All (qubit) decoherences: Complete characterization and physical implementation

Mário Ziman[1,2] and Vladimír Bužek[1,3]
[1]Research Center für Quantum Information, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia
[2]Quniverse, Líščie údolie 116, 841 04 Bratislava, Slovakia
[3]Abteilung Quantenphysik, Universität Ulm, 89069 Ulm, Germany
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We investigate decoherence channels that are modeled as a sequence of collisions of a quantum system (e.g., a qubit) with particles (e.g., qubits) of the environment. We show that collisions induce decoherence when a bipartite interaction between the system qubit and an environment (reservoir) qubit is described by the controlled-U unitary transformation (gate). We characterize decoherence channels and in the case of a qubit we specify the most general decoherence channel and derive a corresponding master equation. Finally, we analyze entanglement that is generated during the process of decoherence between the system and its environment.

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I. INTRODUCTION

One of the most distinctive features of quantum systems is their ability to "exist" in superpositions of mutually exclusive (orthogonal) states [1]. Providing a quantum system has been prepared in a pure state $|\Psi\rangle$ then we can write $|\Psi\rangle = \Sigma_k c_k |\psi_k\rangle$, where $|\psi_k\rangle$ are orthonormal vectors that compose a basis ($\langle\psi_l|\psi_k\rangle = \delta_{kl}$). All bases are unitarily equivalent and we can express the same state different bases. In fact, we can always select a basis such that $|\Psi\rangle$ is a basis vector so in its matrix representation the vector $|\Psi\rangle$ is represented by a single diagonal element. According to quantum postulates for the isolated system any evolution is governed by unitary transformations and the original information about the state preparation of the quantum system is preserved. As soon as an interaction with an environment comes into the play (the quantum system is open) the situation becomes dramatically different and the state is no longer described by the single diagonal element in some basis. Depending on properties of the environment and the character of the interaction our system evolves nonunitarily and its state is, in general, described by a statistical mixture. Among various possible dynamics of an open quantum system interacting with its environment a specific role is played by a process in which the off-diagonal elements of the original state $\varrho = |\Psi\rangle\langle\Psi|$ in some basis are continuously suppressed in time, i.e.,

$$\varrho \rightarrow \varrho_{t\rightarrow\infty} = \mathrm{diag}[\varrho]. \tag{1.1}$$

This is a process of *decoherence* during which some of the information about the initial state of the quantum system might be irreversibly lost [2–4]. The basis in which the decoherence takes place is specified by properties of the environment and the character of the interaction [4]. There are at least two aspects of quantum decoherence that keep it in the center of interests in multiple investigation related to foundations of quantum mechanics and in quantum information processing. The first aspect is, that decoherence is presently viewed as a mechanism via which classicality emerges from the realm of quantum (see e.g. [2–6]). In this context it is of paramount importance to specify the basis (the so-called pointer basis [4]) in which the decoherence takes place. In the field of quantum information the decoherence is an evil—it degrades quantum resources (superpositions of states and quantum entanglement) that are needed for quantum information processing [7]. The degradation of resources is caused by random interactions (errors) between a quantum system under consideration (e.g., a qubit or a quantum register) with its environment. If nothing else then these two facets of quantum decoherence are enough to justify an investigation of decoherence channels (transformations).

As mentioned above the decoherence is caused by (unavoidable) interactions between the system and its environment. Consequently, the whole process of decoherence can be completely described within the framework of the quantum theory as a unitary process that governs the joint evolution of the quantum systems and its environment[1] [2–4]. There are plentiful theoretical models describing the decoherence within the framework of the standard quantum theory that have been in accordance with various experiments [9,10]. These models either use Hamiltonian evolution of the composite system-plus-enviroment structure (the Hamiltonian itself is time-independent).

Alternatively, the description of decoherence can be based on a simple collision-like model, i.e. a sequence of interactions between the object under consideration and particles from environment leads to decoherence. These models allow us to study *microscopic* dynamics of open systems, in which the flow of information from the system to the environment and creation of entanglement can be analyzed. In fact, collision models are equivalent to more general models of causal memory channels [11]. In this case, the memory is represented by the system under decoherence, whereas the reservoir (environment) plays the role of input/output systems.

In the present paper we will focus our attention on collisionlike models of decoherence of qubits. Our first aim is to completely classify all possible decoherence channels of a qubit. The second task is to show that all decoherence maps

[1]Another possibility would be to include decoherence into the basic dynamical equation, i.e., to add a non-Hamiltonian part into the Schrödinger equation [8]. However, the modifications of the basic quantum dynamical law are out of scope of this paper.

of qubits can be modeled as sequences of collisions. The paper is organized as follows: Sections II and III are devoted to a description of general properties of all decoherence channels. In Sec. IV we present a generic collisionlike model. In Sec. V the master equations for collision models are derived and *all* possible master equations describing decoherence of a qubit are presented. In Sec. VI we analyze how entanglement is created during a sequence of collisions. Finally, in Sec. VII we summarize our results and formulate some open problems.

II. DECOHERENCE CHANNELS

The aim of this section is to classify all possible completely positive trace-preserving maps (quantum channels) that describe quantum decoherence. Let us denote by $\mathcal{D}$ the set all maps $\mathcal{E}$ satisfying the decoherence conditions, i.e.,

$$\langle e_k|\mathcal{E}[\varrho]|e_k\rangle = \langle e_k|\varrho|e_k\rangle \text{ for all } k, \tag{2.1}$$

$$|\langle e_k|\mathcal{E}[\varrho]|e_l\rangle| < |\langle e_k|\varrho|e_l\rangle| \text{ for all } k \neq l, \tag{2.2}$$

with $\mathcal{B}=\{|e_k\rangle\}$ being the *decoherence basis*. For our purposes it is useful to fix one basis $\mathcal{B}$ and to analyze all decoherences (forming the set $\mathcal{D}_\mathcal{B}$) with respect to this basis. The general decoherence maps are then just unitary rotations of elements from $\mathcal{D}_\mathcal{B}$, that correspond to a change of the decoherence basis. In particular, if $\mathcal{E}$ is a decoherence map, then also $\mathcal{E}'=\mathcal{U}_1\mathcal{E}\mathcal{U}_2$ is such a map. We used the notation $\mathcal{U}_j[\varrho]=U_j\varrho U_j^\dagger$ with U_j unitary operators. From the definition it is clear that decoherence channels are unital (they preserve the total mixture, i.e., $\mathcal{E}[I]=I$) and are not strictly contractive (they might have more than a single fixed point).

