure state of the form $\rho_{12} = |00\rangle\langle 00|$. Keeping only terms of the order of ϵ the distance between the input ρ_{12} and the output $\Lambda_1 \otimes \Lambda_2[\rho_{12}]$ gives

$$D_{12}^p(\Lambda_1 \otimes \Lambda_2[\rho_{12}], \rho_{12}) = \left(\left[k_1\left(1-\frac{1}{d_1}\right) + k_2\left(1-\frac{1}{d_2}\right)\right]^p + \left[\frac{k_1}{d_1}\right]^p (d_1-1) + \left[\frac{k_2}{d_2}\right]^p (d_2-1)\right)^{1/p}. \tag{C.1}$$

Case study: trace distance. The trace distance is defined with the help of the 1-norm so that $p = 1$. The two contraction parameters k_1 and k_2 read $k_1 = \epsilon/[2(1 - 1/d_1)]$ and $k_2 = \epsilon/[2(1 - 1/d_2)]$. Using these relations in equation (C.1) the distance between the input and the output reads

$$D_{12}(\Lambda_1 \otimes \Lambda_1[\rho_{12}], \rho_{12}) = 2\epsilon. \tag{C.2}$$

Case study: Hilbert–Schmidt distance. The Hilbert–Schmidt distance is defined with the help of the 2-norm so that $p = 2$. The two contraction parameters k_1 and k_2 read $k_1 = \epsilon/\sqrt{1 - 1/d_1}$ and $k_2 = \epsilon/\sqrt{1 - 1/d_2}$. Using these relations in equation (C.1) the distance between the input and the corresponding output reads

$$D_{12}(\Lambda_1 \otimes \Lambda_1[\rho_{12}], \rho_{12}) = \sqrt{2 + 2\sqrt{(1 - 1/d_1)(1 - 1/d_2)}}\,\epsilon. \tag{C.3}$$

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