Instability and Entanglement of the Ground State of the Dicke Model

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Using tools of quantum information theory we show that the ground state of the Dicke model exhibits an infinite sequence of instabilities (quantum-phase-like transitions). These transitions are characterized by abrupt changes of the bi-partite entanglement between atoms at critical values κ_j of the atom-field coupling parameter κ and are accompanied by discontinuities of the first derivative of the energy of the ground state. We show that in a weak-coupling limit ($\kappa_1 \leq \kappa \leq \kappa_2$) the Coffman-Kundu-Wootters inequalities are saturated, which proves that for these values of the coupling no intrinsic multipartite entanglement (neither among the atoms nor between the atoms and the field) is generated by the atom-field interaction. We show that in the strong-coupling limit the entangling interaction with atoms leads to a highly sub-Poissonian photon statistics of the field mode.

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Interactions between quantum objects lead to correlations that may have no classical analogue. These purely quantum correlations, known as entanglement, play a fundamental role in modern physics and have already found their applications in quantum information processing and communication [1]. A degree of quantum entanglement depends on the physical nature of interacting objects and on the character of their mutual coupling. There is another physical phenomenon that shares many of the features with the entanglement—quantum phase transitions (QPT) [2]. These transitions occur at zero temperature and they are induced by the change of a coupling parameter. Recently several authors have reported on a close relation between quantum critical phenomena and appearance of entanglement. In particular, scaling of entanglement close to a quantum phase transition in an Ising model has been reported [3]. In this Letter we report on an infinite sequence of instabilities (i.e., a sequence of quantum-phase-like transitions) and its connection to *bi-partite* atomic entanglement in the so-called Dicke model (DM) [4–8]. The QPT in the DM was first rigorously described by Hepp and Lieb [9] (see also Ref. [10]) while the instability of the ground state has been reported by Narducci *et al.* [6]. In the paper by Hepp and Lieb [9] only a thermodynamics limit of large-N and weak-coupling κ has been studied. It has been shown that in this limit at $\kappa = \omega/\sqrt{N}$ the ground state of the model exhibits the second-order QPT. QPT in the DM have been studied by Hillery and Mlodinow [11] and Rzazewski *et al.* [12] in the limit of large N and weak coupling. QPT and its relation to a chaos have been studied by Emary and Brandes [13]. Schneider and Milburn [14] (see also Refs. [15,16]) studied appearance of entanglement and its connection to quantum phase transitions in the DM with just two atoms. The connection between the atom-field entanglement and the QPT in the thermodynamical limit of the DM has recently been studied by Lambert *et al.* [17]. However, in all these studies the sequence of ground-state instabilities (quantum-phase-like transitions) and the behavior of bi-partite atomic concurrence have been completely overlooked. In this Letter we report on this remarkable feature of quantum entanglement in the Dicke model both in the weak coupling as well as the strong-coupling regimes.

Let us assume a model of N identical two-level atoms (spin-$1/2$ particles), at positions $\vec{r}_1, \ldots, \vec{r}_N$ coupled to a single-mode electromagnetic field via the electric-dipole interaction originally proposed by Dicke [4] (see also Refs. [5–8]). In the rotating-wave approximation the model Hamiltonian reads

$$H = \frac{\hbar\omega_A}{2}\sum_{j=1}^{N}\sigma_j^z + \hbar\omega_F a^\dagger a + \hbar\kappa\sum_{j=1}^{N}(\mathrm{e}^{i\vec{r}_j\cdot\vec{k}_j}a^\dagger\sigma_j^- + \mathrm{e}^{-i\vec{r}_j\cdot\vec{k}_j}a\sigma_j^+), \qquad (1)$$

where a and $a^\dagger$ are the field annihilation and creation operators, respectively. In what follows we will assume that the atomic frequency ω_A is resonant with the field frequency ω_F and $\hbar = 1$. All the atoms are assumed to be coupled to the field with the same coupling κ and the position dependent phase factors can be included into definition of the Pauli matrices σ via local unitary transformations (see, e.g., Ref. [11]). As shown by Tavis and Cummings [5] one of the integrals of motion of the DM corresponding to the total excitation number $P = a^\dagger a + \sum_{j=1}^{N}\sigma_j^+\sigma_j^-$ partitions the total Hilbert space of the DM into a direct sum of subspaces labeled by different excitation numbers.