Denoting by $\mathcal{D}_\mathcal{B}$ the set of all decoherence maps with respect to a fixed basis $\mathcal{B}$ we can write $\mathcal{D}=\cup_\mathcal{B}\mathcal{D}_\mathcal{B}$. Each decoherence map $\mathcal{E}\in\mathcal{D}$ belongs only to one class $\mathcal{D}_\mathcal{B}$. Elements of $\mathcal{D}_\mathcal{B}$ and $\mathcal{D}_{\mathcal{B}'}$ are unitarily related, i.e.,

$$\mathcal{D}_{\mathcal{B}'} = \{\mathcal{E}'|\mathcal{E}'[\varrho] \coloneqq \mathcal{E}[U\varrho U^\dagger], \mathcal{E}\in\mathcal{D}_\mathcal{B}, \mathcal{B}'=U\mathcal{B}\} = \mathcal{D}_{U\mathcal{B}}.$$

This defines a new decoherence class only if $\mathcal{B}'\neq\mathcal{B}$. That is, the unitary operation U does not commute with all projectors $|e_k\rangle\langle e_k|$, or equivalently the basis $\mathcal{B}$ is not an eigenbasis of the transformation U. If $[U,|e_k\rangle\langle e_k|]=0$ for all k then from a given $\mathcal{E}\in\mathcal{D}_\mathcal{B}$ we obtain different decoherence maps within the fixed set $\mathcal{D}_\mathcal{B}$.

A. Qubit decoherences

In what follows we will analyze the case of qubit decoherence channels. In this case the set $\mathcal{D}$ has surprisingly simple form. We will use the so-called left-right notation, in which the evolution map is represented by a 4×4 matrix [12]. Let us choose the following operator basis

$$S_0 = I,$$

$$S_1 = |\psi\rangle\langle\psi^\perp| + |\psi^\perp\rangle\langle\psi|,$$

$$S_2 = i|\psi\rangle\langle\psi^\perp| - i|\psi^\perp\rangle\langle\psi|,$$

$$S_3 = |\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp|, \tag{2.3}$$

where $\mathcal{B}=\{|\psi\rangle,|\psi^\perp\rangle\}$ is the decoherence basis. The elements of S-basis satisfy the same properties as the Pauli operators, because $S_j=W\sigma_jW^\dagger$ with W being a unitary operation. In this basis the operators (states) take the form of four-dimensional vectors $\varrho=\frac{1}{2}(I+\vec{r}\cdot\vec{S})\leftrightarrow\vec{r}_\varrho=(1,\vec{r})$, where $r_j=\mathrm{Tr}[\varrho S_j]$. The evolution $\mathcal{E}$ is described by 4×4 matrix with elements given by the equation $\mathcal{E}_{kl}=\frac{1}{2}\mathrm{Tr}(S_k\mathcal{E}[S_l])$. Because of the trace-preservation we have $\mathcal{E}_{00}=1$ and $\mathcal{E}_{01}=\mathcal{E}_{02}=\mathcal{E}_{03}=0$. Consequently, we obtain the Bloch sphere representation [7] of the state space, in which the states are illustrated as points (three-dimensional real vectors $\vec{r}$) lying inside a sphere with a unit radius. The action of $\mathcal{E}$ corresponds to an affine transformation of the Bloch vector $\vec{r}$, i.e., $\vec{r}\to\vec{r}'=T\vec{r}+\vec{t}$, where $T_{jk}=\mathcal{E}_{jk}$ (for $j,k=1,2,3$) and $t_j=\mathcal{E}_{j0}$. The translation vector $\vec{t}$ describing the shift of the Bloch sphere (including its center, i.e., the total mixture) is related to the unitality of the channel. For unital maps $\vec{t}=\vec{0}$.

Diagonal elements of the state ϱ are in this case associated with the mean value $z=\mathrm{Tr}[\varrho S_3]$. The conservation of the diagonal elements implies that the corresponding components of ϱ are preserved. Combining the unitality with this property we find the following form for decoherence maps

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.4}$$

from where it follows that the set of all possible qubit decoherence maps is at most four-parametric.

Each unital map can be written as [12]

$$\mathcal{E}[\varrho] = R_{U_1}\Phi_\mathcal{E}R_{U_2}[\varrho] = U_1\Phi_\mathcal{E}[U_2\varrho U_2^\dagger]U_1^\dagger, \tag{2.5}$$

where R_{U_1},R_{U_2} are orthogonal rotations corresponding to unitary transformations U_1,U_2; $\Phi_\mathcal{E}=\mathrm{diag}\{1,\lambda_1,\lambda_2,\lambda_3\}$ and λ_j are the singular values of the matrix $\mathcal{E}$. In fact, the above relation is the singular-value decomposition of the matrix $\mathcal{E}$. The conditions of the complete positivity restricts the possible values of λ_j. In particular, the allowed points $\vec{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ must lie inside a tetrahedron with vertices that have coordinates (1, 1, 1), $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$, respectively.

Applying these facts to the decoherence map under consideration [$\mathcal{E}$ from Eq. (2.4)] we obtain that $\Phi_\mathcal{E}=\mathrm{diag}\{1,\lambda_1,\lambda_2,1\}$, i.e., $\lambda_3=1$. Let us note that in this case we use unitaries that do not change the decoherence basis, so we are still dealing with all decoherences that belong to a fixed basis $\mathcal{B}$. The condition of complete positivity restricts the values to the points $\vec{\lambda}=(\lambda,\lambda,1)$ with $-1\leq\lambda\leq 1$, i.e., to a line connecting the two vertices of the tetrahedron representing the identity ($\lambda=1$) and the unitary rotation S_3 ($\lambda=-1$). Consequently, the general decoherence channel $\mathcal{E}\in\mathcal{D}_\mathcal{B}$ reads

$$\mathcal{E}=\begin{pmatrix}1&0&0&0\\0&c_1&s_1&0\\0&-s_1&c_1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\lambda&0&0\\0&0&\lambda&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&c_2&s_2&0\\0&-s_2&c_2&0\\0&0&0&1\end{pmatrix}, \tag{2.6}$$

