# Quantum information approach to the Ising model: Entanglement in chains of qubits

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Simple physical interactions between spin-1/2 particles may result in quantum states that exhibit exotic correlations that are difficult to find if one simply explores state spaces of multi-partite systems. In particular, we present a detailed investigation of the well known Ising model of a chain (ring) of spin-1/2 particles (qubits) in a transverse magnetic field. We present explicit expressions for eigenstates of the model Hamiltonian for arbitrary number of spin-1/2 particles in the chain in the standard (computer) basis and we investigate quantum entanglement between individual qubits. We analyse bi-partite as well as multi-partite entanglement in the ground state of the model. In particular, we show that bi-partite entanglement between pairs of qubits of the Ising chain (measured in term of a concurrence) as a function of the parameter $\lambda$ has a maximum around the point $\lambda = 1$ and it monotonically decreases for large values of $\lambda$. We prove that in the limit $\lambda \to \infty$ this state is locally unitary equivalent to an $N$-partite Greenberger-Horn-Zeilinger state. We also analyse a very specific eigenstate of the Ising Hamiltonian with a zero eigenenergy (we denote this eigenstate as the $X$-state). This $X$-state exhibits the "eXtreme" entanglement in a sense that an arbitrary subset $A$ of $k \leq n$ qubits in the Ising chain composed of $N = 2n + 1$ qubits is maximally entangled with the remaining qubits (set $B$) in the chain. In addition we prove that by performing local operation just on the subset $B$ one can transform the $X$-state into a direct product of $k$ singlets shared by the parties $A$ and $B$. This property of the $X$-state can be utilised for new secure multi-partite communication protocols.

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## I. INTRODUCTION

Those multi-partite quantum systems which are fundamental objects of statistical and solid state physics, have been found interesting also from a perspective of quantum information processing. These systems often exhibit multi-partite entanglement that can be used either for quantum information processing or quantum communication. Amongst such systems a distinguished role is played by exactly solvable models, such as the Ising model describing a chain of interacting spin-1/2 particles in an external magnetic field. Eigenstates of the corresponding model Hamiltonian can be studied from a perspective of quantum information theory with a good physical motivation: Any quantum computer is a physical device composed of elementary units, qubits, described by a certain Hamiltonian. Consequently, perfect knowledge of the Hamiltonians and their eigenvectors are vital. An important condition the physical system has to fulfil is the possibility of preparation of an *a priori* known initial state. The easiest way to realize this task is to simply let the system evolve into its ground state. Thus the knowledge of the entanglement properties of the ground state or more practically thermal states are necessary. This has been followed by many authors. In particular, various versions of the Heisenberg model (XX, XY, XYZ) have been investigated. Many of these studies concern numerical and analytical investigations primarily focused on the behaviour of *bipartite* entanglement of small number of qubits in ground and thermal states, e.g. Refs. [1, 2, 3, 4, 5, 6, 7, 7, 8, 9, 10, 11, 12]. The notion of "thermal entanglement", i.e. the entanglement of thermal states is introduced, and its properties including threshold temperatures and magnetic field dependence are studied.

Symmetry properties of multi-partite systems have been used to calculate entanglement among their constituents. In Ref. [13] thermal equilibrium states of isotropic two-spin systems are analysed exploiting SU(2) invariance. The results are related to isotropic Heisenberg models. In Ref. [14], analytical expressions for certain entanglement measures are derived using general symmetries of the quantum spin system. Then they are used for the XXZ model in order to calculate concurrence and the critical temperature for disentanglement for finite systems with up to six qubits. It should be noted, that they use the 3-tangle to analyse some multi-partite entanglement aspects of the system, and discuss entanglement sharing in detail.

In Ref. [15] the authors pointed out, that in a finite chain of qubits, the time evolution generated by the Ising Hamiltonian produces "entanglement oscillations", which lead to the presence of GHZ and W type entangled states. A generalisation to 2D and 3D models is also outlined. Discussions of multi-partite entanglement also appear in Refs. [2, 3, 9]. In Ref [16] quantum teleportation is utilised as a tool to reveal the importance of multi-qubit entanglement in a 3 qubit Heisenberg-XX chain.

A central question in the problem of entanglement of more than two systems is that of bounds on entanglement. Three or more quantum systems cannot be arbitrarily entangled in the similar way as they cannot be arbitrarily classically correlated [17]. The state with an *a priori* specified entanglement properties may not exits

at all [18, 19] and therefore the search for a state with given, in a sense optimal entanglement properties, is in general a hard problem. In Refs. [20, 21] the authors have solved such particular problems by minimising the energy of a Hamiltonian. That is, the sought state with a given pairwise entanglement is the ground state of a Hamiltonian with a very clear physical interpretation. Since such states may be useful for quantum information processing it is desirable to known the concrete form of the states that are either optimal, or obey certain bounds. It should be pointed out that the problem of finding entangled webs with given properties has been extensively addressed in Refs. [18, 22] but without any reference to systems described with Hamiltonians. Thus it is interesting to see how this issue can be approached in other exactly solvable models.

Another interesting issue concerning such models of many-body systems is the collective behaviour of these systems under certain conditions known as the critical phenomena. Let us point out that these phenomena has already been studied extensively. On the other hand it has been pointed out only recently [23, 24, 25, 26, 27] that entanglement is the quantity that *may* play a crucial role in the description and understanding of critical phenomena. The central concept of the theory of critical phenomena is the universality - the critical exponents characterising divergences near critical points are the same for all systems belonging to the same universality class. For a special class of one-dimensional magnetic systems it has been shown in Ref. [23] that the bipartite entanglement shows scaling behaviour near the transition point. One should also expect, that precursors of the critical behaviour may appear even in non-critical systems.

In this paper we present a detailed investigation of the well known Ising model of a chain (ring) of spin-1/2 particles (qubits) in a transverse magnetic field (Section II). We present explicit expressions for eigenstates of the model Hamiltonian for arbitrary number of spin-1/2 particles in the chain in the standard (computer) basis and we investigate quantum entanglement between individual qubits (Sections III and IV). We analyse bi-partite as well as multi-partite entanglement in the ground state of the model. In particular, we show that bi-partite entanglement between pairs of qubits of the Ising chain (measured in term of a concurrence defined in Section II) as a function of the parameter $\lambda$ has a maximum around the point $\lambda = 1$. In addition, it monotonically decreases for large values of $\lambda$. We prove that in the limit $\lambda \to \infty$ this state is locally unitary equivalent to an $N$-partite Greenberger-Horn-Zeilinger state (Section IV). We also analyse a very specific eigenstate of the Ising Hamiltonian with a zero eigenenergy (we denote this eigenstate as the $X$-state). This $X$-state exhibits eXtreme entanglement in a sense that a arbitrary subset $A$ of $k \leq n$ qubits in the Ising chain composed of $N = 2n + 1$ qubits is maximally entangled with the remaining qubits (set $B$) in the chain. In addition we prove that by performing local operation just on the subset $B$ one can transform the $X$ state into a direct product of $k$ singlets shared by the parties $A$ and $B$. This property of the $X$ state can be utilised for new secure multi-partite communication protocols. Technical details of some of our calculations are presented in appendices.

## II. SETTING-UP THE SCENE

### A. The Ising model

We consider a model of a linear chain of spin one-half particles forming a circle, placed in a magnetic field where only the z-component of the field is non-zero. Since we are interested in the spin degrees of freedom only, the Hamiltonian of the system is given by

$$H_N = -C_I \sum_{n=1}^{N} \sigma_n^x \otimes \sigma_{n+1}^x + B \sum_{n=1}^{N} \sigma_n^z \,, \qquad (2.1)$$

where $\sigma_n^\alpha, \alpha = x, y, z$ are well known Pauli operators. The first term in the Hamiltonian is the interaction term with coupling constant $C_I$ and the second term corresponds to a free Hamiltonian. The lower index $n$ labels the position of a spin in the chain and $N$ is the overall number of particles. The cyclic boundary conditions

$$\sigma_{N+1}^\alpha = \sigma_1^\alpha \,; \qquad \alpha = x, y, z \qquad (2.2)$$

ensure that the chain forms a circle. The form of the interaction is chosen such that each particle interacts only with its two nearest neighbours.

The Hamiltonian in Eq. (2.1) can be rewritten into a form which is more convenient for numerical calculations:

$$H_N = E \left\{ -\lambda \sum_{n=1}^{N} \sigma_n^x \otimes \sigma_{n+1}^x + \sum_{n=1}^{N} \sigma_n^z \right\} \,, \qquad (2.3)$$

where $E = B$ and $\lambda = C_I/B$ is a dimensionless parameter. Now, instead of two parameters in energy units ($B$ and $C_I$) we have one dimensionless parameter $\lambda$ and one parameter in energy units, $E$, that can be neglected in our further calculations except for the investigation of entanglement in Gibbs states (see Section III C).

### B. Measures of entanglement

In this paper we will use three different measures – the concurrence, the tangle and a measure of an intrinsic three-partite entanglement.

The concurrence [28] is a measure of the bipartite entanglement between two qubits. Let $\rho_{AB}$ be the joint density matrix of the system consisting of qubits $A$ and $B$. The matrix $\rho_{AB}\tilde{\rho}_{AB}$ has four non-negative eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ that are written in a descending order (i.e. $\{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4\}$. The matrix

$\tilde{\rho}_{AB}$ is a spin-flipped version of density matrix $\rho_{AB}$, i.e. $\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho^*_{AB} \sigma_y \otimes \sigma_y$. The concurrence is given by the relation

$$C_{AB} = \max\left\{\left[\sqrt{\lambda_1} - \sum_{i=2}^{4}\sqrt{\lambda_i}\right]; 0\right\} . \qquad (2.4)$$

Let us point out that the state $\rho_{AB}$ is arbitrary, that is, the concurrence is a valid measure of entanglement for two-qubit mixed states as well. The minimal value of the concurrence is zero (in this case two qubit states are prepared in a separable state), while for any $C_{AB} > 0$ two qubits are entangled. The maximal value of $C_{AB} = 1$ corresponds to maximally entangled states that are locally unitary equivalent to Bell states. It has been shown earlier that the concurrence is directly related to the entanglement of formation [28].

On the other hand the tangle has been originally defined for pure states only. (There is an extension to mixed states but the extension is not computationally feasible except for the case of two qubits when the tangle is equal to the square of the concurrence). Consider a multipartite system where one of the subsystems, denoted $A$, is a qubit. The tangle $C_{A\bar{A}}$ between a subsystem $A$ and the rest of the system, denoted as $\bar{A}$ reads

$$C_{A\bar{A}} = 4\,\mathrm{Det}\rho_A = 2(1 - \mathrm{Tr}\rho_A^2) , \qquad (2.5)$$

where $\rho_A$ is the reduced density operator describing a state of the subsystem $A$.

Finally, the intrinsic three-qubit entanglement is defined for pure states only. Consider a system composed of three qubits $A$, $B$ and $C$ and let the system of three qubits be in a pure state. With the help of the tangle and concurrence introduced above we can define pure three-partite entanglement

$$C_{ABC} \equiv C_{A\bar{A}} - C^2_{AB} - C^2_{AC} . \qquad (2.6)$$

In Ref. [17] it has been proven that the definition (2.6) of intrinsic three-qubit entanglement is independent of permutations of particles and shares all properties that a proper measure of entanglement has to fulfil.

Finally, let us mention that measures of entanglement are not unique and different measures might result in different ordering of states. For the case of bipartite systems $AB$ prepared in a pure state the measure of entanglement is in fact any suitable function of the eigenvalues of the reduced density matrix of either of the two subsystems $\rho_A$ or $\rho_B$ Ref. [29]. For example the well known Von Neumann entropy

$$S(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A , \qquad (2.7)$$

defines a bipartite measure of entanglement usable for arbitrarily dimensional systems.

## III. THREE SPIN-1/2 PARTICLES

In order to understand entanglement properties of the Ising chain under consideration it makes sense to start with a relatively simple example of three spin-1/2 particles (qubits). Even this simple system exhibits interesting properties and their understanding will guide us in general case of an arbitrary number of qubits.

The Ising Hamiltonian with three spin-1/2 particles can be directly diagonalized and energy levels easily calculated. In what follows, we will call spin-1/2 particles as qubits since the Hilbert space $\mathcal{H}$ of a spin one-half particle is two-dimensional. Let us note that the simplest example is the case of two qubits, that is $N = 2$. However, this trivial example has already been investigated in Refs. [1] and [5]. The case of three qubits is also interesting on account of the fact that, besides intrinsic bipartite entanglement, three qubits can also share three-partite entanglement. In the case of three qubits being in a pure state this intrinsic three-qubit entanglement can be easily calculated with the help of Eq. (2.6). Finally, even such a simple example nicely illuminates main results concerning multi-partite entanglement where most results can be generalised to the case with an arbitrary number $N$ of qubits (spin-1/2 particles) in the chain.