When the number of excitations p is smaller than the number of atoms N, then the corresponding subspace of the Hilbert space is spanned by $p+1$ vectors $|\{e^s, g^{\otimes(N-s)}\}\rangle_A|p-s\rangle_F$ with $s = 0, \ldots, p$, where $|\{e^s, g^{\otimes(N-s)}\}\rangle_A$ is a completely symmetric state of s atoms in the upper level $|e\rangle$ and $(N-s)$ atoms in the lower level

TABLE I. The number of excitations p in the ground state of the DM with N atoms changes abruptly at critical values κ_j. The larger the coupling is the larger the number of excitations in the ground state. The energy of the ground state E is continuously decreasing with the increase of the coupling κ. For different regions of κ characterized by different excitation numbers, the ground-state energy is presented in Table I and in Fig. 1. At critical values κ_j, the first derivatives of the energy $\partial E/\partial\kappa$ exhibit discontinuities while the bi-partite concurrence reflecting entanglement between pairs of atoms is changed abruptly (discontinuously).

p	κ/ω	$E^{(p)}$	$C^{(p)}$
0	$0 \leq \kappa \leq \kappa_1$	$E^{(0)} = -\frac{N}{2}\omega$	$C^{(0)} = 0$
1	$\kappa_1 \leq \kappa \leq \kappa_2$	$E^{(1)} = \frac{2-N}{2}\omega - \kappa\sqrt{N}$	$C^{(1)} = \frac{1}{N}$
2	$\kappa_2 \leq \kappa \leq \kappa_3$	$E^{(2)} = \frac{4-N}{2}\omega - \kappa\sqrt{2(2N-1)}$	$C^{(2)} = \frac{4N-5-2\sqrt{2N^2-5N+4}}{N(2N-1)}$
3	$\kappa_3 \leq \kappa \leq \kappa_4$	$E^{(3)} = \frac{6-N}{2}\omega - \kappa\sqrt{5(N-1)+\sqrt{(4N-5)^2+8N}}$	...
4	$\kappa_4 \leq \kappa \leq \kappa_5$	$E^{(4)} = \frac{8-N}{2}\omega - \kappa\sqrt{10N-15+3\sqrt{17-12N+4N^2}}$	...

$|g\rangle$, while $|p-s\rangle_F$ describes a Fock state with $(p-s)$ photons. The DM Hamiltonian (1) in this subspace is represented by a 3-diagonal $(p+1)\times(p+1)$ matrix, the eigenvalues $E_j^{(p)}(\kappa)$ of which are linear functions of the coupling κ, i.e., $E_s^{(p)}(\kappa) = K_s^{(p)}\kappa + E_0^{(p)}$, where $E_0^{(p)} = (p - N/2)\omega$ and $K_s^{(p)}$ are parameters to be determined from the eigenvalue problem. The lowest energy eigenstate in the given subspace of the Hilbert space is determined by the smallest (negative) $K_{\min}^{(p)}$. In what follows we will omit the subscript min, and when we use $E^{(p)}(\kappa)$ and $K^{(p)}$ we refer to the energy of the ground state for a given value of κ.

The character of the ground state of the DM (1) with the smallest eigenenergy depends on the coupling κ. At the critical values of the coupling κ_j the number of excitations p in the ground state of the system, the bi-partite concurrence C between atoms as well as the von Neumann entropy of the field mode do exhibit discontinuities. These effects can be clearly seen from Table I and Figs. 1 and 2. Specifically, let us consider eigenenergies of the ground state with 0 and 1 excitations. Their explicit expressions are $E^{(0)} = -\frac{N}{2}\hbar\omega$ and $E^{(1)} = \frac{2-N}{2}\omega - \kappa\sqrt{N}$. It is clear that for $\kappa/\omega < 1/\sqrt{N}$ the energy $E^{(0)}$ is smaller than $E^{(1)}$, so the state $|g^{\otimes N}\rangle$ with zero excitations is the ground state of the DM. But for $\kappa/\omega > 1/\sqrt{N}$ this state is not a ground state of the DM anymore since $E^{(1)} < E^{(0)}$. That is $\kappa_1 = \omega/\sqrt{N}$ is the critical point at which the number of excitations in the ground state of the DM is abruptly changed. Analyzing the inequalities between eigenenergies of lowest energy eigenstates in subspaces of the Hilbert space with different excitations numbers, we can determine critical values of the coupling parameter κ. The first few critical values can be expressed in a simple analytical form:

$$\kappa_1/\omega = N^{-1/2}; \qquad \kappa_2/\omega = [\sqrt{4N-2} - \sqrt{N}]^{-1};$$
$$\kappa_3/\omega = \left[\sqrt{5(N-1) + \sqrt{(4N-5)^2 + 8N}} - \sqrt{4N-2}\right]^{-1}. \tag{2}$$

In a thermodynamical limit of large N (while $p \ll N$) all critical points "merge" together, i.e., $\kappa_j \simeq \omega/\sqrt{N}$ (where $j = 1, \ldots, p$) and we recover the result of Hepp and Lieb [9].