where $s_j=\sin\varphi_j$ and $c_j=\cos\varphi_j$ represent rotations R_{U_j} around the z axis by an angle φ_j. From here it follows that a general decoherence map $\mathcal{E}$ takes the form

$$\mathcal{E}=\begin{pmatrix}1&0&0&0\\0&\lambda\cos(\varphi_1+\varphi_2)&\lambda\sin(\varphi_1+\varphi_2)&0\\0&-\lambda\sin(\varphi_1+\varphi_2)&\lambda\cos(\varphi_1+\varphi_2)&0\\0&0&0&1\end{pmatrix} \tag{2.7}$$

and, consequently, it is specified only by two real parameters $a=\lambda\cos(\varphi_1+\varphi_2)$ and $b=\lambda\sin(\varphi_1+\varphi_2)$, i.e.,

$$\mathcal{E}=\begin{pmatrix}1&0&0&0\\0&a&b&0\\0&-b&a&0\\0&0&0&1\end{pmatrix}. \tag{2.8}$$

As a result we obtain that any map $\mathcal{E}$ of the above form with the numbers a,b satisfying the condition $a^2+b^2\leq 1$ is completely positive. Therefore we can conclude that the set of all decoherence maps of a qubit is characterized just by two parameters. Moreover, to obtain the decoherence (to secure the suppression of off-diagonal terms) the inequality must be strict, i.e., $a^2+b^2<1$. Otherwise the map $\mathcal{E}$ desribes a unitary rotation around the z axis. Defining the rotation map

$$R_\varphi=\begin{pmatrix}\cos\varphi&\sin\varphi\\-\sin\varphi&\cos\varphi\end{pmatrix} \tag{2.9}$$

and using the relation $\varphi=\varphi_1+\varphi_2$, we can write the most general decoherence channel ($\mathcal{E}\in\mathcal{D}_\mathcal{B}$) in a very compact form

$$\mathcal{E}^n=\begin{pmatrix}1&0&0&0\\0&&&0\\&&\lambda R_\varphi&\\0&&&0\\0&0&0&1\end{pmatrix}. \tag{2.10}$$

This form is suitable for our purposes, because the powers of the map $\mathcal{E}$ read

$$\mathcal{E}^n=\begin{pmatrix}1&0&0&0\\0&&&0\\&&\lambda^n R_{n\varphi}&\\0&&&0\\0&0&0&1\end{pmatrix}. \tag{2.11}$$

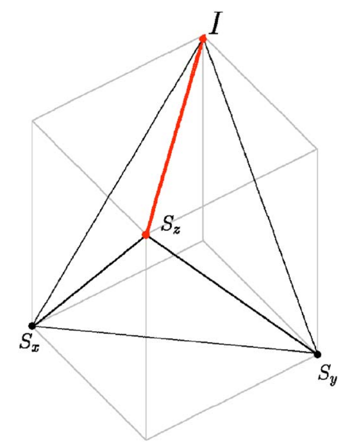

FIG. 1. (Color online) The cube corresponds to all positive unital trace-preserving maps. The condition of complete positivity confines quantum channels into the tetrahedron with (generalized) Pauli matrices as vertices. In this picture the set of decoherence channels $\mathcal{D}_\mathcal{B}$ forms a line connecting the points I and S_z.

III. STRUCTURAL PROPERTIES OF DECOHERENCE CHANNELS

In this section we will briefly review structural properties of the set of all possible decoherence completely positive maps $\mathcal{E}$. Let us denote this set by $\mathcal{D}$.

Convex structure

The set of all decoherence maps $\mathcal{D}$ is not convex, i.e., a convex combination of two decoherence channels $\mathcal{E}_\mu=\mu\mathcal{E}_1+(1-\mu)\mathcal{E}_2$ is not again a decoherence channel. This is true except the case when the decoherence bases of $\mathcal{E}_1,\mathcal{E}_2$ coincide, i.e., the set $\mathcal{D}_\mathcal{B}$ is convex. The extremal points of $\mathcal{D}_\mathcal{B}$ correspond to unitary transformations. However, these are not elements of $\mathcal{D}_\mathcal{B}$, because they do not fulfill the second decoherence condition (2.2).

We have already mentioned that for qubits the set of all possible $\Phi_\mathcal{E}$ channels form a tetrahedron and up to unitary transformations each channel belongs to this tetrahedron. Those channels that correspond to decoherence maps form a line connecting the points (1, 1, 1) and $(-1,-1,1)$. From this picture (see Fig. 1) the convexity of $\mathcal{D}_\mathcal{B}$ is transparent and also the extremal points can be easily identified as unitary channels. It follows that each decoherence map can be written as a convex sum of only two unitary channels. In fact all maps $\Phi_\mathcal{E}$ for which one of the λ's equals to unity and all the others are the same define a decoherence with respect to some basis. This means that all edges of the tetrahedron correspond to decoherence channels. It illustrates that the set $\mathcal{D}$ as a whole is not convex, but is composed of a continuous number of "convex" subsets $\mathcal{D}_\mathcal{B}$ corresponding to each orthonormal basis $\mathcal{B}$.

Composition

A composition of two decoherence channels $\mathcal{E}=\mathcal{E}_1\circ\mathcal{E}_2$ is not, in general, a decoherence channel. So the set $\mathcal{D}$ is not closed under the operation of multiplication. The channel $\mathcal{E}$

belongs to $\mathcal{D}$ only if the decoherence bases of $\mathcal{E}_1$ and $\mathcal{E}_2$ coincide, i.e., again only the sets $\mathcal{D}_{\mathcal{B}}$ are closed under the composition.

Classical capacity

The decoherence basis is preserved by the decoherence map. Therefore it is possible to exploit these bases states to transmit the maximally possible amount of information, i.e., the capacity achieves its maximum $C=\log_2 d$ with $d=\dim\mathcal{H}$.

Tensor product

The tensor product of two decoherence maps describes a decoherence. However, $\mathcal{D}_{12}\neq\mathcal{D}_1\otimes\mathcal{D}_2$, because the decoherence basis of $\mathcal{E}_1\otimes\mathcal{E}_2$ is always separable. The open problem is whether the whole set $\mathcal{D}_{12}$ can be obtained from the sets $\mathcal{D}_1,\mathcal{D}_2$ by global unitary rotations. Properties of decoherence channels under tensor products is an interesting topic, which is related to our ability of controlling the decoherence. For example, how the decoherence of a sub-system affects characteristics of the whole system?