The Hamiltonian Eq. (2.3) of the Ising model with only three qubits in the chain reads

$$H_3 = -\lambda\sigma_1^x \otimes \sigma_2^x - \lambda\sigma_2^x \otimes \sigma_3^x - \lambda\sigma_1^x \otimes \sigma_3^x + \sigma_1^z + \sigma_2^z + \sigma_3^z . \qquad (3.1)$$

Note that the Hamiltonian $H_3$ is permutationally invariant, unlike the Hamiltonians $H_N$ for $N > 3$ qubits. All Hamiltonians $H_N$ are obviously translationally invariant. What is not so obvious is the fact that the Hamiltonian is invariant under the inversion of the order of particles. The particles in the chain are labelled with $n = 1, \ldots N$. Now, if we relabel them as $n \to N - n + 1$ the Hamiltonian remains unchanged and thus is invariant under the inversion of the order. For the case of $N = 3$ the two transformations, translation and inversion of the order, together with an arbitrary combination of the two yield in fact all possible permutations of the particles in the chain. Thus it follows that the Hamiltonian $H_3$ is permutationally invariant.

The knowledge of the symmetry of the Hamiltonian is utmost important, as it plays a crucial role in the process of finding its eigenvalues and eigenvectors. It is well known that for every symmetry $S$ there exists a unitary or an anti-unitary operator $T_S$ such that the corresponding Hamiltonian commutes with $T_S$

$$[T_S, H] = 0 . \qquad (3.2)$$

As a result of this commutation relation the two operators $H$ and $T_S$ have common set of eigenvectors. It means, that there is one set of vectors (basis of the corresponding Hilbert space) which are eigenvectors of $H$ as well as the operator $T_S$. Moreover, any non-degenerate eigenstate of $H$ has to be invariant under the action of the operator $T_S$. On the other hand, any eigenstate which is not invariant under the action of the operator $T_S$ is degenerate. In what follows the knowledge of symmetries

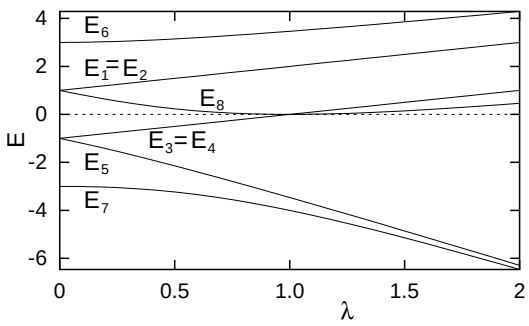


FIG. 1: The spectrum of the Hamiltonian $H_3$. We present the dependence of eigenenergies as a function of the coupling parameter $\lambda$. The energy levels $E_1 = E_2$ and $E_3 = E_4$ are degenerate. The other four energy levels are non-degenerate. The ground state corresponding to the state with the lowest energy in our notation is represented by the seventh level $E_7$ for all values of the parameter $\lambda$.

of $H_N$ will help us to find some particularly interesting states of the spin-chain under consideration.

### A. Spectrum of the Hamiltonian

The spectrum of the Hamiltonian $H_3$ can be easily calculated directly by diagonalising the Hamiltonian $H_3$. The Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ of three qubits is eight dimensional and the Hamiltonian $H_3$ has eight eigenvalues (see Appendix B), shown in Fig. 1 as functions [33] of the coupling constant $\lambda$. Two of them are double degenerated, while the remaining four are not, apart from several isolated values of the parameter $\lambda$. The ground state of the system for any finite value of $\lambda$ is non-degenerate, and in our notation it is the seventh state $|e_7\rangle$. When the parameter $\lambda$ is infinite, which corresponds to the zero value of the external magnetic field $B$, the Hamiltonian $H_3$ has only one free parameter, and can be expressed as

$$H_3(\lambda = \infty) = -C_I \left(\sigma_1^x \otimes \sigma_2^x + \sigma_2^x \otimes \sigma_3^x + \sigma_1^x \otimes \sigma_3^x\right).$$

The two lowest states (denoted as $|g_1\rangle$ and $|g_2\rangle$) in the energy spectrum become degenerate in this case. These states read

$$|g_1\rangle = \frac{1}{2}(|000\rangle + |110\rangle + |011\rangle + |101\rangle) ,$$
$$|g_2\rangle = \frac{1}{2}(|111\rangle + |100\rangle + |010\rangle + |001\rangle) .$$

Here the first one is the limit of the state $|e_7\rangle$ when $\lambda$ tends to infinity and the second one is the limit of the state $|e_5\rangle$. We know that any linear combination of the two vectors is an eigenvector with the same energy and, consequently, can be identified as a ground state. However, there is one exceptional linear combination. If we demand the ground state of the system with the parameter $\lambda = \infty$ to be the limit of the ground state when $\lambda = \infty$, then the appropriate choice for the ground state is $|g_1\rangle$.

As we will see in Section IV A, the point $\lambda = 1$ turns out to be rather interesting. There is a particular eigenstate of the Hamiltonian which has quite interesting behaviour of entanglement around $\lambda = 1$. However, for $N$ large it is rather difficult to identify this specific state among $2^N$ eigenstates of the Hamiltonian $H_N$. Having calculated the spectrum, the state can be easily identified with the help of the level crossing at the point $\lambda = 1$. The special state with the remarkable properties is in general non-degenerate but at the point $\lambda = 1$ becomes degenerate

$$E_8(\lambda = 1) = E_{3,4}(\lambda = 1) = 0 ,$$

and crosses the degenerate levels $E_3$ and $E_4$. What is important is the fact that this type of level crossing is independent of $N$ (we might say universal), there is the same type of level crossing for $N$ being an arbitrary odd number.

### B. Entanglement properties

Our main goal is to analyse the entanglement properties of the model. Let us begin with the Ising chain of three qubits in the ground state and examine entanglement as a function of the parameter $\lambda$. We will use the three different measures of entanglement: The concurrence, the tangle and a measure of the intrinsic three-partite entanglement, as introduced in Section II B. An important aspect is the comparison of specifically bipartite, and multi-partite entanglement.

The bipartite entanglement between individual qubits, the entanglement between a qubit and the rest of a system, and an intrinsic three-partite entanglement for the ground state are shown in Fig. 2. They are quantified by concurrence, tangle, and the intrinsic three-partite entanglement of Eq. (2.6), respectively. Due to the fact that any non-degenerate state shares all symmetries of the corresponding Hamiltonian, the entanglement of the ground state between an arbitrary pair of qubits has the same dependence on the parameter $\lambda$ and it holds that

$$C_{12}(\lambda) = C_{13}(\lambda) = C_{23}(\lambda) , \tag{3.3}$$

Moreover, the same holds for bipartite entanglement between a given qubit and the rest of the system so that

$$C_{1\bar{1}}(\lambda) = C_{2\bar{2}}(\lambda) = C_{3\bar{3}}(\lambda) , \tag{3.4}$$

where $\bar{X}$ denotes a system of two qubits with the qubit on the $X$-th position omitted and $C_{X\bar{X}}$ is the entanglement shared between the qubit on the $X$-th position and the rest of the Ising chain (remaining two qubits).

The solid line in the picture Fig. 2 shows bipartite entanglement between an arbitrary pair of qubits. For $\lambda = 0$ the concurrence, i.e. the pairwise entanglement is

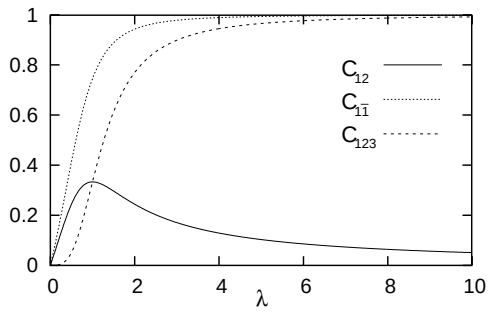


FIG. 2: The entanglement in the ground state of the Ising chain with three qubits $N = 3$. In the figure we present three different types of the entanglement as a function of the parameter $\lambda$: the bipartite entanglement between the first and the second qubit $C_{12}$ (solid line); the bipartite entanglement between the first qubit and the remaining two qubits $C_{1\bar{1}}$ (dotted line) and the intrinsic three-partite entanglement $C_{123}$ (dashed line).

zero. As $\lambda = C_I/B$ it means that the case $\lambda = 0$ corresponds to the absence of the interaction, $C_I = 0$. Consequently, the ground state of the system is such that all spins are aligned along the same direction, the direction of the magnetic field, and are not entangled. When we turn on the interaction, the constant $C_I$ is no longer zero and the spins become entangled. As we increase the value of the interaction constant $C_I$ (or equivalently decrease the value of the magnetic field $B$ so that the ratio $C_I/B$ increases) the qubits become more and more entangled. This holds up to the value of $\lambda = 1$ where the two qubit entanglement reaches its maximum. Further increase of the parameter $C_I$ (or decrease of $B$) causes degradation of the entanglement and in the limit $\lambda$ goes to infinity the entanglement is zero. As we have chosen the ground state for $\lambda = \infty$ to be the limit of the ground state when $\lambda$ tends to infinity, the concurrence at the point $\lambda = \infty$ is zero. It means that when the magnetic field is zero, the ground state of the system is such that all pair concurrencies are zero and there is no entanglement in any pair of qubits.

The entanglement of a given qubit $X$ and the rest of the system $\bar{X}$ expressed in terms of the tangle is, on contrary, a non-decreasing function of $\lambda$. At the point $\lambda = 0$ the entanglement is zero on account of the same reason as the entanglement between an arbitrary pair of qubits. When the parameter $\lambda$ is non-zero (i.e. the interaction constant $C_I$ is non-zero) the qubits are entangled. That is, any chosen individual qubit is entangled with the rest of the system. The stronger the interaction (the larger the value of the parameter $\lambda$), the stronger the qubits are entangled with the system. In the limit $\lambda \to \infty$ (infinitely strong interaction) the qubits become maximally entangled and the tangle, measuring the amount of entanglement between a given qubit and remaining two qubits, reaches its maximum value. For the case of our specific choice of the ground state for $\lambda = \infty$ the tangle is maximal and equals to unity.

As we have already pointed out the reason why we have described the case of three qubits in such detail is that the entanglement behaves in the same manner for an arbitrarily large $N$. But the case of three qubits is special for a different reason too. In the case of just three qubits being in a pure state we are able to calculate the intrinsic three-partite entanglement using Eq. (2.6). The dashed line in the Fig. 2 shows the dependence of the three-qubit entanglement on the parameter $\lambda$. We can see that the dependence of the intrinsic three-partite entanglement on the parameter $\lambda$ is very similar to the dependence on the same parameter of the entanglement between a given qubit and remaining two qubits (Fig. 2). It seems that for a strong interaction the three-partite entanglement is the largest contribution provided we express the entanglement between a single qubit and remaining two qubits (rest of the system) as a sum of two and three-qubit entanglement (see Eq. (2.6) and comments therein). This result suggests the following physical picture: *"When the system of interacting spin-1/2 particles is in the ground state then the interaction causes entanglement of qubits such that each qubit is entangled with the rest of the system. For the system of N spin-1/2 particles the N-partite entanglement will be dominant when the interaction between the particles is very strong compared to the magnitude of the magnetic field."* This conjecture, proven to be valid in the case of three qubits, will be further examined in following sections where the general case of a chain with an arbitrary number of qubit will be analysed.

### C. Entanglement in Gibbs ensembles

In this subsection we will continue to investigate the 3-qubit Ising model. We will analyse the entanglement properties of thermal states of three qubits interacting according to the Ising Hamiltonian. The ground state of the system is probably the most important state and through the study of those states we acquire a lot of information about the corresponding system itself. Beside being the states with the lowest energy we know that the ground states are associated with zero temperature and that they are related to fundamental properties of Hamiltonians. However, there are other states which are equally relevant for the physical description of the system. The temperature of a system we measure in our laboratories is always non-zero. Keeping in mind the third law of thermodynamics and the impossibility of reaching the absolute zero temperature we can conclude that in practice there is always a non-zero probability for finding the system under study in one of the excited states. Of course, the probability depends on the temperature but as far as the temperature is non-zero, no matter how big the gap in the energy between the ground state and the first excited state is, the probability is non-zero as well. Consequently, it is interesting to study entanglement in systems in thermal equilibrium, i.e. in their

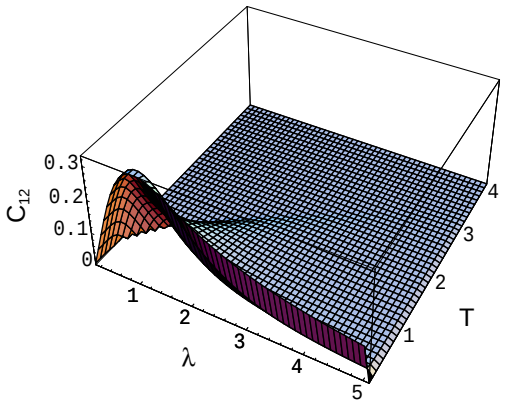


FIG. 3: The bipartite entanglement between the 1st and the 2nd qubit as a function of the temperature $T$ and the parameter $\lambda$. The system of three qubits is in the thermal state Eq. (3.5).

