Nonmaximally entangled bases and their application in entanglement purification via swapping

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(Received 17 March 2004; published 21 March 2005)

Four basis vectors of the Hilbert space of two qubits have the property that if three of them are product states, then the fourth one has to be a product state as well. We address the following situation: Consider a set of orthogonal vectors, each exhibiting a certain degree of entanglement. What is the bound on entanglement of the rest of the basis vectors to form a complete orthonormal basis? Specifically, we present an orthonormal basis, the Ξ basis in the Hilbert space of two qubits, with one product state and three equally entangled states. The maximum of the so available entanglement is quantified. A close-to-optimal protocol is presented for entanglement purification via entanglement swapping of two-qubit states. It is based on a suitably chosen nonmaximally entangled basis and carried out in a single step without any ancillas. A similar application of the Ξ basis is examined. In this latter case, all the involved entangled states have different and nonorthogonal Schmidt decompositions and, except for some possibly resulting states, none of them are maximally entangled. Entanglement of single pair purification is not conserved on average in this case.

DOI: 10.1103/PhysRevA.71.032331 PACS number(s): 03.67.Mn, 03.67.Hk

I. INTRODUCTION

Measurements with entangled eigenstates are clearly a standard part of most quantum information processing (QIP) protocols. The Bell measurement is the key ingredient of, e.g., quantum teleportation [1] and entanglement swapping [2]. Furthermore *maximally entangled bases* themselves are essential for optimal quantum information distributors [3].

Though we frequently consider "a maximally entangled basis," less attention has been paid to nonmaximally entangled bases: a set of nonmaximally entangled states forming an orthonormal basis spanning the Hilbert space of the multipartite system.

Measurements have been very frequently considered as tools for concentrating entanglement to a subsystem of a multipartite system, giving rise to the concept of entanglement of assistance, which we will find useful in our present considerations too. However, we investigate a complementary question: we are primarily interested in the entanglement of the basis vectors themselves, though some of our considerations relate to the entanglement of the complementary system too.

Studying entangled bases can be regarded as studying canonical transformations and, thus, real symplectic transformations [4]. According to the Stone–von Neumann theorem, this means also studying joint unitary transformations, which indeed transform bases into each other. For Galois quantum systems (those with Hilbert spaces of prime-power dimension), this aspect has been studied in detail by Vourdas [5], who has classified their unitary symplectic transforms into local and entangling ones.

There are certainly limitations on the entanglement of the basis vectors. This gives rise to the concept of unextendible product bases (UPB's) [6]: an incomplete orthonormal basis formed by product state vectors, which cannot be supplemented with additional product-state elements. For instance, in the case of two qubits one can specify such a set of five orthogonal state vectors that it is impossible to find any additional product states orthogonal to these. UPB's became the subject of a considerable literature as they have remarkable properties which have implication for the theory of bound entangled states and local distinguishability.

In the case of two qubits, having three product states in a basis implies that the fourth one is also a product. The question may arise what happens if we do not require all the basis elements to be maximally entangled or product states. What quantitative statements can be formed concerning entanglement of the basis elements? The problem is similar to the question of distributed entanglement in multipartite systems, which cannot be arbitrarily entangled. In their case, inequalities limiting the pairwise entanglement of the parties can be derived [7]. One of the main intentions of this paper is to show an example of a similar quantitative limitation for entangled bases in the case of two qubits. A special partially entangled basis will be also introduced as a part of this consideration, which is of rater different nature from the usually considered ones. This basis will find its actual application in a protocol in the second part of this paper.

Besides the relevance of nonmaximally entangled bases from the kinematical point of view, the actual use of them in QIP protocols is of particular interest. Nonmaximally en-

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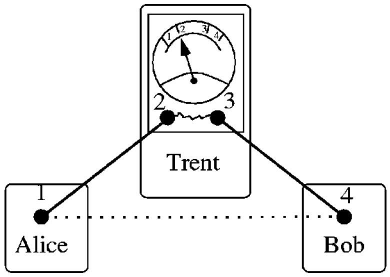

FIG. 1. The entanglement swapping protocol. Alice has qubit 1, and Trent has qubits 2 and 3, while Bob has qubit 4. Initially 1-2 and 3-4 are entangled. Trent carries out a joint measurement on 2-3, and as a result, 1-4 become entangled. Their state is known by Trent from the measurement result.

tangled *states* have been extensively studied, e.g., from the point of view of application for quantum teleportation [8–13], quantum key distribution [14], and entanglement swapping [9,12]. Considering the whole basis on the other hand implies that *all* outcomes of a von Neumann measurement are taken into account.

A typical application in which the use of nonmaximally entangled bases can be profitable is entanglement purification via entanglement swapping, a way of concentrating entanglement via local operations [15] introduced by Bose *et al.* [16]. In this procedure, Trent shares one partially entangled pair with Alice and one with Bob (see Fig. 1). Then, as in the case of entanglement swapping, he carries out a Bell measurement on his systems. With some probability, the systems at Alice and Bob are left in a maximally entangled state, the identity of which is determined by the measurement result. In Ref. [16] it is shown that the success probability depends on the *entanglement of single pair purification*, which is a kind of entanglement measure: the maximum probability of obtaining a Bell state by local operations and classical communication from a single partially entangled pair originally in a pure state [17]. This quantity is found to be conserved on average by the protocol: its ensemble average over the resulting states is the same as its value for the initial state. By applying local unitary transformations with ancillary qubits, the procedure can be made optimal in the sense that the possibly resulting nonmaximally entangled states are transformed into maximally entangled ones with the maximum available probability [18,19].

Though the possibility of using a nonmaximally entangled measurement in this protocol was to some extent discussed in Ref. [9], some physically interesting details are still to be revealed. We show here for the case of four qubits that, in some cases, by choosing a suitable nonmaximally entangled basis a close-to-optimal purification of entanglement via swapping can be achieved in one step.

Recently it has been reported by Sen *et al.* [20] that a certain superadditivity of nonclassicality can be observed in multipartite entanglement swapping when Werner states are initially shared. In certain cases, even though initially none of the shared states violate local realism, the resulting state does so indeed. Somewhat in the same spirit, we will present a scenario of purification of entanglement swapping for qubits, where the *only* maximally entangled state "involved" is the resulting one. All the other states—the shared states and the eigenstates of the measurement—are nonmaximally entangled. It is based upon the special nonmaximally entangled basis which we introduce in the framework of the the kinematical considerations in the first part of the paper.