IV. COLLISION MODEL

In what follows we will study whether an arbitrary decoherence channel can be implemented via a sequence of bipartite collisions. Each of the collisions is described by a unitary transformation U. Our task will be to derive all possible unitary transformations that force the system to decohere. Our analysis will be performed only for qubits, but up to technical details all results hold for qudits.

Let us consider that initially the system qubit is decoupled from an environment (reservoir) that is modeled as a set of qubits, i.e., $\Omega_{in}=\varrho\otimes\Xi_{\text{res}}$. Moreover, we will simplify the model by assuming that initially the reservoir qubits are in a factorized state $\Xi_{res}=\xi^{\otimes N}$ and each reservoir qubit interacts with the system qubit just once. In addition we assume that reservoir qubits do not interact between themselves. Under such conditions the evolution of the system qubit is induced by the sequence of maps $\mathcal{E}_1=\cdots=\mathcal{E}_N\equiv\mathcal{E}$. In particular, the state of the system after the nth interaction equals

$$\varrho^{(n)}=\mathcal{E}_n\ldots\mathcal{E}_1[\varrho]=\mathcal{E}^n[\varrho], \tag{4.1}$$

where $\mathcal{E}[\varrho]=\text{Tr}_{res}[U(\varrho\otimes\xi)U^\dagger]$. We will refer to this picture as to a collision model. The system qubit collides with reservoir qubits.

In order to obtain the decoherence channel, i.e.,

$$\varrho\to\varrho^{(n)}=\begin{pmatrix}\varrho_{00} & \varrho_{12}^{(n)}\\ \varrho_{21}^{(n)} & \varrho_{11}\end{pmatrix}$$

with $\varrho_{12}^{(n)}=[\varrho_{21}^{(n)}]^*\to 0$ for n goes to infinity, we have to ensure that the map $\mathcal{E}$ preserves diagonal elements of each state ϱ in a given (decoherence) basis.

In order to preserve the diagonal elements of pure states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ (decoherence basis) the bipartite unitary transformation U must necessarily satisfy the relations

$$|00\rangle\to|0\psi\rangle,$$
$$|01\rangle\to|0\psi^\perp\rangle,$$
$$|10\rangle\to|1\phi^\perp\rangle,$$
$$|11\rangle\to|1\phi\rangle. \tag{4.2}$$

In what follows we will prove our main result that the class of possible bipartite interactions that induce decoherence in collision models coincides with the set of all *controlled-U transformations* (the so-called U-processors as introduced in Ref. [13]), where the system under consideration plays the role of the control and the reservoir particle is a target. Certainly, we have to identify those transformations for which the off-diagonal elements of the system density operator do vanish in the limit of infinitely many collisions with reservoir particles.

The unitary bipartite transformation (the controlled-U operation) defined by the relations (4.2) can be rewritten into the following operator form

$$U=|0\rangle\langle 0|\otimes V_0+|1\rangle\langle 1|\otimes V_1, \tag{4.3}$$

where V_0,V_1 are unitary rotations of a reservoir qubit. In particular, $V_0=|\psi\rangle\langle 0|+|\psi^\perp\rangle\langle 1|$ and $V_1=|\phi^\perp\rangle\langle 0|+|\phi\rangle\langle 1|$. Thus the initial state $\Omega=\varrho\otimes\xi$ of a bipartite system evolves according to a transformation

$$\Omega\to\Omega'=U\Omega U^\dagger=\sum_{j,k=0}^{1}\varrho_{jk}|j\rangle\langle k|\otimes V_j\xi V_k^\dagger \tag{4.4}$$

and by performing the partial trace over the reservoir qubit we obtain the induced map

$$\begin{aligned}\varrho\to\varrho'=\mathcal{E}[\varrho]&=\text{Tr}_p\Omega'\\ &=\sum_{j,k=0}^{1}\varrho_{jk}\text{Tr}[V_j\xi V_k^\dagger]|j\rangle\langle k|\\ &=\text{diag}[\varrho]+\varrho_{01}\langle X\rangle_\xi|0\rangle\langle 1|+\varrho_{10}\langle X^\dagger\rangle_\xi|1\rangle\langle 0|,\end{aligned}$$

where $X=V_1^\dagger V_0$ and $\langle X\rangle_\xi=\text{Tr}[X\xi]$ stands for the mean value of the operator X in the state ξ.

Applying the transformation $\mathcal{E}$ in a sequence of n collisions the state of the system qubit is described by the density operator

$$\varrho^{(n)}=\mathcal{E}^n[\varrho]=\text{diag}[\varrho]+\varrho_{01}\langle X\rangle_\xi^n|0\rangle\langle 1|+\varrho_{10}\langle X^\dagger\rangle_\xi^n|1\rangle\langle 0|,$$

from where we can conclude, that providing $|\langle X\rangle_\xi|<1$ and $|\langle X^\dagger\rangle_\xi|<1$ the off-diagonal terms vanish. However, because $XX^\dagger=X^\dagger X=I$, i.e., X is unitary, its eigenvalues are just complex square roots of the unity. Therefore, for the eigenvectors of X the off-diagonal terms do not tend to zero.

The fact that for convex combinations of the eigenvectors the off-diagonal elements still vanish might sound counterintuitive. But it can be seen from the following consideration: Let us denote by $e^{i\varphi}$ and $e^{i\eta}$ the eigenvalues of X associated with the eigenvectors $|f_1\rangle$ and $|f_2\rangle$, respectively. Then the mean value $\langle X\rangle_\xi$ for the convex combination $\xi=a|f_1\rangle\langle f_1|+(1-a)|f_2\rangle\langle f_2|$ equals $\langle X\rangle_\xi=e^{i\varphi}a+e^{i\eta}(1-a)$. The condition

$|\langle X\rangle_\xi|<1$ can be rewritten as the inequality $2a(1-a)[1-\cos(\varphi-\eta)]<0$, which is satisfied only if $\cos(\varphi-\eta)\neq 1$, or $a\neq 1$ and $a\neq 0$. The latter property means that ξ is the eigenstate. The first property requires $\varphi=\eta$, i.e., the operator X is proportional to the identity, $X=e^{i\varphi}I$. However, under this assumption $V_1=e^{i\varphi}V_0$, i.e., we have no interaction and $U=(e^{i\varphi}|0\rangle\langle 0|+|1\rangle\langle 1|)\otimes V_1$. Hence, we can conclude that whenever the reservoir state is not an eigenstate of X and the interaction is not trivial, the described collision model with controlled-U interaction forces the system to decohere.