"natural" state, and investigate the dependence of the entanglement on temperature.

The density operator corresponding to a thermal state of a quantum system at the temperature $T$ is usually given by the relation

$$\rho(T) = \sum_i w_i |e_i\rangle\langle e_i| \ , \tag{3.5}$$

where $|e_i\rangle$ is an energy eigenstate (eigenstate of the Hamiltonian $H_3$), $w_i$ are weights or probabilities defined as

$$w_i = K e^{-E_i/T} \ , \tag{3.6}$$

where we assume the Boltzmann constant to be equal to unity and we sum over all energy eigenstates. The constant $K$ in Eq. (3.6) is a normalisation so the sum of probabilities $w_i$ equals to unity

$$\sum_i w_i = 1 \ . \tag{3.7}$$

In Fig. 3 we have plotted the entanglement between the first and second qubit in a three-qubit system at temperature $T$. Repeating the same line of arguments, taking into account the symmetry of the Gibbs state at the temperature $T$, we know that $C_{12} = C_{13} = C_{23}$ and Fig. 3 shows us the dependence of the entanglement on temperature for an arbitrary pair of qubits. For nearly zero values of the temperature the entanglement behaves in a similar way as in the case of the system in the ground state. Increasing the temperature the two qubits become less and less entangled. In the high temperature limit the entanglement is practically equal to zero. It has a very simple explanation. If the temperature is high enough all probabilities $w_i$ are almost equal and the state of the system $\rho$ is proportional to the identity operator [34], i.e. it is the total mixture. Consequently, the state of an arbitrary pair of qubits is proportional to the identity as well and the two qubits are not entangled.

In our case an increase of the temperature always causes degradation of the entanglement. Thus we may conclude that to maximalize the entanglement it is convenient to keep the temperature as low as possible. It follows that under certain conditions, one way of increasing the entanglement can be lowering of the temperature. At the end let us note that there are quantum models where an increase of the temperature can cause an increase of entanglement (see for instance Ref. [1]).

### D. Quantum entanglement at $\lambda = 1$

Performing an analysis of the entanglement for the whole set of eigenstates of the Hamiltonian $H_3$, we have found one particular eigenstate with rather peculiar behaviour of the entanglement. Namely, the entanglement of this state as a function of the parameter $\lambda$ is non-analytic at the point $\lambda = 1$.

Let us remind the reader that at the end of Section III A we have mentioned a level crossing. That is, at the point $\lambda = 1$ there is an energy level crossing and one of the non-degenerate eigenstates becomes degenerate. What is remarkable is the fact that the eigenstate with non-analytic behaviour of entanglement is the same state mentioned in Section III A in connection with the level crossing. However, while we have discovered the state through our analysis of the entanglement of the eigenvectors for three qubits, in the general case of arbitrary odd number of qubits we have followed the reverse path : we have identified the state by exploiting the level crossing at the point $\lambda = 1$ [35].

In what follows we will denote the state exhibiting this very intriguing behaviour as the "*X-state*" (since it exhibits e*X*treme entanglement around $\lambda = 1$ - for details see Sec. IV.B). In our earlier notation, it is the eighth state $|e_8\rangle$. To remind the reader the state has the following form:

$$|X\rangle \equiv |v_8\rangle = K_8 \left[ \frac{1 - E_8 - 2\lambda}{\lambda} |000\rangle + |011\rangle + |101\rangle + |110\rangle \right] \ , \tag{3.8}$$

where $K_8$ is a normalisation constant and $E_8$ is the energy corresponding to the eigenstate $|e_8\rangle = |X\rangle$. The X-state is a non-degenerate eigenstate of the Hamiltonian (except for a finite number of values of $\lambda$) and thus shares all symmetries of the Hamiltonian $H_3$ in the same way as the ground state. That is $C_{12} = C_{13} = C_{23}$ and $C_{1\bar{1}} = C_{2\bar{2}} = C_{3\bar{3}}$. The bipartite entanglement between the first and second qubit $C_{12}$ and between the first qubit and remaining two qubits $C_{1\bar{1}}$ are shown in Fig. 4. We from from the figure that the concurrence between two qubits in the system exhibit non-analytical behavior at $\lambda = 1$.

Certainly, the reason behind this non-analyticity cannot be a phase transition. We know that the Ising model has a quantum phase transition at the point $\lambda = 1$ but for that the chain must be infinite and the temperature

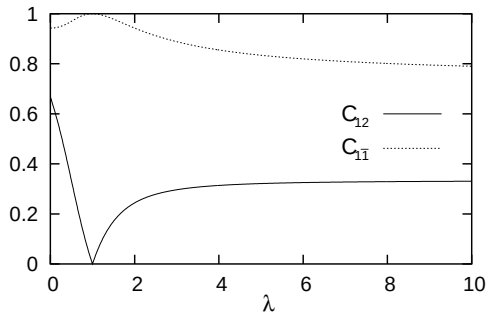


FIG. 4: The entanglement in the X-state of the Ising chain with three qubits $N = 3$. In the figure we show two different types of entanglement as a function of the parameter $\lambda$: the bipartite entanglement between the first and the second qubit $C_{12}$ (solid line) and the bipartite entanglement between the first and remaining two qubits $C_{1\bar{1}}$ (dotted line).

must be zero [30]. That is, we can observe a phase transition only if there is an infinite number of particles in the chain and the system must be in the ground state. From this point of view there cannot be a direct link between the found non-analyticity and the phase transition.

The other question is the relation between the entanglement and a change of symmetries in the system. The change of symmetries of a Hamiltonian can have a significant effect on the correlation properties of the eigenstates. In our case we know that the phase transition is accompanied with a symmetry change at the corresponding point. Similar change of symmetry is observed even in the case of a finite dimensional Ising chain at the point of the level crossing. This suggests that the change of the symmetry at the point $\lambda = 1$ may in general be reflected in the behavior of the entanglement - the quantum part of the correlations.

The X-state is interesting not only on account of the non-analyticity but mainly for the fact that it exhibits remarkable quantum correlations. At the point $\lambda = 1$ the entanglement in an arbitrary pair of qubits is zero but the entanglement between a given qubit and the remaining two qubits is maximal (c.f. Fig. 4). If we calculate the entanglement length introduced in Ref. [1], it is zero. But we know that each qubit is maximally entangled with the rest of the system so there is a sort of a "long range" entanglement [36]. In other words, as each qubit is maximally entangled with the rest of the system and the pairwise entanglement is zero, we have an intrinsic multi-partite entanglement. Moreover, since the system consists of only three qubits the only possible multi-partite entanglement is a three-partite entanglement $C_{123}$ and at the point $\lambda = 1$ the three-partite entanglement reaches its maximal possible value.

Tu sum up we can conclude that in the case of the finite dimensional Ising model, there is an energy eigenstate, the X-state, for which entanglement exhibits a rather special behaviour at the point, where the infinite Ising chain has a phase transition.

## IV. GENERAL CASE OF $N$ SPINS

So far we have considered the particular case of three qubits. Despite its simplicity the case of three qubits shares many features of the general case of a chain with an arbitrary number of qubits. This has helped us to formulate basic theorems and to identify states which are particularly interesting with respect to the entanglement.

Let us consider a chain with $N$ qubits where $N$ is arbitrarily large. The Hilbert space of $N$ qubits is $2^N$-dimensional and the corresponding Hamiltonian $H_N$ has $2^N$ eigenvectors and eigenvalues. Despite the very possibility to calculate any eigenvector or eigenvalue (recall that the model under study is exactly solvable) it is not feasible to perform the calculation for all eigenvectors (eigenvalues) and to analyse them afterwards. Therefore, we have used the results of the previous section and beside the ground state as an important state, we have analysed the X-state. Of course, prior to that we have to find or identify the X-state among $2^N$ eigenstates of the Hamiltonian $H_N$. At this point we can take the advantage of our detailed knowledge on the spectrum we have at hand, as the level crossing at the point $\lambda = 1$ studied in the previous section is crucial in identifying the sought state.

Our main goal is to analyse the entanglement properties of the states under consideration. In addition to the dependence on the number of qubits $N$ in the chain we will also consider the dependence on the "distance" of qubits. For $N > 3$ there are more possibilities how to create pairs of qubits and beside the nearest neighbours, a pair can be created from the next nearest neighbours and etc. Since the Ising model is not permutationally invariant unlike in the special case studied so far, we can expect that the entanglement will vary with the distance between qubits.

### A. The ground state

The ground state of the system for different values of $N$ can be calculated using the formalism developed in Appendix A and Appendix C. Due to the complicated form of the state itself, we do not quote the explicit expression, it may be found in the above mentioned Appendices. In the following, we describe the entanglement properties of the state instead.

The entanglement shared between pairs of nearest neighbour qubits in terms of the concurrence is plotted in Fig. 5 for different values of $N$. The shape of different curves, corresponding to different number of qubits in the chain, is very similar. At the point $\lambda = 0$ the values of all curves are zero, increasing the parameter $\lambda$ the entanglement (measured in the concurrence) increases and around the point $\lambda \approx 1$ reaches a maximum. This maximum depends on the number of qubits $N$, but with increasing $N$ the concurrence converges to a specific value that even for $N > 5$ is almost constant. Increasing the parame-

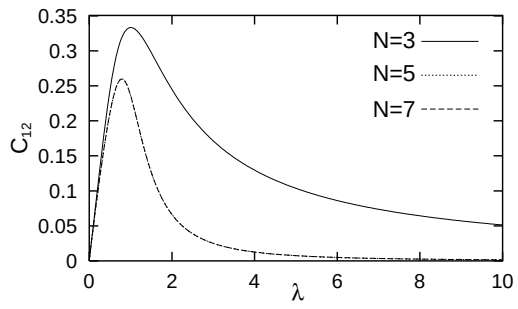


FIG. 5: The entanglement between the nearest neighbours as a function of the parameter $\lambda$ and the number of qubits in the chain $N$. The system is in the ground state and the number of qubits in the chain is $N = 3, 5, 7$.

ter $\lambda$ further, the entanglement decreases and finally, in the limit $\lambda \to \infty$ the entanglement tends to zero. For $\lambda \to \infty$, the ground state is degenerate. Similarly, as in the case $N = 3$, we may choose a particular ground state the $\lambda \to \infty$ limit of which becomes the ground state of the Hamiltonian for $\lambda \to \infty$ (c.f Appendix C)

$$|\psi_N\rangle_{\lambda=\infty} = K_N \sum_{\{i,j,\ldots\}e} |\{i,j,\ldots\}\ldots\rangle \,, \tag{4.1}$$

where $K_N$ is a normalisation constant, $\{i, j, \ldots\}$ denote positions of the qubits that are up and $\{i, j, \ldots\}e$ means summation over all states of the standard basis with an even number of qubits up. (We use the term "up" for a qubit if it is in the state $|1\rangle$ and down if it is in the state $|0\rangle$). From the construction of the state $|\psi_N\rangle_{\lambda=\infty}$, it follows that the entanglement between arbitrary two qubits is zero while the entanglement between a given qubit and all remaining qubits is maximal.