Regarding the conservation of entanglement of single pair purification described in Ref. [16] for the case of Bell measurements, we find that in our schemes using nonmaximally entangled states, it shows a somewhat different behavior. Namely, for the nearly optimal case presented here, entanglement of single-pair purification is conserved on average not only for the resulting states, but also for the eigenstates of the measurement. On the other hand, in the scheme where no maximally entangled states are included, this quantity is *not* conserved.

This paper is organized as follows. In Sec. II we revise some of the properties of the two-qubit Bell basis. In Sec. III the search for a special nonmaximally entangled basis is performed and an inequality for the entanglement of its elements is found. In Sec. IV the nearly optimal purification of entanglement via swapping is introduced, involving four qubits and a specially chosen nonmaximally entangled basis. In Sec. V, the application of the special basis found in Sec. III for purification via swapping is examined. It is shown that this basis can be also used in a similar, though not optimal, scenario. Moreover, there is a qualitative difference between the optimal purification and this latter one. In Sec. VI results are summarized and conclusions are drawn.

II. A CLOSER LOOK AT THE BELL BASIS

Consider a system of two qubits described by $\mathcal{H}=\mathcal{H}_1 \otimes \mathcal{H}_2$, where the two-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ describe the first and second qubits, respectively. If one were to choose an arbitrary entangled state in $\mathcal{H}$, one first may fix an orthonormal basis (ONB) $(|a\rangle_1, |b\rangle_1)$ on $\mathcal{H}_1$ and another $(|a\rangle_2, |b\rangle_2)$ on $\mathcal{H}_2$. Then the entangled state reads

$$|\Psi_{\text{ent}}\rangle = \sqrt{p_a}|a\rangle_1 \otimes |a\rangle_2 + \sqrt{p_b}|b\rangle_1 \otimes |b\rangle_2, \tag{1}$$

where $p_a, p_b \in]0,1[, p_a + p_b = 1$. This is the Schmidt form of an entangled state; any entangled state can be written so. The state is maximally entangled if $p_a = p_b$. It is important to note that in the case of two qubits, the Schmidt decomposition of a nonmaximally entangled state is unique up to a possible redefinition of $|a\rangle_1$ and $|a\rangle_2$ (or $|b\rangle_1$ and $|b\rangle_2$) by multiplying them with an opposite phase factor [21]. In what follows we will omit the symbols "$\otimes$" from tensor products and use the short notation $|a\rangle_1 \otimes |b\rangle_2 = |ab\rangle$.

In order to construct an ONB on $\mathcal{H}$, consisting of maximally entangled states, one might choose first the computational basis $(|0\rangle, |1\rangle)$ on each subspace. Thus we obtain the first basis element

$$|\Phi^{(+)}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2}$$

One may then chose $-|1\rangle$ instead of $|1\rangle$ on one of the subspaces to obtain the second basis vector

$$|\Phi^{(-)}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad (3)$$

and finally, interchanging $|0\rangle$ and $|1\rangle$ in $\mathcal{H}_2$ we complete the basis by the vectors

$$|\Psi^{(\pm)}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \quad (4)$$

Note, however, that the so-defined Bell basis divides $\mathcal{H}$ into two orthogonal subspaces, to $\mathcal{H}_\Phi$ spanned by $|00\rangle$, $|11\rangle$ and $\mathcal{H}_\Psi$ spanned by $|01\rangle$, $|10\rangle$. This separation is mainly due to the fact that the Schmidt decomposition of the Ψ states and that of the Φ states are orthogonal in the sense that all the constituting vectors of one of the decompositions are orthogonal to all those in the other.

In many studies concerning partially entangled pure states, the maximal entanglement is "deteriorated" by keeping the same Schmidt decomposition as that of the Bell basis and altering the coefficients. Though in arguments related to a *single* nonmaximally entangled state one might perform all calculations in the respective Schmidt basis, if a set of nonmaximally entangled states—e.g., *a whole ONB*—is taken into account, this cannot be done anymore. Thus an ONB of the form

$$C_0|00\rangle + C_3|11\rangle, \quad C_0'|00\rangle + C_3'|11\rangle,$$

$$C_1|01\rangle + C_2|10\rangle, \quad C_1'|01\rangle + C_2'|10\rangle \quad (5)$$

(where $C_0^*C_0'+C_3^*C_3'=0$ and $C_1^*C_1'+C_2^*C_2'=0$) is not the most general one. And though, as we shall see in Sec. IV, such a basis can be useful in some cases, in Sec. V we will show that more general bases may behave in a qualitatively different way.

III. Ξ BASIS

We now consider the problem of finding a basis with certain predefined entanglement properties. Particularly, let us chose a product state—e.g., $|11\rangle$. (It can be shown that choosing another basis element yields essentially the same results.) We require the remaining three basis vectors to be equally entangled. They need not be product states, but they obviously cannot be all maximally entangled. The question arises then, what is the maximal entanglement of the remaining three vectors? Notice that the presence of the product vector and the requirement of equal entanglement of the remaining states *a priori* exclude the choice of the Schmidt decompositions similar to the Bell basis. This makes our otherwise *ad hoc* assumption of having three equally entangled states physically interesting: the so-constructed basis should be of significantly different character than that in Eq. (5).

A. Entanglement measures

After the above formulation of our problem, we briefly describe two entanglement measures that we shall use in this consideration. For arbitrary (even mixed) states of two qubits, the maybe most prevalently used measure of entanglement is the *concurrence* introduced by Hill and Wootters [22]. Having the two-qubit system in the state described by the density matrix ϱ, the concurrence is calculated as

$$\mathcal{C} = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \quad (6)$$

where the λ's are the eigenvalues of the matrix

$$\hat{F} = \sqrt{\sqrt{\rho}(\hat{\sigma}_y \otimes \hat{\sigma}_y)\rho^*(\hat{\sigma}_y \otimes \hat{\sigma}_y)\sqrt{\rho}} \quad (7)$$

in decreasing order. The asterisk stands for complex conjugation of the elements of the matrix in the computational basis.