It is straightforward to show that unitary interactions $U=|0\rangle\langle 0|\otimes V_0+|1\rangle\langle 1|\otimes V_1$ induce maps of the left-right form [see Eq. (2.8)] with the parameters

$$a=\frac{1}{2}(\langle X\rangle_\xi+\langle X^\dagger\rangle_\xi),$$

$$b=\frac{i}{2}(\langle X\rangle_\xi-\langle X^\dagger\rangle_\xi), \tag{4.5}$$

or, equivalently, $\langle X\rangle_\xi=\lambda e^{i\varphi}$. So given a decoherence map $\mathcal{E}$ one can, in principle, find an interaction U and an initial state of the reservoir qubits ξ, such that the desired decoherence process is implemented via a sequence of collisions.

V. MASTER EQUATION

In this section we will derive a master equation that describes the decoherence process induced by collisions of the system qubit with reservoir particles. Although the studied decoherence model is intrinsically discrete, we will show that we can perform a continuous-time approximation that enables us to write down the master equation (see, e.g. [14]).

As shown in the previous section the collision model is described by a set of maps $\mathcal{E}_n=\mathcal{E}^n$ that form a discrete semigroup, i.e., $\mathcal{E}_n\mathcal{E}_m=\mathcal{E}_{n+m}$ for all integer m,n and $\mathcal{E}_0=\mathcal{I}$. The question is whether we can introduce a continuous one-parametric set of transformations $\mathcal{E}_t$ such that $\mathcal{E}_{t_n}=\mathcal{E}_n$ for $t_n=n\tau$ (τ is a time scale roughly corresponding to the time interval between two interactions). It turns out that a simple relation $n\to t/\tau$ can be used to accomplish the task. The obtained continuous set of transformations $\mathcal{E}_t$ will be used to derive the generator $\mathcal{G}$ of the dynamics by using a simple formula $\mathcal{G}_t=\dot{\mathcal{E}}_t\mathcal{E}_t^{-1}$.

With the help of results from Sec. III [namely, Eq. (2.11)] we can directly write

$$\mathcal{E}_t=\begin{pmatrix}1&0&0&0\\0&&&0\\&\lambda^t R_{t\varphi}&&\\0&&&0\\0&0&0&1\end{pmatrix} \tag{5.1}$$

where for simplicity we set the time scale $\tau=1$. It is easy to see that the one-parametric set of transformations $\mathcal{E}_t$ possesses the semigroup property, i.e., $\mathcal{E}_t\mathcal{E}_s=\mathcal{E}_{t+s}$ for all real t,s. It means that the generator and the associated master equation will be of the Lindblad form [15], i.e. the process under consideration is Markovian.

The corresponding generator reads

$$\mathcal{G}=\begin{pmatrix}0&0&0&0\\0&\ln\lambda&-\varphi&0\\0&\varphi&\ln\lambda&0\\0&0&0&0\end{pmatrix}, \tag{5.2}$$

where we used the identity $\dot{R}_{t\varphi}=\varphi R_{t\varphi+\pi/2}$ and $(d/dt)\times[\lambda^t R_{t\varphi}]\lambda^{-t}R_{-t\varphi}=\ln\lambda R_0+\varphi R_{\pi/2}$. This step can be performed only if λ is non-negative (i.e., when the logarithm is defined), which, in general, is not the case. The parameter λ belongs to the open interval $(-1,1)$. Consequently, it seems that the generator cannot be derived in all cases. However, using the equality $-\lambda R_\varphi=\lambda R_{(\varphi+\pi)}$ for λ nonnegative, one can write $|\lambda|^t R_{t(\varphi+\pi)}$ instead of $\lambda_t R_{t\varphi}$ in the expression for $\mathcal{E}_t$ with $\lambda<0$. Then the generator is slightly different and contains the term $\varphi+\pi$ instead of φ, and $\ln|\lambda|$ instead of $\ln\lambda$. This is not a problem, because in terms of parameters of the collision model $\langle X\rangle_\xi=\lambda e^{i\varphi}$, i.e., the parameter $\lambda=|\langle X\rangle_\xi|$ is always positive. Therefore we can consider the generator $\mathcal{G}$ as the most general one.

The general master equation in Lindblad form reads

$$\dot{\varrho}_t=\mathcal{G}[\varrho_t]=-i[H,\varrho_t]+\frac{1}{2}\sum_{a,b}c_{ab}([S_a,\varrho_t S_b]+[S_a\varrho_t,S_b]).$$

If the numbers c_{ab} are time-independent and form a positive matrix, then the generated evolution is Markovian and satisfies the semigroup property. To find the values of the coefficients c_{ab} we will use the following relations (see Ref. [14])

$$h_1=\frac{[\mathcal{G}]_{32}-[\mathcal{G}]_{23}}{4},\ h_2=\frac{[\mathcal{G}]_{13}-[\mathcal{G}]_{31}}{4},\ h_3=\frac{[\mathcal{G}]_{21}-[\mathcal{G}]_{12}}{4},$$

$$e_{23}=\frac{[\mathcal{G}]_{10}}{4},\ e_{31}=\frac{[\mathcal{G}]_{20}}{4},\ e_{12}=\frac{[\mathcal{G}]_{30}}{4}, \tag{5.3}$$

and

$$d_{11}=\frac{[\mathcal{G}]_{11}-[\mathcal{G}]_{22}-[\mathcal{G}]_{33}}{4},\ d_{12}=\frac{[\mathcal{G}]_{12}+[\mathcal{G}]_{21}}{4},$$

$$d_{22}=\frac{[\mathcal{G}]_{22}-[\mathcal{G}]_{11}-[\mathcal{G}]_{33}}{4},\ d_{23}=\frac{[\mathcal{G}]_{23}+[\mathcal{G}]_{32}}{4},$$

$$d_{33}=\frac{[\mathcal{G}]_{33}-[\mathcal{G}]_{11}-[\mathcal{G}]_{22}}{4},\ d_{13}=\frac{[\mathcal{G}]_{13}+[\mathcal{G}]_{31}}{4}, \tag{5.4}$$

where $[\mathcal{G}]_{kl}$ correspond to matrix elements of the generator $\mathcal{G}$, $c_{ab}=d_{ab}-ie_{ab}$ and $H=\Sigma_a h_a S_a$. Note that d_{ab} form a symmetric matrix and e_{ab} is an antisymmetric matrix.