On the other hand, in the strong-coupling regime ($p \geq N$) when the Hamiltonian is represented by the $(N+1)\times(N+1)$ matrix (the size of which does not depend on the particular value of p), the energy of the ground state of the DM again linearly depends on κ, and for a given $p \gg N$ it

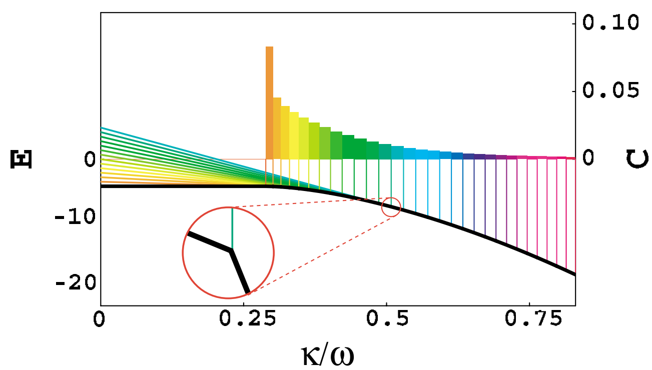

FIG. 1 (color online). The energy E (measured in units of $\hbar\omega$) of the ground state of the DM with $N = 12$ atoms as a function of the scaled coupling parameter κ/ω. We see a sequence of instabilities (quantum-phase-like transitions) that are marked by vertical lines corresponding to different values of the coupling κ_j. The first interval of values of the coupling parameter $0 \leq \kappa \leq \kappa_1$ is characterized by the excitation number equal to zero. The second interval $\kappa_1 \leq \kappa \leq \kappa_2$ is characterized by one excitation, etc. The first derivative of the ground-state energy (black line) is not a continuous function of κ (see the inset that illustrates the behavior of the energy of the ground state around the critical point κ_j). In the figure we plot also a bi-partite concurrence as a function of κ. The concurrence exhibits abrupt changes at "critical" values of κ_j. The maximal value of bi-partite concurrence in the Dicke system appears for $\kappa_1 \leq \kappa \leq \kappa_2$ when the ground state of the system contains just one excitation ($p = 1$). Then, the value of the concurrence decreases with the increase of p. Nevertheless, even for $p \gg 12$ the concurrence is nonzero.

can be approximated as $E^{(p)} = -N\sqrt{p}\kappa + p\omega$. The critical values of κ in the limit $p \gg N$ can be approximated as $\kappa_p \simeq \omega[N(\sqrt{p+1} - \sqrt{p})]^{-1}$.

In our approach we consider distinguishable atoms (located at positions $\vec{r}_1, \ldots, \vec{r}_N$) that interact with a single-mode field. This field mediates quantum correlations between the atoms. In what follows we will study bi-partite atomic entanglement in the ground state of the Dicke system that for a given κ is represented by a superposition $|\Xi^{(p)}\rangle = \sum_{s=0}^{p} A_s|\{e^{\otimes s}; g^{\otimes(N-s)}\}\rangle_A|p-s\rangle_F$. The composite states $|\{e^{\otimes s}; g^{\otimes(N-s)}\}\rangle$ of this superposition exhibit entanglement. For instance, the state $|\{e; g^{\otimes(N-1)}\}\rangle$ is the state (the so-called entangled web) with maximal bi-partite entanglement between an arbitrary pair of atoms (see Koashi *et al.* [18]).