In case of pure bipartite states $|\Psi\rangle$, the typical measure of entanglement is the *entropy of entanglement*:

$$E(|\Psi\rangle) = -\operatorname{tr}(\sigma \log_2 \sigma), \quad (8)$$

where σ is the reduced density matrix describing one of the subsystems. Considering qubits, this can be expressed with either of the eigenvalues κ and $1-\kappa$ of σ:

$$E(|\Psi\rangle) = h(\kappa) = -\kappa \log_2 \kappa - (1-\kappa)\log_2(1-\kappa), \quad (9)$$

where we have introduced h, the binary entropy function. Exploiting the fact that $\operatorname{tr}\sigma=1$ we have, for one of the eigenvalues,

$$\kappa = \frac{1 + \sqrt{1 - 4\det\sigma}}{2}. \quad (10)$$

According to the result of Hill and Wootters [22], however, the entropy of entanglement can be expressed using the concurrence $\mathcal{C}$ as

$$E(|\Psi\rangle) = h\left(\frac{1 + \sqrt{1 - \mathcal{C}^2}}{2}\right). \quad (11)$$

Comparing Eqs. (9)–(11) we find that

$$\mathcal{C}^2(|\Psi\rangle) = 4\det\sigma, \quad (12)$$

the *square of the concurrence* (or *tangle*), can be very easily calculated with the help of Eq. (12) for pure two-qubit states, as noted in Ref. [7]. We shall use this squared concurrence and concurrence itself in our following considerations.

Consider now two qubits in a mixed state ρ again. This state can be expressed as a convex combination of pure-state ensembles in many ways: there exist many sets $\{p_i, |\phi_i\rangle\}$ such that

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|. \quad (13)$$

One may define the quantity *concurrence of assistance* [23] as the maximum of average concurrence of such ensembles:

$$\mathcal{C}_{\text{assist}}(\rho) = \max_{\{p_i, |\phi_i\rangle\}} \sum_i p_i \mathcal{C}(\phi_i). \quad (14)$$

This can be interpreted as the maximal average of the concurrence of the states resulting after an optimal measurement on an ancillary system, so that the state of the two original qubits and the ancilla realize a purification of ρ. The average is understood as an ensemble average over the measurement outcomes. (A similar quantity defined using entropy of entanglement is *entanglement of assistance* [24].) Concurrence of assistance is also a very informative quantity for mixed two-qubit states. One of its advantages is that similarly to concurrence, it can be calculated very simply: it is the trace of the matrix $\hat{F}$ in Eq. (7); thus, with the λ's of Eqs. (6) it reads

$$\mathcal{C}_{\text{assist}}(\rho) = \sum_{k=1}^{4} \lambda_k. \tag{15}$$

Now we are in the position of finding an ONB in the orthogonal complement space of $|11\rangle$, consisting of equally entangled vectors. We will follow two routes. First, we will provide the maximum value of the available concurrence from a shorter consideration. Second, we shall give an explicit construction of the basis, which will confirm the concurrence limit given before.

B. Upper bound for the concurrence

Consider the complete mixture of the basis vectors $|00\rangle$, $|01\rangle$, and $|10\rangle$:

$$\rho^{(3)} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Direct calculation of concurrence of assistance according to Eq. (15) gives

$$\mathcal{C}_{\text{assist}}(\rho^{(3)}) = \frac{2}{3}. \tag{17}$$

Consider now that we have three equally entangled basis vectors $|\Xi^{(1)}\rangle$, $|\Xi^{(2)}\rangle$, and $|\Xi^{(3)}\rangle$ spanning the same subspace, with the maximal possible concurrence $\mathcal{C}_0$. Obviously the completeness relation

$$\rho^{(3)} = \frac{1}{3}\sum_{k=1}^{3} |\Xi^{(k)}\rangle\langle\Xi^{(k)}| \tag{18}$$

should hold. But according to the definition of concurrence of assistance in Eq. (14),

$$\mathcal{C}_{\text{assist}}(\rho^{(3)}) = \frac{2}{3} = \frac{1}{3} \times 3 \times \mathcal{C}_0 \tag{19}$$

should hold, which gives, for the maximum available concurrence, $\mathcal{C}_0 = 2/3$.

Thus we have found the upper bound for the concurrence, but the actual basis realizing this limit is still to be constructed.

C. Construction of the basis

In the following we will adopt a constructive approach to the problem. We will not exploit the result in Sec. III B, as it will be found from this consideration too, though less elegantly.

We have to find an ONB in the orthogonal complement space of $|11\rangle$, consisting of states with equal concurrence. Expanding a general state on the computational basis,

$$|\Psi\rangle = C_0|00\rangle + C_1|01\rangle + C_2|10\rangle + C_3|11\rangle, \tag{20}$$

the square of concurrence in Eq. (12) reads

$$\mathcal{C}^2(\Psi) = 4[|C_0|^2|C_3|^2 + |C_1|^2|C_2|^2 - 2\,\mathrm{Re}(C_0 C_1^* C_2^* C_3)]. \tag{21}$$

The orthogonal complement in argument is spanned by $\{|00\rangle, |01\rangle, |10\rangle\}$. We will use the notation

$$|\Psi\rangle = C_0|00\rangle + C_1|01\rangle + C_2|10\rangle \leftrightarrow \Psi = \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} \tag{22}$$

for the $\mathbb{C}^3$ representation of the vectors in this linear subspace. A general vector of unit absolute value in this subspace reads, up to an arbitrary phase factor,

$$\Psi_{\mathcal{C}_0^2}(x,\phi_1,\phi_2) = \begin{pmatrix} \sqrt{1-x} \\ \sqrt{\dfrac{x}{2} \pm \dfrac{1}{2}\sqrt{x^2 - \mathcal{C}_0^2}}\,e^{i\phi_1} \\ \sqrt{\dfrac{x}{2} \mp \dfrac{1}{2}\sqrt{x^2 - \mathcal{C}_0^2}}\,e^{i\phi_2} \end{pmatrix}, \tag{23}$$

where $\mathcal{C}_0^2 \in [0,1]$ and $x \in [\mathcal{C}_0, 1]$. Note that the multiplication of *each element* of the basis with different arbitrary phase factors affects neither the orthogonality nor the entanglement of the basis. We have chosen the seemingly complicated parametrization of Eq. (23) as, in this case,

$$\mathcal{C}^2(\Psi_{\mathcal{C}_0^2}(x)) = \mathcal{C}_0^2; \tag{24}$$

cf. Eq. (21). The signs in the second two coordinates can be chosen at will.

We have to find three mutually orthogonal vectors of the form in Eq. (23) with the same concurrence-square.

1. Conjugate cross product and its concurrence

We define a way to construct orthogonal complement of two linearly independent vectors in $\mathbb{C}^3$ similar to the real three-dimensional case.