*Proof:* The state Eq. (4.1) can be rewritten into a simpler form using the following recurrence relation:

$$|\psi_N\rangle_{\lambda=\infty} \;=\; \frac{1}{\sqrt{2}}\left[|0\rangle|\psi_{N-1}\rangle_{\lambda=\infty} + |1\rangle|\tilde{\psi}_{N-1}\rangle_{\lambda=\infty}\right] \tag{4.2}$$

where $|\tilde{\psi}_{N-1}\rangle_{\lambda=\infty}$ has the same form as $|\psi_{N-1}\rangle_{\lambda=\infty}$ , but instead of summing over all states with an even number of qubits up we sum over all states with an odd number of qubits up. With the help of the Eq. (4.2) it is easy to prove the above statements concerning entanglement. Let $i$ and $j$ denote two arbitrary but mutually different $(i \neq j)$ positions of *a priori* chosen qubits in the chain. Using the relation Eq. (4.2) we rewrite the state vector Eq. (4.1) as follows:

$$\begin{aligned}|\psi_N\rangle_{\lambda=\infty} = \frac{1}{2}\,[\,(\,|00\rangle_{ij} + |11\rangle_{ij}\,)|\psi_{N-2}\rangle_{\lambda=\infty} \\ +\;(\,|01\rangle_{ij} + |10\rangle_{ij}\,)|\tilde{\psi}_{N-2}\rangle_{\lambda=\infty}\Big]\;.\end{aligned} \tag{4.3}$$

The reduced density operator $\rho_{ij}$, of the two qubits at the positions $i$ and $j$ is calculated as a trace over all remaining qubits of the density operator $\rho = |\psi_N\rangle_{\lambda=\infty}\langle\psi_N|$ of the whole system. The state $\rho_{ij}$ expressed as a matrix in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ reads

$$\rho_{ij} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \;. \tag{4.4}$$

The spin-flipped density matrix in this case reads $\tilde{\rho}_{ij} = \rho$ so that $\tilde{\rho}_{ij}\rho_{ij} = \frac{1}{2}\rho_{ij}$. The matrix $\tilde{\rho}_{ij}\rho_{ij}$ has two eigenvalues that are equal and according to Eq. 2.4 the entanglement shared by the two qubits in the $i$-th and $j$-th positions is zero.

The density operator of a single qubit at the $i$-th position $\rho_i$ can be calculated similarly,

$$\rho_i = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \;. \tag{4.5}$$

and it corresponds to a maximally mixed state. Since the whole system is in a pure state, the qubit in the $i$-th position is maximally entangled with the rest of the system.

Let us note, that the state given by Eq. (4.1) is a GHZ state in the basis built by direct products of eigenvectors of $\sigma_x$ of each qubit. Thus the state can be transformed via *local unitary transformations* into the standard form of the GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots\rangle) \;. \tag{4.6}$$

This observation provides us with a very clear explanation of the above mentioned entanglement properties. Moreover, it thoroughly confirms the proposal conjectured at the end of Subsection III B. For $\lambda$ being large but not infinite or, equivalently, for a large coupling constant $C_I$ compared to the absolute value of the magnetic field $B$, the ground state of the system exhibits properties close to the GHZ state. Taking the parameter $\lambda$ larger and larger the state is closer and closer to the GHZ state and for a sufficiently large $\lambda$ we can consider the ground state of the system to be the GHZ state even for $\lambda$ being large but finite. When the state of the system is the GHZ state, the entanglement between any pair of qubits is zero because the reduced density operator describes a separable state [c.f. Eq. (4.4)]. Further, a reduced density operator of a subsystem consisting of $n < N$ qubits is also separable, as one would expect for a GHZ state. It follows that if we consider an arbitrary subsystem there is no entanglement: choosing any set of $n < N$ qubits, the state of the chosen set is separable. Consequently, the state under consideration exhibits only intrinsic $N$-partite entanglement. Recalling the conjecture from Subsection III B we can now confirm the result to be valid for a general case of the $N$-partite chain as well.

Finally, let us note that the ground state (or equivalently the GHZ state) has a long range entanglement.

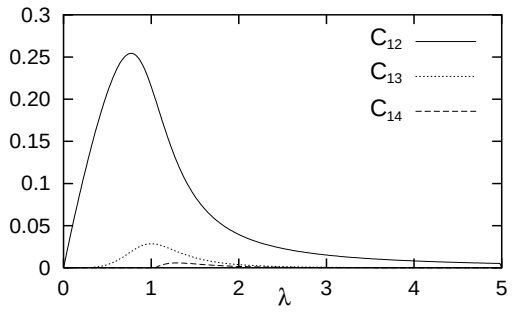


FIG. 6: The bipartite entanglement shared between qubits, expressed in terms of the concurrence $C_{ij}$ where $i$ and $j$ label positions of two qubits, as a function of the parameter $\lambda$ and the distance between the qubits. The system (chain) of seven qubits is in the ground state.

The $N$-partite entanglement is certainly of a long range since it concerns all qubits in the chain, though, for instance, the entanglement length defined in Ref. [1] is zero.

We have also studied how the entanglement depends on the distance between an arbitrary pair of qubits qubits, see Fig. 6. The farther the two qubits are they are less entangled. It means that with increasing the distance (specified by positions of qubits, i.e. the distance between qubits $j_1$ and $j_2$ is represented by the difference $|j_1 - j_2|$) between qubits the entanglement converges to zero. Besides, the peak, or the point where the entanglement is maximal, is shifted to larger $\lambda$'s (see Fig. 6).

### B. General X-state

In the case of the chain composed of three qubits we have found a particular eigenstate of the Hamiltonian $H_3$, the X-state, with a non-analytic behaviour of entanglement. The question naturally arises, whether there exists such a state in the case of more than three qubits. As the Hamiltonian $H_N$ has $2^N$ eigenvectors and eigenvalues it is impossible to analyse the whole spectrum. However, we know that in the three qubit case the X-state is interesting not only on the account of entanglement but also because of energy-level crossing. It is a non-degenerate eigenvector of $H_3$ apart from a single point $\lambda = 1$ where there is a level crossing. Our knowledge of the level crossing can be successfully exploited in identifying the X-state for an arbitrary $N$. We have found that for $N$ odd there is a level crossing at the point $\lambda = 1$ [37] and one of non-degenerate eigenvectors becomes degenerate. Let us note that in the case $N = 3$ the X-state corresponds to the non-degenerate eigenvector while the other energy level is degenerate and corresponds to two vectors. The situation we have now is similar. There is a level crossing of two energy levels at the point $\lambda = 1$. One of them is a non-degenerate energy level; in what follow we will call the state corresponding to that level as the X-state, and the other energy level is degenerate and there are $2^n$ (where $N = 2n + 1$) eigenvectors corresponding to that level (for proofs and more details see Appendix D).

Having successfully identified the X-state, we can analyse its entanglement properties. In contrast with the case of $N = 3$ qubits, for $N > 3$ bipartite entanglement (the concurrence) as a function of the parameter $\lambda$ is analytic. In order to see this we need to know the form of the X-state. It is derived in Appendix A, we quote only the result for $\lambda = 1$ here (see Appendix D and also Ref.[31])

$$|X\rangle_{\lambda=1} = \sum_{\{i,j,\ldots\}_e} |\{i,j,k,\ldots\}\rangle (-1)^{\sum_{i>j} d(i,j)}, \quad (4.7)$$

where $\{i,j,\ldots\}_e$ denote a sum over all sets of indices with an even number of indices in each set, the letters $i,j,\ldots$ in a single set denote positions of qubits in the chain that are up and $|\{i,j,\ldots\}\rangle$ is the corresponding state vector, $d(i,j)$ is a distance between qubits on the $i$-th and $j$-th positions defined below, while the sum over $i < j$ means that we sum over all pairs of qubits counting only once the cases with switched positions of the qubits. The distance $d(i,j)$ of the two qubits is defined as the shortest path on the ring that brings us from the qubit on the $i$-th positions to the qubit on the $j$-th position. In Appendix D we present a complete proof that the state Eq. (4.7) is an eigenstate of the Hamiltonian $H_N$ for $\lambda = 1$. The Appendix D also contains several additional proofs and more details on the $X$-state.

The reduced density operator of two qubits on the $i$-th and $j$-th positions has been calculated from the state Eq. (4.7) by tracing over degrees of freedom of the remaining qubits. The density operator expressed in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ reads

$$\rho_{ij} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}. \quad (4.8)$$

In the same way the density operator of a single qubit in the $i$-th position [38] expressed in the one-qubit basis $\{|0\rangle, |1\rangle\}$ is

$$\rho_i = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (4.9)$$

With the help of the density matrix Eq. (4.8) we have calculated bipartite entanglement between qubits on the $i$-th and $j$-th positions while the density matrix Eq. (4.9) has been used for the calculation of entanglement between a qubit on the $i$-th position and the rest of the system (all remaining qubits). Since the density matrix Eq. (4.8) is a complete mixture, there is no bipartite entanglement between any two qubits. On the other hand as Eq. (4.9) describes a complete mixture and the whole system is in a pure state, entanglement between a given qubit and remaining qubits is obviously maximal: the tangle is equal to one. As the eigenvalues

of the density matrices $\rho_{ij}$ and $\rho_i$ are continuous functions of the parameter $\lambda$ it is easy to check that both the entanglement shared between qubits and the entanglement between a given qubit and the rest of the system are continuous functions of the parameter $\lambda$ around the point $\lambda = 1$. Consequently, the point $\lambda = 1$ is not a point of non-analytical behaviour of entanglement anymore. Moreover, the state Eq. (4.7) is not equivalent to a GHZ state. In other words there does not exist a local unitary transformation which would transform the state Eq. (4.7) into the GHZ state in Eq. (4.6). However, we have found that the X-state Eq. (4.7) has the following remarkable property:

**Theorem 1** *Let $N = 2n+1$ denote the total number of qubits forming the Ising chain where $n$ is an integer and let the system of $N$ qubits be in the X-state Eq. (4.7).*

*The density matrix of any sequence of $n$ neighboring qubits is*

$$\rho = \frac{1}{2^n}\,\mathbb{1}_n\ , \tag{4.10}$$

*where $\mathbb{1}_n$ is the identity operator acting in the $2^n$-dimensional Hilbert space of $n$ qubits.*

An important consequence of the theorem is the fact that if the system is in the X-state then any set of neighboring qubits is perfectly (maximally) entangled with the rest of the system. Consider an arbitrary set of $n$ neighboring qubits. The reduced density operator of the $n$ qubits is Eq. (4.10). Using the entropy Eq. (2.7) as a measure of bipartite entanglement for pure states

$$S = -\sum_{2^n} \frac{1}{2^n} \log \frac{1}{2^n} = n\ \log 2\ , \tag{4.11}$$

we can see that the set of $n$ qubits is maximally entangled with the remaining qubits. That is, if a system of $N = 2n + 1$ qubits is in the X-state and we choose $n$ neighboring qubits then according to Eq. (4.11) we know that the $n$ qubits are perfectly entangled with the remaining $n + 1$ qubits. Moreover, if we choose a subset of say $k$ qubits from the set of $n$ neighboring qubits then the state of the $k$ qubits is

$$\rho = \frac{1}{2^k}\,\mathbb{1}_k\ , \tag{4.12}$$

so the $k$ qubits are perfectly entangled with the rest of the system (all remaining $N - k$ qubits). To sum up the X-state is a highly entangled state and consequently it is a good candidate for a quantum communication between many parties. A rather simple protocol that can serve as an example of its applications described in the next section.

### C. Controlling distribution of entanglement in the X-state

We will present a simple example how the X-state can be exploited for a communication or a secret key distribution in a situation when bipartite entanglement between many parties is needed. The X-state with its remarkable properties can be considered to be a very good resource of communication as any set of $n$ neighboring qubits is maximally entangled with the rest of the system.

First, imagine that $n$ neighboring qubits belong to Alice and the rest ($n+1$ qubits) belongs to Bob. Moreover, let us assume that Alice and Bob want to exploit the entanglement of the X-state for their protocol. But unfortunately, their protocol is designed for qubits, that is to say it uses pairs of maximally entangled qubits. We have shown that the density operator of any pair of qubits is proportional to the identity and thus the two qubits cannot be entangled. It means that Alice and Bob cannot take any two qubits and use them for their protocol. But we know that the $n$ neighboring qubits which belongs to Alice are maximally entangled with Bob's qubits as the entropy in Eq. (4.11) equals to $n\ \log 2$. Such amount of entanglement corresponds to $n$ pairs of maximally entangled qubits. Therefore one may ask whether they are able to create $n$ pairs of maximally entangled qubits only by performing *local* (though multi-qubit) unitary transformations $\mathbf{U}_A^{(n)}$ and $\mathbf{U}_B^{(n+1)}$ on their respective qubits. The answer is positive.