In order to make our discussion quantitative, we recall the definition of the concurrence [19,20] between a pair of qubits (two-level atoms). Let ϱ be a density matrix of the pair of two-level atoms expressed in the basis $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$. Let $\tilde{\varrho}$ be a matrix defined as $\tilde{\varrho} = (\sigma^y \otimes \sigma^y)\varrho^T(\sigma^y \otimes \sigma^y)$, where σ^y is a Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, while T indicates a transposition. Then the concurrence of a bi-partite system reads $C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where λ_i are the square roots of the eigenvalues of the matrix $\tilde{\varrho}\varrho$ in descending order. The values of concurrence range from zero (for separable states) to one (for maximally entangled states of two qubits). The bi-partite entanglement of the entangled web state $|\{e; g^{\otimes(N-1)}\}\rangle$ measured in concurrence takes the value $C = 2/N$. As follows from our investigations (see Table I and Fig. 1) for $\kappa_1 \leq \kappa \leq \kappa_2$ the entanglement between an arbitrary pair of atoms corresponds to the concurrence $C^{(1)} = 1/N$. Increasing the coupling, the value of the concurrence is decreasing but it changes *discontinuously* at critical values κ_j (see Fig. 1). Nevertheless, the total amount of the bi-partite entanglement in the atomic sample measured in terms of the function τ_A is nonzero for arbitrarily large number N of atoms and arbitrarily large coupling (see Fig. 2).

The bi-partite entanglement between atoms is mediated via a single-mode electromagnetic field that interacts with the atoms in the dipole and the rotating-wave approximations. In fact, this interaction leads also to an entanglement of the field mode to the atomic system. In order to measure the degree of entanglement between the field and the atoms, we utilize the von Neumann entropy of the atomic sample $S_A = -\text{Tr}[\rho_A \ln\rho_A]$, where ρ_A is the density operator of the atoms in the ground state $|\Xi^{(p)}\rangle$. It follows from the Araki-Lieb theorem that for a pure atom-field state $|\Xi^{(p)}\rangle_{AF}$ the field entropy S_F is equal to S_A. The larger the entropy is the larger the degree of entanglement. For $p \leq N$ the upper bound on the entropy S_A is given by the number of excitations p, i.e., $S_A^{(\max)} = \ln(p+1)$, while for $p \geq N$ the upper bound is given by the total number of atoms in the system, i.e. $S_A^{(\max)} = \ln(N+1)$. For a given number of atoms, the atomic (field) entropy depends on the number of excitations in ground state of the DM. For very weak coupling, i.e., when $0 \geq \kappa \geq \kappa_1$, the atoms are totally disentangled from the field. In the region of values $\kappa_1 \leq \kappa \leq \kappa_2$ when the ground state is characterized by one excitation, the field mode occupies a Hilbert space spanned by two Fock states $|0\rangle_F$ and $|1\rangle_F$. In this case, the entropy is maximal and equals ln2. With the increase of the coupling κ the entropy of the field is always smaller than $\ln(p+1)$. In fact, for large N and for p large ($p \geq N$), we find that $S_A = \frac{1}{2}\ln(N+1)$, i.e., the entropy is equal to *one-half* of its maximally possible value. That is, the field is not maximally entangled with the atomic sample. The reason for this behavior is that the entanglement cannot be shared freely. To illuminate this issue let us consider the range of κ