Consider two vectors in $\mathbb{C}^3$:

$$\mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \end{pmatrix}. \tag{25}$$

We define the "conjugate cross product" of these vectors to be a vector $\mathbf{C}$ so that

$$(\mathbf{C})_i = (\mathbf{A}\overline{\times}\mathbf{B})_i = \sum_{j,k=0}^{2} \varepsilon_{ijk} A_j^* B_k^*, \tag{26}$$

where ε_{ijk} is the Levi-Cività symbol for indices (0, 1, 2). A straightforward calculation shows that the so-defined $\mathbf{C}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$. [To achieve this, both complex conjugations in Eq. (26) are needed.] Moreover, if the two vectors are orthogonal unit vectors [we use the Euclidean norm, $\|\mathbf{A}\|=\|\mathbf{B}\|=1$ and $(\mathbf{A},\mathbf{B})=0$], then $\mathbf{C}$ is also of unit norm: $\|\mathbf{C}\|=1$. The conjugate cross product is conjugate bilinear.

A useful lemma can be stated regarding the entanglement of the conjugate cross product vector.

Lemma 1. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^3$ and $\mathbf{C}=\mathbf{A}\overline{\times}\mathbf{B}$. Concurrence and, thus, entanglement of $\mathbf{C}$ depend on the coordinates of $\mathbf{A}$ and $\mathbf{B}$ only through their moduli and the expressions $A_0B_0^*$, $A_1B_1^*$, and $A_2B_2^*$ only.

Direct calculation shows that

$$\mathcal{C}^2(\mathbf{C}) = 4(|A_2|^2|B_0|^2 + |A_0|^2|B_2|^2 - 2\,\mathrm{Re}[A_0B_0^*A_2^*B_2]) \times (|A_0|^2|B_1|^2 + |A_1|^2|B_0|^2 - 2\,\mathrm{Re}[A_0B_0^*A_1^*B_1]), \tag{27}$$

from which the statement of lemma 1 follows.

2. Construction of the basis

We construct now the basis with three equally entangled states. Suppose that the required concurrence is $\mathcal{C}_0$. We choose two arbitrary vectors and consider them together with their conjugate cross product as an ONB. Let us chose $x \in [\mathcal{C}_0,1]$. Let our first basis vector be a generic vector

$$\mathbf{E}^{(1)} = \begin{pmatrix} \sqrt{1-x} \\ \sqrt{\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2-\mathcal{C}_0^2}}\,e^{i\alpha_1} \\ \sqrt{\frac{x}{2} \mp \frac{1}{2}\sqrt{x^2-\mathcal{C}_0^2}}\,e^{i\alpha_2} \end{pmatrix} \tag{28}$$

[cf. Eq. (23)]. Let us choose a $y \in [\mathcal{C}_0,1]$ as well. One may now intend to choose a second generic vector

$$\mathbf{E}^{(2)\prime} = \begin{pmatrix} \sqrt{1-y} \\ \sqrt{\frac{y}{2} \pm \frac{1}{2}\sqrt{y^2-\mathcal{C}_0^2}}\,e^{i\beta_1} \\ \sqrt{\frac{y}{2} \mp \frac{1}{2}\sqrt{y^2-\mathcal{C}_0^2}}\,e^{i\beta_1} \end{pmatrix}. \tag{29}$$

However, from lemma 1 it follows that the concurrence of the conjugate cross product of the two vectors depends only on the difference of the complex phases of the coordinates of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)\prime}$. The same holds for their orthogonality and concurrence as well. Thus we may choose instead of $\mathbf{E}^{(2)\prime}$, without the loss of generality our second basis vector to be real:

$$\mathbf{E}^{(2)} = \begin{pmatrix} \sqrt{1-y} \\ \sqrt{\frac{y}{2} \pm \frac{1}{2}\sqrt{y^2-\mathcal{C}_0^2}} \\ \sqrt{\frac{y}{2} \mp \frac{1}{2}\sqrt{y^2-\mathcal{C}_0^2}} \end{pmatrix}. \tag{30}$$

(This corresponds to the mere redefinition of $\alpha_i \to \alpha_i - \beta_i$ in the first basis vector $\mathbf{E}^{(1)}$, yielding an equivalent choice.) Due to lemma 1, setting nonzero arguments for the second two coordinates would alter neither the concurrence of the chosen $\mathbf{E}^{(2)}$ nor that of

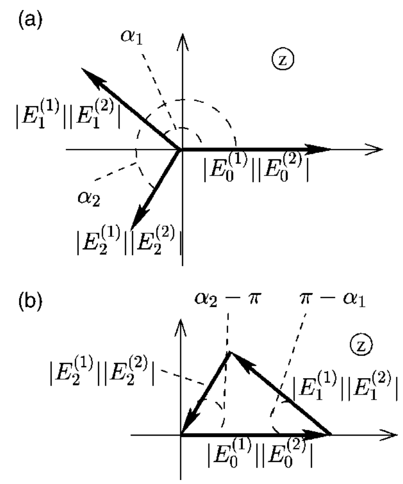

FIG. 2. Triangles on the complex plane, for the determination of the angles in Eq. (32) so that $(\mathbf{E}^{(1)},\mathbf{E}^{(2)})=0$ holds. (a) shows the three complex numbers arising in the scalar product in argument, while (b) is equivalent to (a) but the three vectors are plotted so that the desired triangle is visible.

$$\mathbf{E}^{(3)} = \mathbf{E}^{(1)}\overline{\times}\mathbf{E}^{(2)}. \tag{31}$$

In this way we have four free parameters (x,y,α_1,α_2) instead of six, which is a relevant simplification.

The remaining requirement is that the first two vectors should be orthogonal: $(\mathbf{E}^{(1)},\mathbf{E}^{(2)})=0$. This can be assured by appropriately choosing α_1 and α_2.