Consider a state of $2n+1$ qubits with $n$ pairs of maximally entangled qubits and let Bob's last qubit be in the state $|0\rangle$ (We know that unlike Alice, Bob has got $n+1$ qubits.):

$$|\Xi\rangle = \left(\bigotimes_n \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})\right) \otimes |0\rangle_B\ .$$

Now let us reorder the qubits in such a way that the first $n$ qubits belong to Alice and the remaining $n+1$ qubits belong to Bob. We need to do that because Alice possesses $n$ *neighboring* qubits.

$$|\Xi\rangle = \sum_{\{i,j,\ldots\}} |\{i,j,\ldots\}\rangle_A \otimes \frac{1}{2^{n/2}} |\{i,j,\ldots\}\rangle \otimes |0\rangle \tag{4.13}$$

where a set of indices $\{i,j,\ldots\}$ denotes positions of qubits up in the standard basis vector and we sum over all possible sets of indices. Now we want to find a local unitary transformation $\mathbf{U} = \mathbf{U}_A^{(n)} \otimes \mathbf{U}_B^{(n+1)}$ such that the state $|\Xi\rangle$ transforms into X-state. It follows from Eq. (D13) that it is enough to consider the unitary transformation of the form $\mathbf{U} = \mathbb{1}_A \otimes \mathbf{U}_B^{(n+1)}$ where

$$\mathbf{U}_B^{(n+1)} : |\alpha_{\{i,j\ldots\}}\rangle_{\bar{O}} \to \frac{1}{2^{n/2}} |\{i,j,\ldots\}\rangle \otimes |0\rangle\ ,$$

and the states $|\alpha_{\{i,j\ldots\}}\rangle_{\bar{O}}$ are defined in the Appendix D. After Bob has performed the unitary operation Alice and

Bob share $n$ pairs of maximally entangled qubits and they can begin with their original protocol. This simple example illuminates the remarkable properties of the X-state and its convenient form since only Bob has to perform the local unitary operation.

The situation becomes even more interesting if we replace Alice with $n$ parties $\{A_1, \dots A_n\}$. Now, Bob communicates with $n$ different parties. By performing a local operation he can decide which of his qubits is entangled with a given partner $A_j$. Let us stress that this is only a simple example and more sophisticated protocols are the topic of current research.

## V. SUMMARY

We have performed detailed analytical calculations concerning stationary states of a finite-size Ising chain with cyclic boundary conditions, and their entanglement properties. We have put a special emphasis to a kind of description of multi-partite entanglement.

The primary motivation of our investigation has been an attempt to illuminate the Ising model using tools of quantum information theory. In addition we were studying a possibility whether some properties of the Ising model can be used as a resource of quantum information processing / communication. For this purpose, one of the criteria that should be met is that of the possibility of preparing the system in a suitable initial state. As physical systems tend to occupy their ground states, it is advantageous if the ground state is a suitable initial state for some purposes We have shown that by adjusting the external magnetic field the ground state of the model considered is the well known GHZ state used in several quantum information processing schemes. Consequently, with the ground state of the system well known and having particularly nice properties makes the Ising model a good candidate as a resource for a quantum information processing.

This result also demonstrates the usefulness of the approach to finding an entangled state with pre-defined multi-partite entanglement properties by finding the ground state of a suitably chosen Hamiltonian. The ground state of the Ising model for certain values of the parameter $\lambda$ is a very specific state - the GHZ state. The GHZ state has the property that the entanglement between any set of $n$ qubits where $n < N$ is zero while the $N$-partite entanglement peaks reaching the maximum possible value in the limit $\lambda \to \infty$. It means that in the limit $\lambda \to \infty$ the ground state of the Ising model maximises the $N$-partite entanglement and the state of $N$ qubits with maximum $N$-partite entanglement can be found as a ground state of the Ising model with $N$ qubits in the chain.

Regarding entanglement properties, not only the ground state of the Ising Hamiltonian is found to be interesting. For instance, we have discovered a very specific eigenstate of the Hamiltonian, the X-state. The X-state is *strongly* (extremely) entangled as every set of $k \leq n$ neighboring qubits (where the total number of qubits is $N = 2n + 1$) is maximally entangled with the remaining qubits. An important message of our results is, that multi-partite entanglement plays a crucial role in the understanding of exactly solvable models of quantum statistics. We have also presented a simple example to demonstrate the usefulness of such state in quantum communication.

The X-state is identified via a certain crossing of energy levels at $\lambda = 1$, where a phase transition occurs in the thermodynamic (infinite-qubit) limit. Consequently, there might be some connections between a functional dependence of the entanglement as a function of $\lambda$ and the point of a phase transition. One important lesson one can learn from our investigation is that higher energy eigenstates of the Ising Hamiltonian might carry non-trivial information about quantum correlations of the system under consideration.

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## APPENDIX A: EXACT SOLUTION OF THE $1D$ ISING CHAIN

The Ising model as one of the simplest exactly solvable models has been studied extensively in the literature and there is a chapter on the Ising model in almost every textbook on solid state physics with exactly solvable models. The derivation here will mainly, up to a few minor deviations, follow Ref. [32]. The reason why we summarise the derivation here is the fact that we present a complete solution together with the exact form of the eigenvectors usually omitted in the literature and also to keep our discussion self-content.

Consider the Hamiltonian Eq. (2.3) with the cyclic conditions Eq. (2.2). Our task is to find the eigenvalues and eigenvectors of this Hamiltonian. As a first step we perform several transformations. This is a bit technical part however it needs to be included in the derivation to keep it transparent.

### 1. Transition from variables $\sigma^\alpha$ to $\sigma^\pm$

Firstly we introduce new variables $\sigma_n^\pm$ where

$$\sigma_n^\pm \;=\; \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)\;.$$

The Hamiltonian Eq. (2.3) after the first transformation reads

$$H = -\lambda \sum_{n=1}^{N} (\sigma_n^+ + \sigma_n^-)(\sigma_{n+1}^+ + \sigma_{n+1}^-) + 2 \sum_{n=1}^{N} (\sigma_n^+ \sigma_n^- - \frac{1}{2}\mathbf{1}) \, .$$

### 2. Jordan-Weyl transformation

Secondly, we introduce fermionic variables $c_n^\dagger$ and $c_n$ such that

$$c_n = \exp(i\pi \sum_{j=1}^{n-1} \sigma_j^+ \sigma_j^-)\sigma_n^- \, ,$$
$$c_n^\dagger = \exp(i\pi \sum_{j=1}^{n-1} \sigma_j^+ \sigma_j^-)\sigma_n^+ \, .$$

The variables $c_n^\dagger$ and $c_n$ satisfy the anti-commutation relations. The form of the Hamiltonian after the second transformation is following

$$H = -\lambda \sum_{n=1}^{N-1} \Big( (c_n^\dagger - c_n)(c_{m+1}^\dagger + c_{m+1}) - (-1)^{\hat{N}} (c_N^\dagger - c_N)(c_1^\dagger + c_1) \Big) + 2 \sum_{n=1}^{N} \left( c_n^\dagger c_n - \frac{1}{2} \right) \, ,$$

where $\hat{N} = \sum_{n=1}^{N} c_n^\dagger c_n$ can be interpreted as the operator of number of fermions. The operator $(-1)^{\hat{N}}$ commutes with the Hamiltonian $H$. For that reason it is possible to choose common eigenvectors of the Hamiltonian and the operator $(-1)^{\hat{N}}$. The eigenvalues of the operator $\hat{N}$ are even or odd. If an eigenvector of the operator $\hat{N}$ corresponds to an *even* eigenvalue then the operators $c_{N+1}$ and $c_{N+1}^\dagger$ satisfy the condition

$$c_{N+1}^\dagger = -c_1^\dagger \qquad \textit{anticyclic cond.}$$
$$c_{N+1} = -c_1 \, .$$

On the other hand if an eigenvector of the operator $\hat{N}$ corresponds to an *odd* eigenvalue, the operators $c_{N+1}$ and $c_{N+1}^\dagger$ are defined as

$$c_{N+1}^\dagger = c_1^\dagger \qquad \textit{cyclic cond.}$$
$$x c_{N+1} = c_1 \, .$$

The advantage of the previous choice is that the Hamiltonian has the same form in both cases

$$H = -\lambda \sum_{n=1}^{N} \Big( (c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1}) \Big) + 2 \sum_{n=1}^{N} \left( c_n^\dagger c_n - \frac{1}{2} \right) \, .$$

### 3. Momentum representation

The last transformation is a transition to the momentum representation and specified by the variables $\eta_q^\dagger$ and $\eta_q$. The fermionic operators $c_n^\dagger$ and $c_n$ and the new operators $\eta_q^\dagger$ and $\eta_q$ are related as

$$c_m = \frac{1}{\sqrt{N}} \sum_q e^{iqm} \eta_q \, ,$$
$$c_m^\dagger = \frac{1}{\sqrt{N}} \sum_q e^{-iqm} \eta_q^\dagger \, ,$$

where $q$ depends on the boundary conditions. For cyclic boundary conditions ($\hat{N}$ is odd) we have

$$e^{iqN} = 1 \, ,$$

so that

$$q = \frac{\pi}{N} 2l \quad l = 0, 1, 2 \ldots (N-1) \, .$$

For anti-cyclic boundary conditions ($\hat{N}$ is even)

$$e^{iqN} = -1 \, ,$$

we have

$$q = \frac{\pi}{N}(2l+1) \quad l = 0, 1, 2 \ldots (N-1) \, .$$

The Hamiltonian $H$ has again the same form in both cases and is given by

$$H = \sum_q \Big[ -\lambda \cos q (\eta_q^\dagger \eta_{2\pi-q}^\dagger + 2\eta_q^\dagger \eta_q - \eta_q \eta_{2\pi-q}) - \lambda i \sin q (\eta_q^\dagger \eta_{2\pi-q}^\dagger + \eta_q \eta_{2\pi-q}) + 2 (\eta_q^\dagger \eta_q - \frac{1}{2}) \Big] \, .$$

It is important to note that the operator $\hat{N}$ can easily be rewritten using the new variables $\eta_q$ and $\eta_q^\dagger$

$$\hat{N} = \sum_n c_n^\dagger c_n = \sum_q \eta_q^\dagger \eta_q \, . \tag{A1}$$

It follows that the number of fermions described with $c_m$ and $c_m^\dagger$ is the same as the number of other fermions described with the operators $\eta_q$ and $\eta_q^\dagger$.

### 4. Eigenvalues and eigenvectors of $H_q$

In what follows we calculate eigenvalues and eigenvectors of the Hamiltonian $H$. Firstly, it is convenient to

reorder the contributions to the sum Eq. (A1). Specifically, choose one $q$, and let us define $H_q$ (we note, that it is not possible to perform this step when $q = 0$ or $q = \pi$. In that case see the end of this paragraph.)

$$\begin{aligned} H_q &= -\lambda \cos q(2\eta_q^\dagger \eta_q + 2\eta_{2\pi-q}^\dagger \eta_{2\pi-q}) \\ &\quad -\lambda i \sin q(2\eta_q^\dagger \eta_{2\pi-q}^\dagger + 2\eta_q \eta_{2\pi-q}) \\ &\quad +2(\eta_q^\dagger \eta_q + \eta_{2\pi-q}^\dagger \eta_{2\pi-q} - 1) \ . \end{aligned}$$

The Hamiltonian $H$ is a sum of $H_q$

$$H = \hat{\sum_q} H_q$$

where we sum over half of the q's since $H_q = H_{2\pi-q}$. The point in rewriting the Hamiltonian as a sum of $H_q$ is that it is easy to calculate the eigenstates and eigenvalues of $H_q$. For each $H_q$ we define the following basis

$$\begin{aligned} |\Phi_1\rangle &= \eta_q^\dagger |0> , \\ |\Phi_2\rangle &= \eta_{2\pi-q}^\dagger |0> , \\ |\Phi_3\rangle &= |0> , \\ |\Phi_4\rangle &= \eta_{2\pi-q}^\dagger \eta_q^\dagger |0> . \end{aligned}$$

In this particular basis the Hamiltonian $H_q$ is a $4 \times 4$ matrix

$$\begin{pmatrix} -2\lambda\cos q & 0 & 0 & 0 \\ 0 & -2\lambda\cos q & 0 & 0 \\ 0 & 0 & -2 & -2\lambda i \sin q \\ 0 & 0 & 2\lambda i sin q & 2 - 4\lambda \cos q \end{pmatrix} ,$$

and the four eigenvalues of $H_q$ are

$$\begin{aligned} a_1 &= -2\lambda \cos q \ , \\ a_2 &= -2\lambda \cos q \ , \\ a_3 &= 2(-\lambda \cos q + \sqrt{(\lambda-1)^2 + 2\lambda(1-\cos q)}) \ , \\ a_4 &= 2(-\lambda \cos q - \sqrt{(\lambda-1)^2 + 2\lambda(1-\cos q)}) \ . \end{aligned}$$