Using these expressions one finds that the nonvanishing parameters are

$$h_3=\frac{1}{2}\varphi,\quad d_{33}=-\frac{1}{2}\ln\lambda \tag{5.5}$$

and the corresponding master equation reads

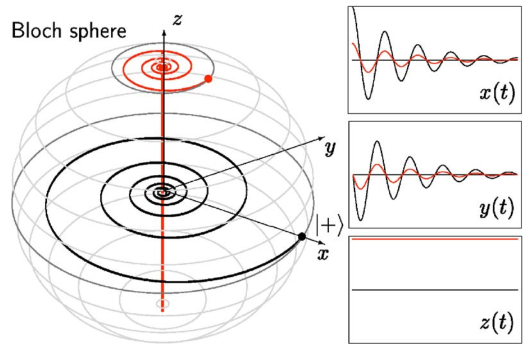

FIG. 2. (Color online) The decoherence of a qubit governed by Eq. (5.6). The Bloch sphere that represents the initial state space of a qubit is mapped into the line connecting the decoherence basis states. On the right the evolution of Bloch-vector components for two different initial states is depicted.

$$\dot{\varrho}_t = - i\frac{\varphi}{2}[S_3,\varrho_t] - \frac{\ln \lambda}{2}(S_3\varrho_t S_3 - \varrho_t). \qquad (5.6)$$

A typical evolution driven by this equation is depicted in Fig. 2.

Let us now address the following question: Is there any other master equation describing a decoherence of a qubit? The preservation of the S_z component (determined by the decoherence basis) together with the unitality of the transformation implies that

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (5.7)$$

The corresponding matrix $C=\frac{1}{2}[c_{ab}]$ then reads

$$C = \frac{1}{4}\begin{pmatrix} a-d & c+b & 0 \\ c+b & d-a & 0 \\ 0 & 0 & -a-d \end{pmatrix}. \qquad (5.8)$$

This matrix is positive only when $a=d$ and $b=-c$. Moreover, a must be negative. These restrictions leave only a single element that does not vanish, namely, $c_{33}=-a/2$. Consequently, the Hamiltonian part takes nonvanishing value for $h_3=b/2$. Therefore the family of *all* master equations describing the decoherence is only two-parametric

$$\dot{\varrho}_t = - i\frac{b}{2}[S_3,\varrho_t] - \frac{a}{2}(S_3\varrho_t S_3 - \varrho_t). \qquad (5.9)$$

This general master equation is of the same form as the one derived for the collision model (5.6). The parameters λ,φ are related to the parameters of the underlying unitary interaction via the formula $\langle X\rangle_\xi=\lambda e^{i\varphi}$. Let us note the constraint $\lambda=|\langle X\rangle_\xi| \in [0,1]$, since X is unitary. Therefore $\ln\lambda \leqslant 0$ as it is required by the condition on possible values of a.

VI. ENTANGLEMENT IN DECOHERENCE VIA COLLISIONS

We start with definitions of entanglement quantities that we will evaluate. Let us denote the joint state of the system of $N+1$ qubits (the system qubit and N reservoir qubits) by Ω. The bipartite entanglement shared between a pair of qubits j and k can be quantified in terms of the concurrence [16]

$$C_{jk} = \max\{0,\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \qquad (6.1)$$

where λ_j are decreasingly ordered square roots of the eigenvalues of the matrix $R_{jk}=\varrho_{jk}\sigma_y\otimes\sigma_y\varrho^*_{jk}\sigma_y\otimes\sigma_y$ and $\varrho_{jk}=\mathrm{Tr}_{\overline{jk}}\Omega$ is the state of two qubits under consideration.

The case of multi-partite entanglement is a more complex phenomenon and there is no unique way of its quantification. Fortunately, for *pure* multiqubit systems there is an accepted method of characterization (identification) of intrinsic multipartite entanglement. Specifically, let us consider how strongly the jth qubit is correlated with the rest of qubits in the multipartite system. This degree of entanglement can be quantified via the so-called tangle (see Ref. [17])

$$\tau_j = 4\det\varrho_j = 2(1-\mathrm{Tr}\varrho_j^2), \qquad (6.2)$$

where $\varrho_j=\mathrm{Tr}_{\bar{j}}\Omega$ is the state of the jth qubit. Then we evaluate bipartite concurrences between the given jth qubit and any other qubit in the system, i.e., we evaluate N quantities C_{jk}.

Wootters and his coworkers have found (see Ref. [17]) that for pure three-qubit states the inequalities

$$\sum_{j\neq k}[C_{kj}]^2 \leqslant \tau_k, \quad \forall\, k=1,2,3, \qquad (6.3)$$

hold. In addition they have conjectured that such inequalities also hold for any number of qubits. This conjecture [to so-called Coffman-Kundu-Wootters (CKW) inequality] has been recently proved by Osborne [18]. These inequalities quantify the property which is known as the *monogamy of entanglement* (the entanglement cannot be shared freely in multipartite systems).

As a consequence of the CKW inequality one can define a measure of intrinsic multipartite entanglement Δ_j as

$$\Delta_j = \tau_j - \sum_{k\neq j}\tau_{jk}, \qquad (6.4)$$

where we have used the notation $\tau_{jk}=[C_{jk}]^2$. It is important to note that in the multi-partite case (in particular for more than three qubits) the differences $\Delta_j := \tau_k - \Sigma_{j\neq k}\tau_{jk}$ take different values for different j. Therefore, a weighted sum $\Delta=(1/N)\Sigma_j\Delta_j$ is an appropriate measure of an intrinsic multipartite entanglement. Based on this quantity we can argue that there are multipartite entangled states for which the entanglement has purely bipartite origin, as for example the family of W states [20] that saturate the CKW inequalities, i.e., $\Delta=0$.