To have $(\mathbf{E}^{(1)},\mathbf{E}^{(2)})=0$, the complex numbers $E_0^{(1)*}E_0^{(2)}$, $E_1^{(1)*}E_1^{(2)}$, and $E_2^{(1)*}E_2^{(2)}$ should form a triangle on the complex plane (see Fig. 2). An appropriate choice is

$$\alpha_1 = \pi - \arccos\left(\frac{|E_0^{(1)}|^2|E_0^{(2)}|^2 + |E_1^{(1)}|^2|E_1^{(2)}|^2 - |E_2^{(1)}|^2|E_2^{(2)}|^2}{2|E_0^{(1)}||E_0^{(2)}||E_1^{(1)}||E_1^{(2)}|}\right),$$

$$\alpha_2 = \pi + \arccos\left(\frac{|E_0^{(1)}|^2|E_0^{(2)}|^2 - |E_1^{(1)}|^2|E_1^{(2)}|^2 + |E_2^{(1)}|^2|E_2^{(2)}|^2}{2|E_0^{(1)}||E_0^{(2)}||E_2^{(1)}||E_2^{(2)}|}\right). \tag{32}$$

This follows from elementary geometry: application of the cosine law to the triangle in Fig. 2(b). [Note that $-\alpha_1$ and $-\alpha_2$ would be also an appropriate choice for the complex phases. However, according to Eq. (27), only the cosine of the alphas is relevant; thus, we do not lose generality here.] Equations (32) are of course valid only if $|E_0^{(1)}||E_0^{(2)}|$, $|E_1^{(1)}||E_1^{(2)}|$, and $|E_2^{(1)}||E_2^{(2)}|$ satisfy the triangle equations $|E_i^{(1)}||E_i^{(2)}|+|E_j^{(1)}||E_j^{(2)}| \geq |E_k^{(1)}||E_k^{(2)}| (i \neq j \neq k)$. Otherwise no solution exists.

Thus, given the required concurrence $\mathcal{C}_0$ and two numbers $x,y \in [\mathcal{C}_0,1]$, we can construct vectors $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$ using Eqs. (28), (30), and (32), if they exist. From these we can calculate $\mathbf{E}^{(3)}$ according to Eq. (31) and evaluate its concurrence according to Eq. (27). We are looking for the case $\mathcal{C}(\mathbf{E}^{(3)})$

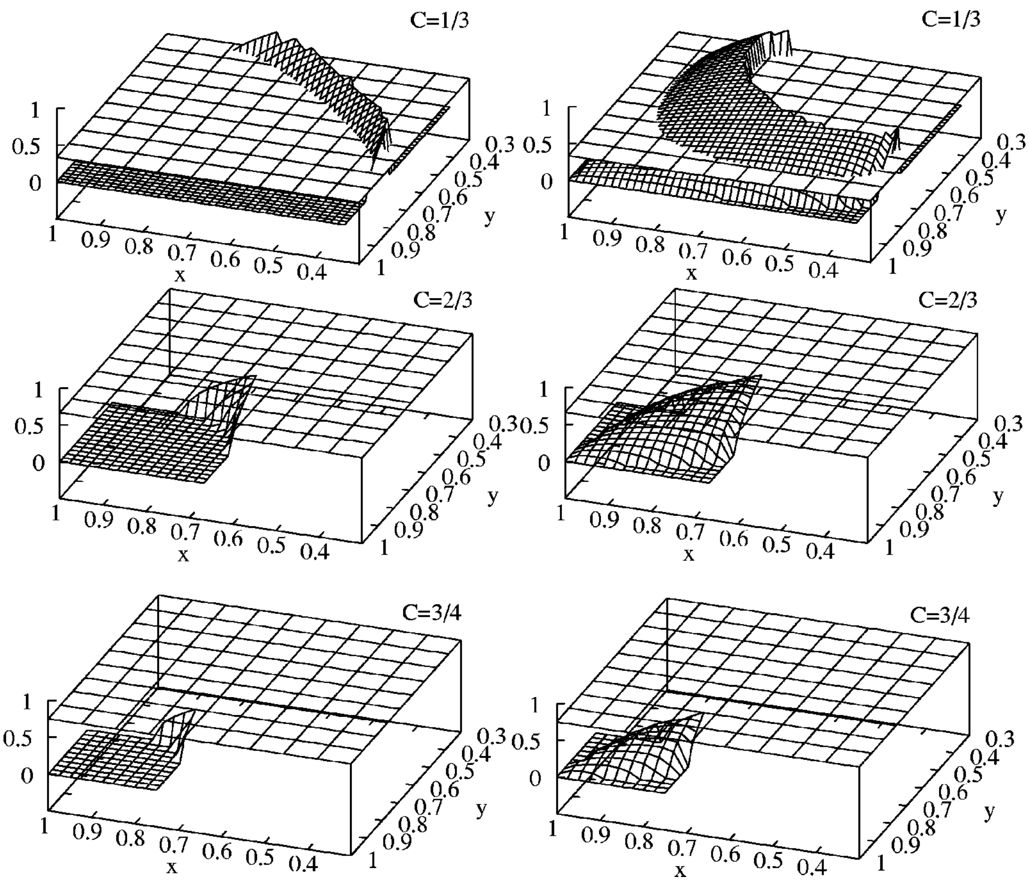

FIG. 3. Concurrence of $\mathbf{E}^{(3)}$ plotted against x and y, for certain representative values of concurrence $\mathcal{C}_0$ of the first two vectors ($\mathbf{E}^{(1,2)}$). All of the displayed quantities are dimensionless. The left column of figures corresponds to the choice of the same sign in Eqs. (28) and (30), while the right column is for the case of different signs. Rows of figures correspond to $\mathcal{C}_0=1/3,2/3,3/4$, respectively. In the first row, surfaces are opaque, while in the second two they are transparent. The horizontal planes in the figures represent the $\mathcal{C}(\mathbf{E}^{(3)})=\mathcal{C}_0$ limit: we are searching for the point of their intersection with the function graph; here, all the three vectors are equally entangled. For $\mathcal{C}_0<2/3$ this is a curve turning into a single point at $\mathcal{C}_0=2/3$. For $\mathcal{C}_0>2/3$, the intersection is empty. Zero concurrence is assigned to those points where there is no solution. This is the reason of the abrupt change to zero for smaller values of x and y in the first row. We have verified the above-described behavior of these functions by making a computer animation with $\mathcal{C}_0$ as the "time variable."

$=\mathcal{C}_0$. Unfortunately, the equations are trigonometrical, and it is hard to draw further analytical consequences. We may however numerically evaluate and plot $\mathcal{C}(\mathbf{E}^{(3)})(x,y)$. This is done in Fig. 3.

A distinction should be made according to the selection of signs in Eqs. (28) and (30). There are four possibilities of choosing the signs, but the value of $\mathcal{C}(\mathbf{E}^{(3)})(x,y)$ depends only on whether we choose the same or different sign in Eqs. (28) and (30). That is, if we use the upper signs in both vectors, we obtain the same $\mathcal{C}(\mathbf{E}^{(3)})(x,y)$ values as in the case of using the lower sign in both vectors. If we use the upper in one of them and the lower in the other, we obtain another value. Thus we have to consider two inequivalent possibilities: equal signs and different signs.