The eigenvectors corresponding to the eigenvalues $a_1$, $a_2$, $a_3$ and $a_4$ in the defined basis reads

$$\begin{aligned} |a_1\rangle &= |\Phi_1\rangle \ , \\ |a_2\rangle &= |\Phi_2\rangle \ , \\ |a_3\rangle &= d_3|\Phi_3\rangle + e_3|\Phi_4\rangle \ , \\ |a_4\rangle &= d_4|\Phi_3\rangle + e_4|\Phi_4\rangle \ , \end{aligned}$$

where

$$\begin{aligned} d_j &= -\frac{2\,\lambda\, i\, \sin q}{a_j + 2}\ e_j \ , \\ e_j &= \frac{1}{\sqrt{1 + \frac{4\lambda^2 (\sin q)^2}{(a_j+2)^2}}} . \end{aligned}$$

At this moment we postpone the derivation of the eigenvectors and eigenstates of the Hamiltonian $H$ to the next paragraph and instead we discuss the cases when $q = 0$ (or $q = \pi$) [39]. The form of operator $H_0$ ( $H_\pi$ ) is

$$\begin{aligned} H_0 &= -2\,\lambda\,\eta_0^\dagger\,\eta_0 + 2\,\eta_0^\dagger\, eta_0 - 1 \qquad q = 0 \\ (\quad H_\pi &= +2\,\lambda\,\eta_\pi^\dagger \eta_\pi + 2\,\eta_\pi^\dagger \eta_\pi - 1 \qquad q = \pi \ ). \end{aligned}$$

In this case we define the basis as follows

$$\begin{aligned} |\Phi_{q=0}\rangle &= \eta_0^\dagger |0> \qquad [\ |\Phi_\pi\rangle = \eta_\pi^\dagger |0> \ ] \\ |\Phi_0\rangle &= |0> \qquad [\ |\Phi_0\rangle = |0\rangle \ ] \ . \end{aligned}$$

The operator $H_0$ ($H_\pi$ ) in this particular basis is a $2 \times 2$ matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -2\lambda + 1 \end{pmatrix} \qquad \left[ \begin{pmatrix} -1 & 0 \\ 0 & 2\lambda + 1 \end{pmatrix} \right] ,$$

The matrix has two eigenvalues

$$\begin{aligned} a_1 &= -1 \qquad [\ a_1 = -1 \ ] \ , \\ a_2 &= -2\lambda + 1 \qquad [\ a_2 = 2\lambda + 1 \ ] \ , \end{aligned}$$

with the corresponding eigenvectors given by

$$\begin{aligned} |a_1\rangle &= |\Phi_0\rangle \qquad [\ |a_1\rangle = |\Phi_0\rangle \ ] \ , \\ |a_2\rangle &= |\Phi_{q=0}\rangle \qquad [\ |a_2\rangle = |\Phi_\pi\rangle \ ] \ . \end{aligned}$$

### 5. Eigenvalues and eigenvectors of H: Example

According to the previous paragraph the Hamiltonian $H$ can be rewritten as a sum of the operators $H_q$. Since each operator $H_q$ is a sum of terms with an even number of fermionic operators and two different $H_q$'s contain different fermionic operators we conclude that the operators $H_q$ commute. Essentially it means that they have common eigenvectors. As an example let us consider the case $N = 5$ and let $(-1)^{\hat{N}}$. Then one of the energy eigenstates is

$$|e_H\rangle = \eta_0^\dagger \eta_{2\pi/5}^\dagger \eta_{4\pi/5}^\dagger |0\rangle$$

with the corresponding eigenvalue

$$E_H = -1 - 2\lambda \cos(\frac{2\pi}{5}) - 2\lambda \cos(\frac{4\pi}{5}).$$

We have to keep in mind here that not all the combinations are the eigenstates of $H$. For example if $(-1)^{\hat{N}} = -1$ then the state

$$|\varphi\rangle = \eta_{2\pi/5}^\dagger \eta_{4\pi/5}^\dagger |0\rangle \ , \qquad \text{(A2)}$$

is not an eigenstate of $H$ even though this state is an eigenvector of $H_0$, $H_{2\pi/5}$ and $H_{4\pi/5}$. The state Eq. (A2)

has even number of fermions and consequently for $\hat{N}$ even the Hamiltonian is not

$$H = H_0 + H_{2\pi/5} + H_{4\pi/5} \ ,$$

but

$$H = H_{\pi/5} + H_{3\pi/5} + H_{\pi} \ .$$

Recall that $q$'s depend on the boundary conditions which in turn depend on the number of fermions.

## APPENDIX B: THREE SPINS: EIGENVALUES AND EIGENVECTORS OF THE HAMILTONIAN $H_3$.

For completeness we review the complete spectrum of the Hamiltonian $H_3$. The Hilbert space corresponding to the system of three qubits is eight dimensional and the Hamiltonian $H_3$ has eight eigenvalues,

$$\begin{aligned}
E_{1,2} &= \lambda + 1 \ , \\
E_{3,4} &= \lambda - 1 \ , \\
E_5 &= 1 - \lambda - 2\sqrt{1 + \lambda + \lambda^2} \ , \\
E_6 &= 1 - \lambda + 2\sqrt{1 + \lambda + \lambda^2} \ , \\
E_7 &= -1 - \lambda - 2\sqrt{1 - \lambda + \lambda^2} \ , \\
E_8 &= -1 - \lambda + 2\sqrt{1 - \lambda + \lambda^2} \ .
\end{aligned}$$

The eigenvalues $E_1 = E_2$ and $E_3 = E_4$ are degenerate for all values of parameter $\lambda$ while the other four are (apart from a finite number of points) not. The eigenvectors corresponding to the eigenvalues read

$$\begin{aligned}
|e_1\rangle &= \frac{1}{\sqrt{2}} \left[ |110\rangle - |011\rangle \right] \ , \\
|e_2\rangle &= \frac{1}{\sqrt{2}} \left[ |101\rangle - |011\rangle \right] \ , \\
|e_3\rangle &= \frac{1}{\sqrt{2}} \left[ |100\rangle - |001\rangle \right] \ , \\
|e_4\rangle &= \frac{1}{\sqrt{2}} \left[ |010\rangle - |001\rangle \right] \ , \\
|e_5\rangle &= K_5 \left[ \frac{E_5 + 1 - \lambda}{E_5 - 3 - \lambda} |111\rangle + |001\rangle + |010\rangle + |100\rangle \right] \ , \\
|e_6\rangle &= K_6 \left[ \frac{E_6 + 1 - \lambda}{E_6 - 3 - \lambda} |111\rangle + |001\rangle + |010\rangle + |100\rangle \right] \ , \\
|e_7\rangle &= K_7 \left[ \frac{1 - E_7 - 2\lambda}{\lambda} |000\rangle + |011\rangle + |101\rangle + |110\rangle \right] \ , \\
|e_8\rangle &= K_8 \left[ \frac{1 - E_8 - 2\lambda}{\lambda} |000\rangle + |011\rangle + |101\rangle + |110\rangle \right] \ .
\end{aligned}$$

## APPENDIX C: GROUND STATE FOR $\lambda = \infty$

The case $\lambda = \infty$ corresponds to the physical situation with zero magnetic field $B$. For the given value of the parameter $\lambda$ the Hamiltonian $H_N$ has the following form:

$$H = -C_I \sum_{i=1}^{N} \sigma_i^x \otimes \sigma_{i+1}^x \ .$$

The ground state of the Hamiltonian is degenerate, the energy of the ground state is $E_g = -NC_I$ and two energy states related to the energy $E_g = -NC_I$ are

$$\begin{aligned}
|\bar{g}_1\rangle &= |\bar{0}\bar{0}\ldots\rangle \ ; \\
|\bar{g}_2\rangle &= |\bar{1}\bar{1}\ldots\rangle \ ,
\end{aligned} \tag{C1}$$

where $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are eigenstates of $\sigma^x$. Of course not only these two states are eigenstates of the Hamiltonian. Any linear combinations of these states is also an eigenstate. However, we are searching for eigenstates that are the limit of the eigenstates of the Hamiltonian (2.3) when $\lambda$ tends to infinity. At this point we use the knowledge of the exact solution (see Appendix A), and more specifically the fact that any eigenstate of the Hamiltonian is a linear combination of standard basis vectors with either even or odd number of qubits up. (Standard basis corresponds to state vectors that are eigenstates of all $\sigma_i^z$. The fact that a state is a linear combination of standard basis vectors with either odd or even number of qubits up is equivalent to the condition that the state is an eigenstate of the operator $(-1)^{\hat{N}}$.) Therefore we select two particular linear combinations, such that they are sums of state vectors of standard basis with either even or odd number of qubits up. The linear combinations of the two vectors in Eq. (C1) which satisfy the condition are

$$\begin{aligned}
|g_1\rangle &= \frac{1}{\sqrt{2}} (|\bar{0}\bar{0}\ldots\rangle + |\bar{1}\bar{1}\ldots\rangle) \ , \\
|g_2\rangle &= \frac{1}{\sqrt{2}} (|\bar{0}\bar{0}\ldots\rangle - |\bar{1}\bar{1}\ldots\rangle) \ ,
\end{aligned}$$

where the first one is a linear combination of standard basis vectors with an even number of qubits up and the second one is a linear combination of standard basis vectors with an odd number of qubits up

$$\begin{aligned}
|g_1\rangle &= K_N \sum_{\{i,j,\ldots\}_e} |\{i,j,\ldots\}\rangle \ , \\
|g_2\rangle &= K_N \sum_{\{i,j,\ldots\}o} |\{i,j,\ldots\}\rangle \ ,
\end{aligned}$$

and the constant $K_N$ is a normalisation constant. It is easy to show that these are the only two possible linear combinations that satisfy the condition, and on top of that it is clear from the construction that each of the states $|g_1\rangle$ and $|g_2\rangle$ is a GHZ state.

## APPENDIX D: THE X-STATE

The expression for the X-state reads

$$|X\rangle = \sum_{\{i,j,\ldots\}_e} |\{i,j,k,\ldots\}\rangle (-1)^{\sum_{i>j} d(i,j)} \ , \tag{D1}$$

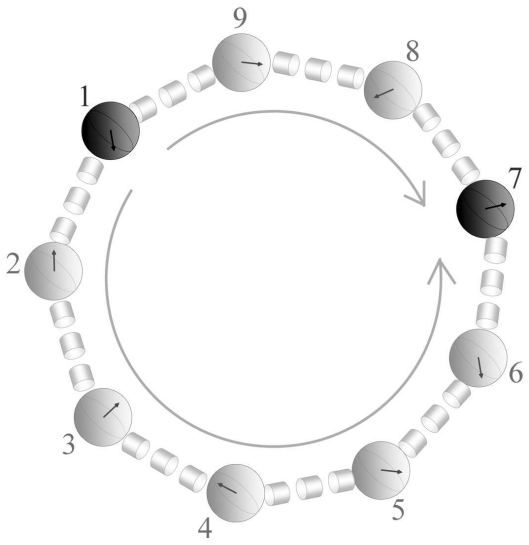


FIG. 7: The ring of 9 qubits. The arrows denote two possible paths from the first qubit to the seventh qubit. The shorter path is the distance between the two qubits $d(1,7)$.

where $\{i,j,\ldots\}_e$ is a set of indices with an even number of indices in the set so that we sum over all sets of indices with an even number of indices, the letters $i,j,\ldots$ denote positions of the qubits in the chain that are up, $d(i,j)$ is the distance of the qubits on the $i$-th and $j$-th positions defined below and the sum over $i<j$ means that we sum over all pairs of qubits up counting only once the cases with switched positions of the qubits. Let us note that the state in Eq. (D1) is not normalised to unity.

The distance $d(i,j)$ of the two qubits is defined as the shortest path on the ring that brings us from the qubit on the $i$-th positions to the qubit on the $j$-th position. As the qubits form a circle, there are always two paths we can go without going backward and we can always choose the shortest one. To make clear what the distance defined above is, let us have a look at a simple example. Let $N$ be 9 so that overall number of qubits in the ring is nine as in Fig. 7. Further let $i=1$ so that it denotes the first position and $j=7$ so that it denotes the seventh position. Then the shortest path is going from first to ninth position as the two are neighboring positions then from the ninth to the eighth and finally from the eighth to the seventh position. Consequently, the distance $d(1,7)$ in this particular case is 3.