Let us assume that the system qubit is initially prepared in the state $|\chi\rangle=a|0\rangle+b|1\rangle$ and each qubit of the reservoir is in a pure state $|\psi\rangle$, i.e., the joint initial state is $|\Omega_0\rangle=|\chi\rangle\otimes|\psi\rangle^{\otimes N}$. After n collisions governed by bipartite controlled unitary operations (4.3) the whole system evolves into the state

$$|\Omega_n\rangle = [a|0\rangle \otimes |V_0\psi\rangle^{\otimes n} + b|1\rangle \otimes |V_1\psi\rangle^{\otimes n}] \otimes |\psi\rangle^{(N-n)}. \quad (6.5)$$

In order to be able to evaluate the entanglement quantities we have to specify all two-qubits and single-qubit density operators. In particular, for $k \leqslant n$, $j \leqslant k$ the bipartite states are given by expressions

$$\varrho_{0k}(n) = |a|^2 |0\psi_0\rangle\langle 0\psi_0| + |b|^2 |1\psi_1\rangle\langle 1\psi_1| + ab^* |\langle\psi_0|\psi_1\rangle|^{(n-1)} \times |0\psi_0\rangle\langle 1\psi_1| + \text{c.c.}, \quad (6.6)$$

$$\varrho_{jk}(n) = |a|^2 |\psi_0\psi_0\rangle\langle\psi_0\psi_0| + |b|^2 |\psi_1\psi_1\rangle\langle\psi_1\psi_1|', \quad (6.7)$$

where we used the notation $|\psi_0\rangle = V_0|\psi\rangle$ and $|\psi_1\rangle = V_1|\psi\rangle$. The single qubit states are as follows:

$$\varrho_0(n) = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1| + ab^* |\langle\psi_0|\psi_1\rangle|^n |0\rangle\langle 1| + \text{c.c.}$$

describes the system qubit after nth collision, and

$$\varrho_k(n) = |a|^2 |\psi_0\rangle\langle\psi_0| + |b|^2 |\psi_1\rangle\langle\psi_1| \quad (6.8)$$

describes the kth qubit of the reservoir after the collision with the system qubit. Evaluation of the tangles is straightforward and results in expressions

$$\tau_0(n) = 4|a|^2|b|^2(1 - |\langle\psi_0|\psi_1\rangle|^{2n}), \quad (6.9)$$

$$\tau_k(n) = 4|a|^2|b|^2|\langle\psi_0|\psi_1^\perp\rangle|^2, \quad (6.10)$$

$$\tau_{0k}(n) = 4|a|^2|b|^2|\langle\psi_0|\psi_1\rangle|^{2(n-1)}|\langle\psi_0|\psi_1^\perp\rangle|^2, \quad (6.11)$$

$$\tau_{jk}(n) = 0. \quad (6.12)$$

One can directly verify the validity of the CKW inequalities

$$\sum_{k=0,k\neq j}^{N} \tau_{jk}(n) = \tau_{j0}(n) = 4|ab|^2|\langle\psi_0|\psi_1\rangle|^{2(n-1)}|\langle\psi_0|\psi_1^\perp\rangle|^2 \leqslant 4|ab|^2|\langle\psi_0|\psi_1^\perp\rangle|^2 = \tau_j(n), \quad (6.13)$$

$$\sum_{k=1}^{N} \tau_{0k}(n) = n \times 4|ab|^2|\langle\psi_0|\psi_1\rangle|^{2(n-1)}|\langle\psi_0|\psi_1^\perp\rangle|^2 \leqslant 4|ab|^2(1 - |\langle\psi_0|\psi_1\rangle|^{2n}) = \tau_0(n), \quad (6.14)$$

where we have used the relations $|\langle\psi_0|\psi_1\rangle| \leqslant 1$ and $|\langle\psi_0|\psi_1^\perp\rangle|^2 = 1 - |\langle\psi_0|\psi_1\rangle|^2$. (See Fig. 3.)

In the limit of large number of interactions ($n \to \infty$) all two-qubit correlations vanish (i.e., finally there is no bipartite entanglement between qubits in reservoir), but the entanglement between the system qubit and the whole reservoir converges to a finite value

$$\tau_0 \to 4|ab|^2, \quad (6.15)$$

$$\tau_{0k} \to 0. \quad (6.16)$$

It means that after the process of decoherence the system qubit is not entangled with the reservoir via bipartite entanglements, but is entangled to the reservoir via multipartite correlations. The final state belongs to the family of Greenbeger-Horn-Zeilinger states that exhibit purely multipartite correlations.

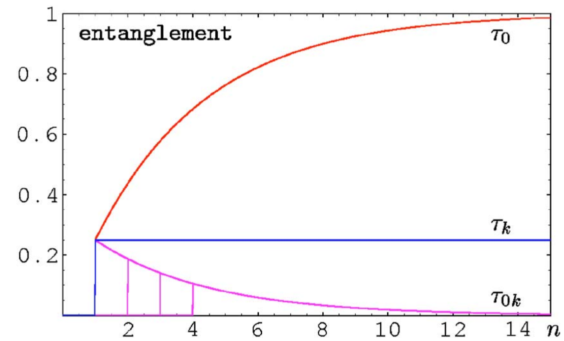

FIG. 3. (Color online) The behavior of entanglement as a function of number n of collisions between the system qubit and reservoir qubits. The degree of entanglement between the system qubit and n reservoir qubits after n collisions is given by τ_0—it increases with the number of collisions (time) to a steady-state value. On the contrary, all reservoir qubits after their interaction with the system qubit are entangled with the constant degree of entanglement (see the tangle τ_k). The bipartite entanglement τ_{0k} (the square of the concurrence C_{0k}) is zero until the kth reservoir qubit collides with the system qubit. After the collision the entanglement takes a nonzero value, though it decreases due to subsequent collisions of the system qubit with other reservoir qubits. It is interesting to note that all $\tau_{0k}(n)$ for $n \geqslant k$ are described by the same function. We assume the following initial state of the system qubit: $|\psi\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|\langle\psi_0|\psi_1\rangle|^2 = 0.75$.