To facilitate the comparison between the figures we have used the same scaling of the axes for all $\mathcal{C}_0$ values. In the $[\mathcal{C}_0,1]\times[\mathcal{C}_0,1]$ domain where the function can be defined at all, concurrence of value 0 is assigned to the points where Eqs. (32) cannot be satisfied due to the violation of triangle inequality.

Though the numerical values are different for the case choosing the same signs than those when we choose different signs, the behavior of the functions is similar. One may observe that the figures are symmetrical in x and y and that concurrence is a decreasing function of both x and y. From this it follows that the maximal concurrence $\mathcal{C}(\mathbf{E}^{(3)})=\mathcal{C}_0$ is achieved at $x=y$.

For small values of $\mathcal{C}_0$, there is a nonzero set of points where $\mathcal{C}(\mathbf{E}^{(3)})=\mathcal{C}_0$. Increasing $\mathcal{C}_0$, at $\mathcal{C}_0=\frac{2}{3}$ we have just one single point where the three concurrences are equally $\frac{2}{3}$, and this is $x=y=\frac{2}{3}$. This is true for both of the inequivalent selections of signs. For $\mathcal{C}_0>\frac{2}{3}$, $\mathcal{C}(\mathbf{E}^{(3)})<\mathcal{C}_0$ holds in case of both sign selections. The result is in accordance with the maximum value of the concurrence expected according to Sec. III B.

The basis with the maximal achievable entanglement is

$$|\Xi^{(1)}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle),$$

$$|\Xi^{(2)}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + e^{i(2\pi/3)}|01\rangle + e^{-i(2\pi/3)}|10\rangle),$$

$$|\Xi^{(3)}\rangle = \frac{1}{\sqrt{3}}(i|00\rangle + e^{-i(\pi/6)}|01\rangle + e^{-i(5\pi/6)}|10\rangle),$$

$$|\Xi^{(4)}\rangle = |11\rangle. \tag{33}$$

(Here, for simplicity we have reordered the vectors so that the real one is the first.)

We can conclude that the maximal concurrence achievable by supplementing a product state to a basis by equally entangled vectors is $\frac{2}{3}$, which has been shown in two independent ways. This is realized by the basis in Eq. (33), which will be referred to as the Ξ *basis* in what follows. Note that the entangled elements of the Ξ basis are "equal-weight superpositions" of the three product states.

IV. NEARLY OPTIMAL ENTANGLEMENT PURIFICATION VIA SWAPPING IN ONE STEP

After the purely kinematical considerations of the previous section, we continue with a seemingly different problem. Consider the usual entanglement swapping scenario, depicted in Fig. 1. Suppose that qubits 1-2 and 3-4, which are initially entangled and shared by Alice and Trent and by Bob and Trent, respectively, are in the nonmaximally entangled state

$$|\Psi_{12\mathrm{opt}}\rangle = |\Psi_{34\mathrm{opt}}\rangle = \cos\phi|00\rangle + \sin\phi|11\rangle, \tag{34}$$

where ϕ is real. We investigate only this "symmetric" scenario, where the shared pairs are in the same pure state throughout this paper.

Trent, aware of the value of ϕ, has a measuring apparatus realizing a nondegenerate measurement, with eigenstates constituting the following *nonmaximally entangled basis*:

$$|\Theta^{(1)}\rangle = |\Psi^{(+)}\rangle, \quad |\Theta^{(2)}\rangle = |\Psi^{(-)}\rangle,$$

$$|\Theta^{(3)}\rangle = \frac{1}{\sqrt{\cos(\phi)^4 + \sin(\phi)^4}}[\sin(\phi)^2|00\rangle + \cos(\phi)^2|11\rangle],$$

$$|\Theta^{(4)}\rangle = \frac{1}{\sqrt{\cos(\phi)^4 + \sin(\phi)^4}}[\cos(\phi)^2|00\rangle - \sin(\phi)^2|11\rangle]. \tag{35}$$

(Note that the third and fourth states are similar to those obtained in the case of a Bell measurement [cf. Eqs. (2c) and (2d) of Ref. [16]], except that their coordinates are interchanged.)

After Trent carries out the measurement on qubits 2 and 3, in case of three of the possible measurement outcomes, qubits 1 and 4 are left in maximally entangled states. In the fourth case, a nonmaximally entangled state is obtained. The resulting states with the respective probabilities are

$$|\Psi^{(1)}_{14\mathrm{opt}}\rangle = |\Psi^{(+)}\rangle, \quad p^{(1)}_{\mathrm{opt}} = \frac{1}{4}\sin(2\phi)^2,$$

$$|\Psi^{(2)}_{14\mathrm{opt}}\rangle = |\Psi^{(-)}\rangle, \quad p^{(2)}_{\mathrm{opt}} = \frac{1}{4}\sin(2\phi)^2,$$

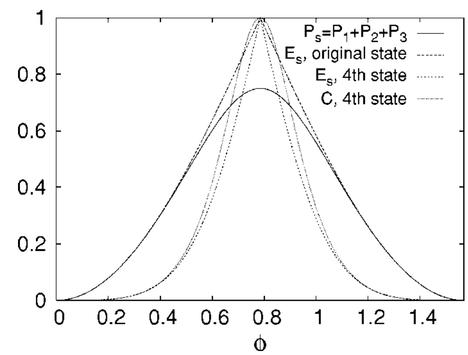

FIG. 4. Probabilities and entanglement measures relevant in the nearly optimal swapping: the joint probability P_s of obtaining either of the first three measurement results (that is, a maximally entangled swapped state), entanglement of single pair purification E_s of the original state and the possibly resulting nonmaximally entangled state, and the concurrence of the latter, as a function of parameter ϕ of the original state in Eq. (34). Note that at $\phi=\pi/4$, the effective success probability is 1 as the swapped state corresponding to the fourth measurement outcome is maximally entangled. All of the plotted quantities are dimensionless.

$$|\Psi^{(3)}_{14\mathrm{opt}}\rangle = |\Phi^{(+)}\rangle, \quad p^{(3)}_{\mathrm{opt}} = \frac{-4\cos(4\phi) + \cos(8\phi) + 3}{16\cos(4\phi) + 48},$$

$$|\Psi^{(4)}_{14\mathrm{opt}}\rangle = \frac{1}{\sqrt{\cos\phi^8 + \sin\phi^8}}[\cos(\phi)^4|00\rangle - \sin(\phi)^4|11\rangle],$$

$$p^{(4)}_{\mathrm{opt}} = \frac{\cos(8\phi) + 28\cos(4\phi) + 35}{16\cos(4\phi) + 48}. \tag{36}$$

In the case of the first three measurement outcomes, a Bell state, thus a maximally entangled state is obtained. The information on which of the possible states is obtained is known by Trent from the measurement result.