### 1. Proof that the X-state is an eigenstate of the Hamiltonian with zero energy

In what follows we show that the state is the eigenstate of the Hamiltonian with zero energy or equivalently, that the following relation holds

$$H|X\rangle = 0\,.$$

If we divide the Hamiltonian into the free Hamiltonian and the interaction Hamiltonian $H=H_I+H_0$, then the last equation can be rewritten as

$$H_I|X\rangle = -H_0|X\rangle\,. \tag{D2}$$

The task now is to show that the two vectors - one on the left and the other on the right side of the last equation are equal. As we know, the equality of two vectors follows from the equality of their components in any complete basis. Actually what we will prove is the equality of the components of the two vectors in the standard basis (computational basis) i.e.

$$(H_I|X\rangle)_i = -(H_0|X\rangle)_i\,. \tag{D3}$$

Take one vector of the standard basis that is included in the sum given by Eq. (D1) and denote it as $|v\rangle$. We show that the $v$-th components obey Eq. (D3) [40]

*a.* All vectors of the standard basis are eigenvectors of $H_0$. If we denote $K$ to be the number of qubits up in the vector $|v\rangle$ then

$$H_0|v\rangle = (2K-N)|v\rangle\,,$$

and the $v$-th component of $H_0|X\rangle$ is

$$(H_0|X\rangle)_v = (2K-N).s\,, \tag{D4}$$

where $s$ is the phase factor of the vector $|v\rangle$ in the sum in Eq. (D1).

*b.* Now it remains to show that the same holds for $H_I$ except for the sign that must be opposite. The Hamiltonian $H_I$ is a sum of many elements $\mathbf{C}_i$ where

$$\mathbf{C}_i = \sigma_i^x\sigma_{i+1}^x\,.$$

If we want to count the $v$-th component of $H_I|X\rangle$ we need to know the individual contributions from each term $\mathbf{C}_i|X\rangle$. What is the action of the operator $\mathbf{C}_i$? It flips two neighboring spins on the $i$-th and $(i+1)$-th positions. Let us assume that there are $K$ spins up in the vector. If the two spins on the $i$-th and $(i+1)$-th positions are up then the operator $\mathbf{C}_i$ flips them down and there are $K-2$ spins up in the vector $\mathbf{C}_i|v\rangle$. Similarly if the two spins on the $i$-th and $(i+1)$-th positions are down then the operator $\mathbf{C}_i$ flips them so that they are up and consequently, there are $K+2$ spins up in the vector $\mathbf{C}_i|v\rangle$. Otherwise if one spin is up and the other is down then the number of spins up in the vector $\mathbf{C}_i|v\rangle$ equals $K$. At this point it is obvious that neither $H_0$ nor $H_I$ can produce a contribution (vector of the standard basis) with an odd number of qubits up since in Eq. (D1) we sum over all sets of indices with an even number elements in each set. Now we use a little trick, namely

$$\mathbf{C}_i^2 = (\sigma_i^x\sigma_{i+1}^x)^2 = \mathbf{1}\,,$$

so that

$$\mathbf{C}_i = \mathbf{C}_i^{-1}\,,$$

in order to answer the question which vectors from the sum in Eq. (D1) give contributions to the $v$-th element considering only one $\mathbf{C}_i$. Using the last relation the only possible one is

$$- s_i \mathbf{C}_i |v\rangle \ , \tag{D5}$$

where $s_i$ is the coefficient of the state $-\mathbf{C}_i|v\rangle$ in the sum in Eq. (D1) (We have introduced the minus sign in the last equation because of the sign of the operator $\mathbf{C}_i$ in $H_I$.). Since the coefficients of the vectors in the sum Eq. (D1) are plus or minus one there are exactly $N$ contributions to the $v$-th element, as we have $N$ operators $\mathbf{C}_i$, and all of them are plus or minus one. Our task is to find out the sign of each individual contribution and count them. First we divide the contributions into three subsets. Let us denote by $M_0$ the set of all vectors $\mathbf{C}_i|v\rangle$ with two more qubits up compared with the vector $|v\rangle$. If we by denote $K_0$ the number of pairs of neighboring qubits both being down in the vector $|v\rangle$ then the number of elements in the set $M_0$ is $K_0$. Equivalently, let us denote by $M_2$ to be the subset of all vectors $\mathbf{C}_i|v\rangle$ with two more spins down as are in the vector $|v\rangle$. The number of elements in the set $M_2$ is $K_2$ where $K_2$ is the number of pairs of neighboring qubits both being up in the vector $|v\rangle$ . Finally, let us denote by $M_1$ the subset of all vectors $\mathbf{C}_i|v\rangle$ with equal number of qubits up as are in the vector $|v\rangle$. The number of elements set $M_1$ is $K_1$ and equals the number of pairs of neighboring qubits in the state $|v\rangle$ with exactly one qubit up. The following relations hold

$$\begin{aligned} K_1 + K_2 + K_3 &= N \, ; \\ 0.K_0 + 1.K_1 + 2.K_2 &= 2K \, . \end{aligned} \tag{D6}$$

We can rewrite $s_i$ in the following way $s_i = s.k_i$ where $k_i$ is the relative sign of the vector $\mathbf{C}_i|v\rangle$ according to the absolute sign of the vector $|v\rangle$ in the sum Eq. (D1).

Consider vectors belonging to the subset $M_0$. To find out the relative sign $s_i$ between the vector $\mathbf{C}_i|v\rangle$ and vector $|v\rangle$ we need to know the following distances

1. The distance from any qubit up in the vector $|v\rangle$ to the $j$-th position: $d(j,x)$ .
2. The distance from the $(j+1)$-th qubit to any qubit up in the vector $|v\rangle$: $d(j+1,x)$ .
3. The distance of the $j$-th and the $(j+1)$-th qubit that is apparently one: $d(j,j+1) = 1$ .

The relative sign between the two vectors is then

$$(-1)^{1+\sum_x d(j,x)+d(j+1,x)} , \tag{D7}$$

where summing over x means that we sum over all positions of qubits up in the vector $|v\rangle$.

**Theorem 2** *If the qubit in the vector $|v\rangle$ that is equally distant from the $j$-th and the $(j+1)$-th qubit is down then the exponent in Eq. (D7) is odd.*

*Proof:* There is only one qubit in the ring that is equally distant from the qubits on the $j$-th and the $(j+1)$-th positions. If that qubit is down and taking any qubit in the vector $|v\rangle$ that is up and summing the distance from the $j$-th qubit to the given qubit and the distance from the $(j+1)$-th qubit to the same qubit we always get an odd number. As vector $|v\rangle$ contains even number of qubits up the sum is an even but to get the final expression in Eq. (D7) we have to add 1 therefore the exponent is odd.

**Theorem 3** *If the qubit that is equally distant from the $j$-th and the $(j+1)$-th qubits is up in the vector $|v\rangle$ then the exponent in the sum in Eq. (D7) is even.*

*Proof:* Follows from the previous statement.

We know that we have $K_0$ vectors in the set $M_0$. Denote $k_0$ to be the number of such states that they have the qubit which is equally distant from the corresponding $j$-th and $(j+1)$-th position up. Then we may say that the contribution of all vectors from the set $M_0$ to the $v$-th component is

$$- (k_0 - (K_0 - k_0)).s \ , \tag{D8}$$

where the minus sign in front comes from Eq. (D5). It is amazing that considering the other sets namely, $M_1$ and $M_2$, we have come to the same conclusion so that their contributions to the $v$-th component are

$$- (k_1 - (K_1 - k_1)).s \ , \tag{D9}$$

from $M_1$ and

$$- (k_2 - (K_2 - k_2)).s \ , \tag{D10}$$

from $M_2$. Consequently, the $v$-th component of the vector $H_I|p\rangle$ is a sum of Eq. (D8), Eq. (D9) and Eq. (D10) and reads

$$(K_0 + K_1 + K_2 - 2(k_0 + k_1 + k_2)).s \, .$$

Now comes the crucial point. The following equation holds

$$k_0 + k_1 + k_2 = K$$

and together with Eq. (D6) the $v$-th component of $H_I|p\rangle$ is

$$(H_I|X\rangle)_v = (N - 2K).s \tag{D11}$$

*c.* We have proved that the left-hand side of Eq. (D3) is equal to $-(N-2K).s$ and the right-hand side of the equation equals $-(N-2K).s$. In other words the two expressions are equal for a given vector $|v\rangle$. Since we have not specified the vector $|v\rangle$, it holds for any vector (see the discussion above) and we have proved that the X-state is an eigenstate of the Hamiltonian $H$ with zero energy.

### 2. Density matrix of n neighboring qubits

**Theorem 4** *Let $N$ denote the total number of spins forming the chain and $n$ be an integer. Further let $N = 2n+1$ and the system be in the X-state Eq. (D1).*

*Then the density matrix of any sequence of $n$ neighboring qubits is*

$$\rho_O = \frac{1}{2^n}\mathbf{1}_n \,, \tag{D12}$$

*where $\mathbf{1}_n$ is the identity operator acting on the $2^n$-dimensional Hilbert space of $n$ qubits.*

The sequence of $n$ neighboring qubits is a subset of all qubits in the chain such that only two "cuts" are needed to cut out the whole sequence from the chain. In what follows we will denote the set of $n$ neighboring qubits by $O$.

*Proof of the Theorem 4:* We want to show that the density operator of $n$ neighboring qubits is proportional to the identity operator acting on the $2^n$-dimensional Hilbert space. Denote by $|\{i,\dots k\}\rangle_O$ one of the basis state vectors of the system of $n$ qubits from the set $O$ where the set of indices $\{i,\dots\}$ denote positions where the spins are up and all remaining spins are down. First we rewrite the X-state using this new basis as follows

$$|X\rangle = \sum_{\{i,\dots k\}} |\{i,\dots j\}\rangle_O |\alpha_{\{i,\dots j\}}\rangle_{\bar{O}} \,, \tag{D13}$$

where $|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}}$ is a state vector of the remaining $n+1$ qubits not belonging to the set $O$ [41] and we sum over all sets of indices $\{i,\dots k\}$ which means that we sum over all basis vectors of the system of $n$ qubits. Then the Theorem 4 says that

$${}_{\bar{O}}\langle\alpha_{\{k,\dots l\}}|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}} = \frac{K^2}{2^n}\delta_{\{k,\dots l\},\{i,\dots j\}} \,, \tag{D14}$$

where $K$ is the norm of the X-state.

In order to prove Eq. (D14) we need to know the form of the states $|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}}$. From Eq. (D1) we have

$$\begin{aligned} |X\rangle &= \sum_{\{i,j,\dots\}_e} |\{i,j,k,\dots\}\rangle (-1)^{\sum_{i>j} d(i,j)} \\ &= \sum_{\{i,\dots j\}_e} |\{i,\dots j\}\rangle_O \sum_{\{k,\dots l\}e} |\{k,\dots l\}\rangle_{\bar{O}} \\ &\quad (-1)^{d_{\{i,\dots j\}}+d_{\{k,\dots l\}}+d_{\{i,\dots j\},\{k,\dots l\}}} \\ &+ \sum_{\{i,\dots j\}o} |\{i,\dots j\}\rangle_O \sum_{\{k,\dots l\}o} |\{k,\dots l\}\rangle_{\bar{O}} \\ &\quad (-1)^{d_{\{i,\dots j\}}+d_{\{k,\dots l\}}+d_{\{i,\dots j\},\{k,\dots l\}}} \,, \end{aligned}$$

where the sum over $\{i,\dots j\}_e$ means that we sum over all sets of indices with an even number of indices in each set, $\{j,\dots k\}o$ means that we sum over all sets of indices with an odd number of indices in each set, $d_{\{i,\dots j\}} = \sum_{i>j} d(i,j)$ where $i,j \in \{i,j,\dots\}$ and $d_{\{i,\dots j\},\{k,\dots l\}} = \sum_{a\in\{i,\dots j\},b\in\{k,\dots l\}} d(a,b)$. It follows that the states $|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}}$ are of the following form. If the set $\{i,\dots j\}$ contains an even number of indices then

$$\begin{aligned} |\alpha_{\{i,\dots j\}}\rangle_{\bar{O}} &= \sum_{\{k,\dots l\}e} |\{k,\dots l\}\rangle_{\bar{O}} \\ &\times (-1)^{d_{\{i,\dots j\}}+d_{\{k,\dots l\}}+d_{\{i,\dots j\},\{k,\dots l\}}} \,, \end{aligned} \tag{D15}$$

while if the set contains an odd number of indices then

$$\begin{aligned} |\alpha_{\{i,\dots j\}}\rangle_{\bar{O}} &= \sum_{\{k,\dots l\}o} |\{k,\dots l\}\rangle_{\bar{O}} \\ &\times (-1)^{d_{\{i,\dots j\}}+d_{\{k,\dots l\}}+d_{\{i,\dots j\},\{k,\dots l\}}} \,. \end{aligned} \tag{D16}$$

Moreover, the norm of the X-state $K$ can be easily calculated and the result is

$$\begin{aligned} K^2 &= \langle X|X\rangle = \sum_{\{i,\dots j\}_e}\sum_{\{k,\dots l\}_e} 1 + \sum_{\{i,\dots j\}o}\sum_{\{k,\dots l\}o} 1 \\ &= 2^{2n-1}2^n = 2^{2n} \,. \end{aligned}$$

In what follows we consider three different possibilities.