From the above one can see how the entanglement is related to the decoherence. Given the relation $|\langle\psi_0|\psi_1\rangle| = |\langle X\rangle_\psi| = \lambda$ we conclude that the decoherence rate restricts the maximum amount of created entanglement and simultaneously it determines the decrease of entanglement with the number of collisions.

VII. SUMMARY AND CONCLUSIONS

In this paper we have studied qubit decoherence channels as defined by Eq. (1.1). We have presented their complete classification. In addition, we have shown that all decoherence channels can be modeled as collisions of a quantum system with its environment. The bi-partite collisions between the system and reservoir particles are modeled as the controlled-U operations such that the system particle is a control while a reservoir particle is a target. Using the collision model we have derived the most general decoherence master equation in the Lindblad form that describes decoherence. The specific basis in which the decoherence takes place as well as the decoherence rates are specified by properties of the controlled-U operation and the initial state of reservoir particles. We have shown that in the collision model the decoherence is accompanied (or, from a different point of view, one can say that the decoherence is due to) quantum entanglement that is created between the system particle and the reservoir particles. We have derived the explicit expressions for entanglement measures (the concurrence between an arbitrary pair of particles involved in the dynamics and a tangle that characterizes a degree of entanglement between the given particle and the rest of the system). Using these measures and the Coffman-Kundu-Wootters inequalities we have shown that in the case of decohering qubit collisions between this qubit and the reservoir lead to intrinsic multiqubit entanglement of all qubits involved in the process.

We conclude our paper with some remarks.

(i) Even though through the paper we have been paying attention mostly to decoherence of qubits many of the results hold in general. In particular, within the framework of a collision model with the controlled-U bipartite collisions (the system particle plays the role of the control while particles from the reservoir are targets) a decoherence of qudits can be described as well.

(ii) The collision model used in this paper is a discrete one. We have assumed that a collision between two particles is localized in time, so that at a given time instant the controlled-U operation (a bipartite gate) is applied. The sequence of interactions is then labeled by an integer number n and the total dynamics is represented by a discrete semigroup.

As shown in the paper it is straightforward to introduce a continuous time parameter so that the continuous evolution version of the sequence of collision is described by a Markovian process represented by a continuous semigroup. We have derived the corresponding master equation that describes the process of decoherence. More importantly, we have shown that for qubits this master equation describing the decoherence is unique and takes the form (5.9) that can be written as

$$\dot{\varrho}_t = -i[H,\varrho_t] - \frac{1}{2\gamma}[H,[H,\varrho_t]], \qquad (7.1)$$

where we use the notation $H=(b/2)S_3=(\varphi/2)S_3$ and $\gamma=-b^2/2a=\varphi^2/2\ln\lambda$. We note that the double commutator term is well known and usually appears in decoherence models even for higher-dimensional systems. For example, Milburn in his work on intrinsic decoherence (see Ref. [8]) has been derived a generalization of the usual Schrödinger equation exactly in the form (7.1).

(iii) We have shown that the decoherence in the collision model is accompanied (caused) by a creation of entanglement between the system and the reservoir. Unlike in the process of homogenization described in [19–21], in which the created entanglement saturates the CKW inequalities, in the case of decoherence the entanglement results in the Greenberger-Horn-Zeilinger type of correlations [22]. This means that decoherence process (as described by our collision model) does not create an entanglement between the environment particles. Specifically, if we trace over the system qubit (which decoheres) in the nth step of the evolution [see Eq. (6.5)], we find that the environment is in a separable state

$$\omega_{env}(n) = \mathrm{Tr}|\Omega_n\rangle\langle\Omega_n| = [|a|^2(|\psi_0\rangle\langle\psi_0|)^{\otimes n} + |b|^2(|\psi_1\rangle\langle\psi_1|)^{\otimes n}] \otimes |\psi\rangle\langle\psi|^{\otimes(N-n)},$$

where all the parameters are specified in Sec. VI. The decoherence rate λ and the rotation parameter φ can be adjusted by a suitable choice of the interaction U and the state of the reservoir ξ. The collision model reflects microscopic origins of both these parameters that enter the decoherence master equations. The eigenvalues of the Hamiltonian H are given by the value of φ and the parameter γ is specified by both these parameters. The eigenvectors of H form the decoherence basis.

(iv) We have shown explicitly that an arbitrary decoherence channel for a qubit can be represented via the collision model with a particularly chosen controlled-U interaction. However, this result holds for arbitrary dimension (i.e., for qudits) as well. Let us remember th at an arbitrary quantum map $\mathcal{E}$ can be represented as unitary operation on some larger system (this is a content of the Stinespring-Kraus dilation theorem [7]). We have shown that for decoherence channels the collision (represented by a unitary transformation) must be of the form of the controlled-U operation. An open question is whether each decoherence master equation (even for $\dim\mathcal{H}=\infty$) can be derived from the collision model. Knowing a decoherence master equation (i.e., knowing a generator $\mathcal{G}$) it is easy to "fix" a time step $t=\tau$ and define $\mathcal{E}_\tau=\mathcal{E}$. This map is for sure a decoherence channel and can be realized by a collision U. By applying this "elementary" map many times (a sequence of collisions) we obtain a discrete semigroup of the powers of $\mathcal{E}$. The inverse task is trickier, that is, how do we interpolate between these discrete sequences of transformations (parameterized by number of collisions) to obtain a continuously parameterized channel. From a construction of the problem we know that the solution exists (we have started our analysis from the master equation). The question is whether this interpolation for qudit channels can be performed as easily as for qubits, i.e., by replacing the discrete powers of n with continuous parameter t. Nevertheless, given the fact that we have started with a continuous set of channels $\mathcal{E}_t$ and by replacing $t\to\tau$ we obtained $\mathcal{E}_1=\mathcal{E}$. Consequently, it is possible to replace $n\to t/\tau$ to obtain the original continuous semigroup of decoherence channels $\mathcal{E}_t$. As a result we have found that a collision model can be used not only to describe any decoherence master equation, but can also be used to describe any quantum evolution governed by the Lindblad equation. On the other hand, it has to be stressed that collision models describe evolutions that might not be "interpolated" by continuous semigroup of quantum channels.[2]

ACKNOWLEDGMENTS

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[2]Mathematically, this is related to the property of *infinite divisibility* of the matrix $\mathcal{E}$, i.e., to the possibility to calculate all real powers.

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