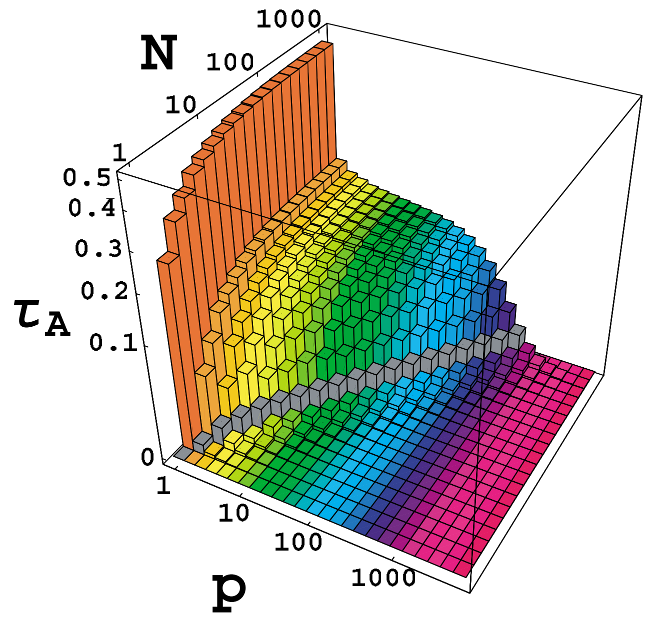

FIG. 2 (color online). The bi-partite concurrence of the ground state of the DM as a function of number of atoms N and the number of excitations p. Since the bi-partite concurrence decreases as $1/N$, we plot the "total atomic bi-partite entanglement" $\tau_A = C^2 N(N-1)/2$, where the factor $N(N-1)/2$ corresponds to the number of possible pairs in the atomic sample. We see that τ_A as a function of N takes the largest value for $p = 1$ and in the large N limit tends to the value $1/2$. We see that even in the strong-coupling case when a $p = N$ arbitrary pair of atoms in the sample is entangled. This is reflected by a nonzero value of τ_A [see the diagonal line in the (p, N) plane of the figure]. In this case, for N large enough the field exhibits sub-Poissonian photon statistics with the mean-photon number $\bar{n} = 2N/3$ and the dispersion $\Delta = \sqrt{5N/26}$. We conclude that the larger the number of excitations p the smaller is the amount of bi-partite entanglement between atoms. In the strong-coupling limit a strong atom-field entanglement is established that is quantified by the von Neumann entropy $S_A = S_F = \frac{1}{2}\ln(N+1)$.

such that $p = 1$. In this case the field mode can be effectively represented as a qubit. Consequently, the system of N two-level atoms and the field mode can be represented as a set of $(N + 1)$ qubits. Recently Coffman *et al.* [21] have conjectured that for pure states the sum of square of concurrencies between a qubit j and any other qubit k is smaller than or equal to the tangle between the given qubit j and the rest of the system, i.e., the Coffman-Kundu-Wootters (CKW) inequalities read

$$\sum_{k=1;k\neq j}^{N} C_{j,k}^2 \leq \tau_{j,\bar{j}} = 4 \det \rho_j, \tag{3}$$

where the sum in the left-hand side is taken over all qubits except the qubit j, while $\tau_{j,\bar{j}}$ denotes the tangle between the qubit j and the rest of the system (denoted as $\bar{j}$). If we assume that the qubit j represents the field mode, then we can find that the tangle between the field and the system of atoms reads $\tau_{FA} = 4 \det \rho_F = 1$, while the concurrencies between the field and each of the atoms is $C_{FA_j} = 1/\sqrt{N}$. From here it directly follows that the ground state of the DM with $p = 1$ saturates the CKW inequalities, which proves that the atom-field interaction as described by the Hamiltonian (1) with small coupling ($\kappa_1 \leq \kappa \leq \kappa_2$) does induce only *bi-partite* entanglement and does not result in intrinsic multipartite quantum correlations. For the moment, it is essentially impossible to generalize this result for other values of p since no measures of entanglement between a qudit (field mode with $p > 1$ excitations) and a set of qubits (atoms) are available and no corresponding generalization of the CKW inequalities is known. Nevertheless, it is interesting to study the strong-coupling limit when $p \gg N$ from a different perspective. In this case the ground state of DM reads $|\Xi^{(p)}\rangle_{AF} = \sum_{s=0}^{N} A_s |\{e^{\otimes s}; g^{\otimes(N-s)}\}\rangle_A |p - s\rangle_F$. For $N \ll p$ the amplitudes A_s can be approximated as $A_s \simeq (-1)^{N+s}\sqrt{\binom{N}{s}2^{-N}}$. We already know that in this case the field mode is highly entangled with the atoms that is reflected by the field entropy $S_F \simeq \frac{1}{2}\ln(N + 1)$. Quantum correlations that are established in the atom-field system lead to highly nontrivial photon statistics of the field mode. Specifically, from above it follows that for $p \gg N$ the mean excitation of the atomic sample is equal to $N/2$. Correspondingly, the mean-photon number of the field mode is equal to $\bar{n} = p - N/2$ with the dispersion of the photon number distribution equal to $\Delta = \sqrt{N}/2$. This means that in the strong-coupling limit entanglement between atoms and the field leads to highly sub-Poissonian photon statistics of the field mode in the ground state of the DM. This is reflected by the Mandel parameter of the field mode, defined as $Q = (\Delta - \bar{n})/\bar{n}$, that is close to -1 for $p \gg N$.

In summary, using tools of quantum information theory we have shown that the ground state of the Dicke model exhibits an infinite sequence of instabilities (quantum-phase-like transitions). These transitions are characterized by abrupt changes of the bi-partite entanglement between atoms at critical values κ_j of the atom-field coupling parameter κ.

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