The probability of obtaining such a state—i.e., the success probability of the entanglement purification—is

$$P_s = p^{(1)}_{\mathrm{opt}} + p^{(2)}_{\mathrm{opt}} + p^{(3)}_{\mathrm{opt}} = 1 - p^{(4)}_{\mathrm{opt}}, \tag{37}$$

which is plotted in Fig. 4. In order to see whether it is an optimal probability, one has to calculate the *entanglement of single-pair purification*, an entanglement measure which is equal to the square of the modulus of the Schmidt coefficient of smaller magnitude [17], using the notation used in Eq. (1):

$$E_s = 2\min(p_a, p_b). \tag{38}$$

This is the maximum probability of obtaining a Bell state by purifying a single entangled pair originally in the given pure state. Since $|\Psi^{(4)}_{14\mathrm{opt}}\rangle$ is in the Schmidt form in Eq. (36) and so is the initial state $|\Psi_{12\mathrm{opt}}\rangle$ in Eq. (34), this can be easily calculated. The quantities $E_s(|\Psi_{12\mathrm{opt}}\rangle)$ and $E_s(|\Psi^{(4)}_{14\mathrm{opt}}\rangle)$ are also plotted in Fig. 4, accompanied by the concurrence of the latter.

The figure shows that for moderate values of entanglement (small and high values of ϕ), the joint success probability mainly coincides with its largest possible value, and the fourth state contains nearly zero entanglement. Thus in such cases, mainly optimal entanglement distillation is achieved in one step. For more entangled initial states (the middle of the figure), the success probability is below the optimal value, but the possibly obtained nonmaximally entangled state is more entangled, as reflected by the value of the two plotted entanglement measures. Therefore this scheme can be regarded as optimal especially in the case of moderately entangled initial states. Note that for a maximally entangled initial state $\phi=\pi/4$ the probability of the first three events is $3/4$, but the success probability is 1 since the fourth resulting state is maximally entangled.

An observation can be made regarding the average of entanglement of single pair purification. As is shown in Ref. [16] the average of E_s of the resulting states for all possible events is preserved in the original entanglement purification via a swapping protocol. It can be verified that it is true in our protocol too:

$$E_s(|\Psi_{12\text{opt}}\rangle) = \sum_{k=1}^{4} p_{\text{opt}}^{(k)} E_s(|\Psi_{14\text{opt}}^{(k)}\rangle). \tag{39}$$

Moreover,

$$E_s(|\Psi_{12\text{opt}}\rangle) = \sum_{k=1}^{4} p_{\text{opt}}^{(k)} E_s(|\Theta^{(k)}\rangle) \tag{40}$$

also holds; that is, the average is preserved for the eigenstates of the measurement—that is, *at Trent's side too*. This is not the case when we use the Bell basis.

Note that the following fact may be also inferred from the results presented in this section. In purification via entanglement swapping, a nonmaximally entangled pure state is turned into a maximally entangled state if the corresponding eigenstate of the measurement that was carried out has a Schmidt decomposition orthogonal to that of the initial state. Orthogonality of the Schmidt decompositions is meant as in the case of the Bell states; cf. Sec. II.

V. ENTANGLEMENT PURIFICATION VIA THE Ξ BASIS

Consider now the same scenario as in the previous section (Fig. 1), but where qubits 1-2 and 3-4 are assumed to be both in the state

$$|\Psi^{(\text{in})}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |10\rangle + |11\rangle). \tag{41}$$

The nondegenerate measurement carried out by Trent will now project onto the Ξ basis in Eq. (33).

The resulting states of qubits 1-4 and the corresponding probabilities are

$$|\Psi_{14\Xi}^{(1)}\rangle = \frac{1}{\sqrt{11}}(|01\rangle + |10\rangle + 3|11\rangle), \quad p_{\Xi}^{(1)} = \frac{11}{27},$$

$$|\Psi_{14\Xi}^{(2)}\rangle = \frac{1}{\sqrt{2}}(e^{-i(\pi/3)}|01\rangle + e^{i(\pi/3)}|10\rangle), \quad p_{\Xi}^{(2)} = \frac{2}{27},$$

$$|\Psi_{14\Xi}^{(3)}\rangle = \frac{1}{\sqrt{2}}(e^{i(\pi/3)}|01\rangle + e^{-i(\pi/3)}|10\rangle), \quad p_{\Xi}^{(3)} = \frac{2}{27},$$

$$|\Psi_{14\Xi}^{(4)}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \quad p_{\Xi}^{(4)} = \frac{4}{9}. \tag{42}$$

Here $|\Psi_{14\Xi}^{(2)}\rangle$ and $|\Psi_{14\Xi}^{(3)}\rangle$ are two maximally entangled states, which are not orthogonal. $|\Psi_{14\Xi}^{(1)}\rangle$ is a state which is equally entangled as the initial state $|\Psi^{(\text{in})}\rangle$, and $|\Psi_{14\Xi}^{(4)}\rangle$ is obviously a product state. Thus using the Ξ basis, state $|\Psi^{(\text{in})}\rangle$ can be distilled with probability $4/27\approx0.1481$. This is done when measurement outcomes 2 and 3 are obtained. If the first outcome is obtained, we are left with a state less entangled than the original one, which can be regarded as a entanglement swapping. The probability for this event is $11/27\approx0.4074$. If the fourth measurement outcome occurs, entanglement is completely destroyed; the probability for this is $4/9 \approx0.4444$. Except for the possibly resulting maximally entangled states, all states occurring in this protocol are partially entangled.