1. The two sets $\{i,\dots j\}$, $\{k,\dots l\}$ are equal. If in the set $\{i,\dots j\}$ is an even number of indices then

   $${}_{\bar{O}}\langle\alpha_{\{i,\dots j\}}|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}} = \sum_{\{k,\dots l\}_e} 1 = 2^n \,,$$

   else if in the set $\{i,\dots j\}$ is an odd number of indices then

   $${}_{\bar{O}}\langle\alpha_{\{i,\dots j\}}|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}} = \sum_{\{k,\dots l\}_o} 1 = 2^n \,.$$

2. There is an even number of indices in the set $\{i,\dots j\}$ while in the set $\{k,\dots l\}$ the number of indices is odd. If the set $\{i,\dots j\}$ contains an even number of indices from Eq.(D15) it follows that state $|\alpha_{\{i,\dots j\}}\rangle_{\bar{O}}$ is a sum of basis state vectors with an even number of spins up. Further if the set $\{k,\dots l\}$ contains an odd number of indices then from Eq. (D16) it follows that the state $|\alpha_{\{k,\dots l\}}\rangle_{\bar{O}}$ is a sum of state vectors with an odd number of spins up and therefore the scalar product Eq. (D14) is zero i.e.

   $${}_{\bar{O}}\langle\alpha_{\{i,\dots j\}}|\alpha_{\{k,\dots l\}}\rangle_{\bar{O}} = 0 \,.$$

3. The case that remains is when in the both sets $\{i,\dots j\}$ and $\{k,\dots l\}$ there is an even number of indices but the two sets are different [42]. The scalar product Eq. (D14) reads as

   $$\begin{aligned} &{}_{\bar{O}}\langle\alpha_{\{i,\dots j\}}|\alpha_{\{k,\dots l\}}\rangle_{\bar{O}} = \\ &\sum_{\{m,\dots n\}_e} (-1)^{d_{\{i,\dots j\}}+d_{\{m,\dots n\}}+d_{\{i,\dots j\},\{m,\dots n\}}} \\ &\quad \times (-1)^{d_{\{k,\dots l\}}+d_{\{m,\dots n\}}+d_{\{k,\dots l\},\{m,\dots n\}}} \\ &= (-1)^{d_{\{i,\dots j\}}+d_{\{k,\dots l\}}} \\ &\times \sum_{\{m,\dots n\}_e} (-1)^{2d_{\{m,\dots n\}}+d_{\{i,\dots j\},\{m,\dots n\}}+d_{\{k,\dots l\},\{m,\dots n\}}} \,. \end{aligned}$$

Since we want to show that the scalar product of the two vectors is zero we drop the overall factor in front of the sum and using the relation $(-1)^{2k} = 1$ we rewrite the last equation as follows

$$\bar{o}\langle\alpha_{\{i,\dots j\}}|\alpha_{\{k,\dots l\}}\rangle_{\bar{O}} = \sum_{\{m,\dots n\}_e} (-1)^{d_{\{i,\dots j\},\{m,\dots n\}}+d_{\{k,\dots l\},\{m,\dots n\}}} .$$

The problem of calculating the scalar product of two vectors has transformed into calculating the distances between the two sets. If an index $a$ is in the both sets of indices $\{i,\dots j\}$ and $\{k,\dots l\}$ then we can neglect it since we sum over distances from $a$ to $\{m,\dots n\}$ twice and the term $(-1)^{2d_{\{a\},\{m,\dots n\}}} = 1$ does not change the sign of the corresponding contributions. Therefore instead of calculating distances $d_{\{i,\dots j\},\{m,\dots n\}}$ and $d_{\{k,\dots l\},\{m,\dots n\}}$ , we create one set of indices

$$\begin{aligned} D(\{i,\dots j\},\{k,\dots l\}) &= \{i,\dots j\}\cup\{k,\dots l\} \\ &- \{i,\dots j\}\cap\{k,\dots l\} , \end{aligned}$$

and then

$$\bar{o}\langle\alpha_{\{i,\dots j\}}|\alpha_{\{k,\dots l\}}\rangle_{\bar{O}} = \sum_{\{m,\dots n\}_e} (-1)^{d_{D(\{i,\dots j\},\{k,\dots l\}),\{m,\dots n\}}} . \qquad \text{(D17)}$$

It is important to note that the set $D(\{i,\dots j\},\{k,\dots l\})$ always contains only even number of indices [43]. The last step is to calculate the distances. In order to calculate these distances we use the following strategy: We choose one of the positions from $D(\{i,\dots j\},\{k,\dots l\})$ such that it is the closest to the qubits not belonging to $O$ and denote the position to be $a$. There are only two qubits that are equally distant from $a$ and none of them belong to $O$. Denote their positions as $b$ and $c$, where the position denoted as $b$ is closer to the set $O$. Let us calculate the distances from $D(\{i,\dots j\},\{k,\dots l\})$ to $b$ and $c$. If the distance is

$$d_{D(\{i,\dots j\},\{k,\dots l\}),\{b\}} = C ,$$

then

$$d_{D(\{i,\dots j\},\{k,\dots l\}),\{c\}} = C + (L-1) , \qquad \text{(D18)}$$

where $L$ is the number of positions in $D(\{i,\dots j\},\{k,\dots l\})$ as

$$\begin{aligned} d(i,c) &= d(i,b)+1 ; \qquad \forall i \in D(\{i,\dots j\}\{k,\dots l\})/\, a \\ d(i,c) &= d(i,c) ; \qquad i = a . \end{aligned}$$

**Example:** To make this clear let us consider a simple example $N = 9$, $n = 4$, $O = \{1,2,3,4\}$ and $D(\{i,\dots j\},\{k,\dots l\}) = \{2,4\}$ see Fig. 8. The position that is the closest to the qubits not belonging to $O$ is 4 so that $a = 4$. The two positions that are equally distant from 4 are 9 and 8. The position 9 is closer to $O$ so that $b = 9$ and $c = 8$. Moreover the distances $d_{D(\{i,\dots j\},\{k,\dots l\}),\{b\}}$ and $d_{D(\{i,\dots j\},\{k,\dots l\}),\{c\}}$ are

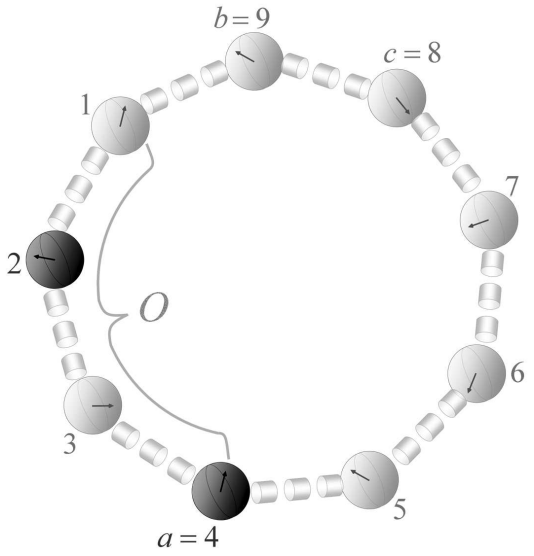


FIG. 8: The ring of 9 qubits. $O = \{1,2,3,4\}$.

$$\begin{aligned} d_{D(\{i,\dots j\},\{k,\dots l\}),\{b\}} &= d(2,9)+d(4,9) = 1+4 , \\ d_{D(\{i,\dots j\},\{k,\dots l\}),\{c\}} &= d(2,8)+d(4,8) = 2+4 . \end{aligned}$$

The number of positions (indices) in $D(\{i,\dots j\},\{k,\dots l\})$ is equal to 2 and consequently the relation Eq.(D18) holds.

Now we have everything necessary to calculate the scalar product Eq. (D17). Choose one arbitrary set $\{m,\dots n\}$ which contains neither $b$ nor $c$. If there is an even number elements in the set $\{m,\dots n\}$ then there are two contributions to the sum in Eq. (D17) namely

$$\begin{aligned} &(-1)^{d_{D(\{i,\dots j\},\{k,\dots l\}),\{m,\dots n\}}} \\ &+(-1)^{d_{D(\{i,\dots j\},\{k,\dots l\}),\{b,c,m,\dots n\}}} , \end{aligned}$$

and these two have opposite signs [see Eq. (D18)]. It follows that these two contributions cancel each other. Equivalently, if there is an odd number of elements in the set $\{m,\dots n\}$ then we have again two contributions

$$\begin{aligned} &(-1)^{d_{D(\{i,\dots j\},\{k,\dots l\}),\{a,m,\dots n\}}} \\ &+(-1)^{d_{D(\{i,\dots j\},\{k,\dots l\}),\{b,m,\dots n\}}} , \end{aligned}$$

with opposite signs and the two contributions are mutually cancelled. It follows that all contributions to the sum in Eq. (D17) mutually cancel and the result is zero.

To conclude, we have proved that

$$\bar{O}\langle\alpha_{\{k,...l\}}|\alpha_{\{i,...j\}}\rangle_{\bar{O}} = 2^n\delta_{\{k,...l\},\{i,...j\}} \ ,$$

and it follows that the density operator of the $n$ neighboring qubits is the identity operator up to a constant factor.

### 3. Consequences

*d. Density matrix of a qubit* Choose one of $N$ qubits forming the Ising chain. Add $n-1$ qubits to the chosen qubit such that the $n$ qubits form a set of $n$ neighboring qubits. Then according to Theorem 4 the density matrix of such system is

$$\rho_n = \frac{1}{2^n}\mathbf{1}_n \ .$$

When we now trace over the $n-1$ qubits that we added to the chosen qubit whose density matrix (state) we want to know, we obtain

$$\rho = \mathrm{Tr}_{n-1}\rho_n = \frac{1}{2}\mathbf{1} \ , \tag{D19}$$

where $\mathbf{1}$ is the identity operator acting in the two-dimensional Hilbert space.

*e. Density matrix of a pair of qubits* The derivation of the density operator of any two qubits follows the same steps as the derivation of the density operator of a single qubit. However, in this case we add only $n-2$ qubits so that the set of $n$ qubits consists only of $n$ neighboring qubits [44]. According to Theorem 4 the density operator of such system is $\rho_n = 1/2^n\mathbf{1}_n$. The corresponding density operator of the two qubits is obtained via tracing over the degrees of freedom of the $n-2$ qubits that we added

$$\rho_{i,j} = Tr_{n-2}\rho_n = \frac{1}{4}\ \mathbf{1}_2 \ , \tag{D20}$$

where $i$, and $j$ denotes the positions of the two *a priori chosen* qubits and $\mathbf{1}_2$ is the identity operator acting in the four-dimensional Hilbert space of the two qubits.


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[33] The energy levels $E_j, j = 1, \dots 8$ are expressed in energy units defined by the parameter $E$ of the Hamiltonian. In what follows this parameter is dropped and the energy levels $E_j$ are taken to be real functions of the parameter $\lambda$.

[34] The state of the system is not proportional to the identity but it is a state which is very close to the total mixture.

[35] Let us note that for $N > 3$ qubits there is more than one level crossing.

[36] It is rather misleading to talk about a "long range" entanglement if there are only three qubits in a chain. However, the study will be extended to many-qubit systems with the same result which will justify our terminology.

[37] For $N$ large there are several level crossings for different values of the parameter $\lambda$. The one we are interested in is at $\lambda = 1$.

[38] The operator can be calculated as a trace of the density matrix Eq. (4.8) over the degrees of freedom of the qubit on the $j$-th position or directly from the state Eq. (4.7).

[39] Actually, we discuss the case $q = 0$ while in the brackets we present the corresponding expressions for $q = \pi$.

[40] We know that in order to prove Eq.(D2) we need to show that Eq. (D3) holds for all components. But the sum in Eq. (4.7) goes over all standard basis vectors with an even number of qubits up. Moreover, as we will see later neither $H_0$ nor $H_I$ can produce a vector with at least one nonzero component of the standard basis vector with odd number of qubits up if the standard basis decomposition of the vector we acted on does not contain a vector with an odd number of qubits up. It follows that it is sufficient to consider only components corresponding to the standard basis vectors with even number of qubits up.

[41] There are $N = 2n + 1$ qubits in the chain and only $n$ of them belong to the set $O$.

[42] The case with an odd number of indices in both sets but when the two sets are different is equivalent to this case.

[43] Let us note that this is also true in the case when there is an odd number of indices in the two sets $\{i, \dots j\}$ and $\{k, \dots l\}$. Therefore from this point the proof is thoroughly identical in both cases.

[44] Let us note that for any pair qubits we are able to choose other $n - 2$ qubits so that the set of $n$ qubits is a set of $n$ neighboring qubits.