There is another important difference between the application of the Ξ basis and the optimal purification described in the previous section. This can be understood by observing the Schmidt decomposition of the Ξ basis and $|\Psi^{(\text{in})}\rangle$, which is to be found in Table I. Namely, all the entangled states in the basis and the state to be purified possess different Schmidt decompositions. Though the coefficients are the same, the vectors are different. The orthogonality present in the case of the optimal purification or the Bell basis is also missing here. This clearly shows that the Ξ basis is essentially different from those of form in Eq. (5)

Knowledge of the Schmidt decompositions also enables us to evaluate entanglement of single-pair purification. In the case of the three nonmaximally entangled basis states, its value is ≈0.2546. Thus our protocol, providing a success probability of 0.1481, is definitely not optimal. In spite of this fact, as it is conceptionally different from the optimal

TABLE I. Vectors of the Schmidt decomposition [cf Eq. (1)] of the entangled elements Ξ basis and that of the state which can be distilled with them. Vectors are represented in the computational basis. $A=(1+\sqrt{5})/(\sqrt{2}\sqrt{5+\sqrt{5}})\approx0.8507$, $B=2/(\sqrt{2}\sqrt{5+\sqrt{5}})\approx0.5257$.

State	Vector for coeff. $\sqrt{p_0}=\frac{1}{3}\left(\frac{\sqrt{6}\sqrt{5+2\sqrt{5}}}{\sqrt{5+\sqrt{5}}}\right)\approx0.9342$	Vector for coeff. $\sqrt{p_1}=\frac{1}{3}\left(\frac{\sqrt{6}\sqrt{5-2\sqrt{5}}}{\sqrt{5-\sqrt{5}}}\right)\approx0.3568$
$\lvert\Xi_1\rangle$	$\begin{pmatrix}A\\B\end{pmatrix}\otimes\begin{pmatrix}A\\B\end{pmatrix}$	$\begin{pmatrix}B\\-A\end{pmatrix}\otimes\begin{pmatrix}-B\\A\end{pmatrix}$
$\lvert\Xi_2\rangle$	$\begin{pmatrix}A\\Be^{-i(2\pi/3)}\end{pmatrix}\otimes\begin{pmatrix}A\\Be^{i(2\pi/3)}\end{pmatrix}$	$\begin{pmatrix}B\\Ae^{i(\pi/3)}\end{pmatrix}\otimes\begin{pmatrix}-B\\Ae^{i(2\pi/3)}\end{pmatrix}$
$\lvert\Xi_3\rangle$	$\begin{pmatrix}Ae^{-i(2\pi/3)}\\B\end{pmatrix}\otimes\begin{pmatrix}iAe^{i(2\pi/3)}\\iB\end{pmatrix}$	$\begin{pmatrix}Be^{i(\pi/3)}\\A\end{pmatrix}\otimes\begin{pmatrix}iBe^{i(2\pi/3)}\\-iA\end{pmatrix}$
$\lvert\Psi^{(\text{in})}\rangle$	$\begin{pmatrix}B\\A\end{pmatrix}\otimes\begin{pmatrix}B\\A\end{pmatrix}$	$\begin{pmatrix}-A\\B\end{pmatrix}\otimes\begin{pmatrix}A\\-B\end{pmatrix}$

one described in the previous section, it is indeed physically interesting.

Regarding conservation of entanglement of single-pair purification on average in the sense as in Eqs. (39) and (40), direct calculation gives $\langle E_s \rangle = 0.1549$ for the resulting states and $\langle E_s \rangle = 0.1415$ for the states of the basis. Both values are smaller than the value 0.2546 of the initial state: entanglement of single-pair purification is decreased on average, in contrast with the other schemes of entanglement purification via swapping where this quantity is preserved. This illustrates the different nature of the Ξ basis.

VI. SUMMARY AND CONCLUSIONS

We have studied nonmaximally entangled bases of two qubits and their application in purification of entanglement in entanglement swapping. As illustrated by the known considerations of unextendible product bases, an orthonormal basis with arbitrary predefined entanglement of its elements does not exist. We have searched for certain limitations on the entanglement of the basis elements.

We have searched for bases in which one of the states is a product and the other three are equally entangled. (All the similarly formulated questions, with different number of product versus equally entangled states are trivial.) We have found that the maximal concurrence (entanglement) of the three equally entangled basis elements can be $2/3$. This is a quantitative inequality limiting the entanglement of the nonmaximally entangled basis under the assumption set. An actual basis satisfying it is the Ξ basis is that in Eq. (33).

The construction of the basis in one of the presented derivations of this inequality relies on the "conjugate cross product," whose advantageous properties enabled us to reduce the number of relevant parameters. It seems to be rather difficult to formulate similar statements for different cases in the same constructive way. If one were to solve the complementary problem of searching a basis with one maximally and three equally entangled states, for instance, the number of relevant parameters cannot be so apparently reduced. Thus the formulation of other, possibly more general inequalities seems to be feasible using the other, nonconstructive approach: namely, the application of concurrence of assistance (or entanglement of formation). This can be a subject of further research.

We have shown that the protocol of entanglement purification via entanglement swapping can be carried out with a rather good efficiency in one step, without any ancillas, with a measurement projecting onto a suitable chosen basis. The protocol has been found very close to optimal for moderately entangled initial states and close to optimal otherwise. The entanglement of single-pair purification is preserved on average both on the measurement's side and for the resulting state.

It has been shown that the Ξ basis can be used for a similar purpose, though with less than the optimal efficiency. However, this case is found to be qualitatively inequivalent to the optimal protocol, and it is pointed out by the examination of the Schmidt decomposition of the Ξ basis that it is of different nature than the usually considered ones. Namely, all the involved states have different and not orthogonal Schmidt decompositions. Another interesting physical feature of the scenario is that maximally entangled states arise from nonmaximally entangled ones only, after a single measurement. In contrast with the other known purification via entanglement swapping schemes, here entanglement of single-pair purification is not preserved on average on either side; it is decreased instead.

It is surprising indeed that even in the "world" of two qubits, new enigmas can be disclosed. In spite of the existence of the mathematical classification of entangling symplectic transformations on Galois quantum systems, the general properties and possible applications of nonmaximally entangled states are not fully exhausted. We believe that the results here illustrate this statement appropriately and might motivate further studies of the topic.

ACKNOWLEDGMENTS

This work was supported by the European Union projects QGATES and CONQUEST and by the Slovak Academy of Sciences via the project CE-PI. M.K. acknowledges the support of National Scientific Research Fund of Hungary (OTKA) under Contract Nos. T043287 and T034484. The authors thank Frank Verstraete for pointing out the usefulness of the concept of concurrence of assistance in this context and also Martin Plesch for useful discussions.

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