Entanglement swapping of noisy states: A kind of superadditivity in nonclassicality

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We investigate the robustness with respect to violation of local realism subsequent to entanglement swapping of partially depolarised states. We consider different configurations of the process of entanglement swapping. The strength of violation of local realism by the state obtained after entanglement swapping, is compared with the one for the input states. We obtain a kind of superadditivity of violation of local realism for Werner states consequent upon entanglement swapping involving Greenberger-Horne-Zeilinger state measurements. This indicates that checking for violation of local realism, in the state obtained after entanglement swapping, can be a *method* for detecting entanglement in the input state of the swapping procedure.

I. INTRODUCTION

Quantum nonseparability, in its operational sense, is the existence of states which cannot be prepared by distant observers acting locally and without any supplementary quantum channel. So it may seem that particles which do not share a common past (i.e., which have not been acted on by an interaction Hamiltonian) cannot be nonseparable (or entangled). Surprisingly however, two particles *can* get entangled even if they do not share a common past. This is achieved in entanglement swapping [1, 2, 3, 4]. The phenomenon was experimentally confirmed in [5].

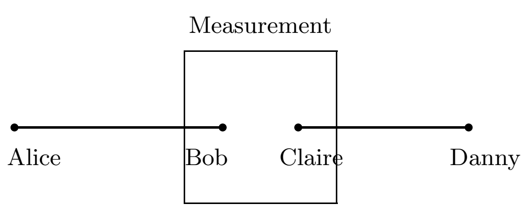

FIG. 1: Entanglement swapping between two states.

Let us first describe very briefly the phenomenon of entanglement swapping. Suppose Alice and Bob share an entangled state. Similarly Claire and Danny also share some entangled state. See Fig. 1. Now the question is as follows: Can it be possible that Alice's and Danny's particles become entangled without an interaction between their particles?

In Refs. [1, 2, 3, 4], the authors have shown that the answer is Yes. If their partners Bob and Claire (whose particles are entangled with the particles of Alice and Danny respectively) come together and make a measurement in a suitable basis and communicate their measurement results clasically (say, by phone call), then Alice's and Danny's particles may become entangled.

A simple example of this phenomenon can be seen if one has two singlets, $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, one of which is shared by Alice and Bob, and the other by Claire and Danny. Now Bob and Claire make jointly a projection measurement (on their parts of the two singlets) in the Bell basis, which is given by (with the + sign applying to states with odd indices)

$$\begin{aligned} |B_{1,2}\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \\ |B_{3,4}\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \end{aligned} \quad (1)$$

It is easy to check that if Bob and Claire communicate (over a classical channel) the result of their measurement to Alice and Danny, they will know that they share one of the Bell states given by Eq. (1). Note, that depending on the measurement results, unitary operations σ_x, σ_y, σ_z, or I may be performed by Alice (or Danny) on her (his) qubit to obtain just a singlet. (I is the identity operator on the qubit Hilbert space, and σ_x, σ_y, σ_z are the Pauli matrices.) The particles of Alice and Danny are completely independent, and nevertheless they share entanglement after Bob and Claire's Bell measurement (and sending its outcome to them). Note that entanglement swapping can be seen as a specific case of teleportation [6]. In the entanglement swapping process, Bob and Claire make a measurement on their systems and send (teleport) the qubit (say, Bob's subsystem) through a channel to Danny. And after communication to Danny, Alice and Danny share an entangled state. Actually, if all the parties agree on the desired output state of the swapping procedure, Bob and Claire can communicate their results only to Danny, and Alice does not need to know the content of the communication.

In this paper, we investigate various entanglement swapping schemes which involve non-perfect initial states. These will be, for simplicity, modeled as partially depolarised states [7]. We shall also investigate to what extent the states resulting out of the entanglement swapping process violate local realism. We address the case where the swapping itself (i.e., the measurement required for swapping) is perfect.

The parent states considered here are the Werner mixtures of certain pure states (say $|\psi\rangle$, shared between n partners) and the white noise [7]:

$$\varrho = p\,|\psi\rangle\,\langle\psi| + (1-p)\varrho_{noise}.$$

The parameter p will be called here visibility. Clearly it shows to what extent the processes that can be described to $|\psi\rangle$ are operationally visible despite the presence of noise. It can be associated with the notion of visibility in multipartite interference experiments. We shall study the relation of the visibility parameter for the initial states, and the states after swapping. This will be done in various configurations:

1. Chain configuration: A chain of entanglement swappings involving initially a sequence of pairs (sharing the parent states). Bell measurements, i.e. measurements projecting onto the Bell states given by Eq. (1), are performed upon two particles of all adjacent pairs (see Fig. 2 for the case of two entanglement swappings with three pairs). This is described in section II.

2. Star configuration: A generalized entanglement swapping involving initially N parent states (each consisting of M particles). An N qubit GHZ state measurement [8] is made on N qubits, each belonging to a different state. This is dicussed in section III. GHZ state measurement projects onto the GHZ basis. The 3-qubit GHZ basis, for example, consists of the states

$$\begin{aligned} G_{1,2} &= \tfrac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle),\\ G_{3,4} &= \tfrac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle),\\ G_{5,6} &= \tfrac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle),\\ G_{7,8} &= \tfrac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \end{aligned} \tag{2}$$

   where again the + sign applies to states with odd indices. Similarly one may define an N-qubit GHZ basis by considering the binary decompositions of $2^N - 1$.

We shall be interested in whether the resulting states in different forms of entanglement swapping are nonclassical. As our bench-mark of nonclassicality, we shall use the threshold value of visibility allowing for violation of suitable Bell inequalities [9, 10]. That is, we compare the critical visibility for violation of local realism of the state obtained after entanglement swapping, with the critical visibility for violation in the input state (parent state) itself. We obtain a kind of superadditivity in violation of local realism, for the case of Werner states in the Hilbert space of dimension $2 \otimes 2$, consequent to entanglement swapping in a specific scenario (section III F). In the concluding section (section IV), we find that such superadditivity exists for other states also in suitably chosen configurations of entanglement swapping. We indicate that checking for violation of local realism in the state obtained after entanglement swapping in suitably chosen configurations, can be an efficient entanglement witness for the input state (of the swapping procedure).

II. CHAIN CONFIGURATION OF ENTANGLEMENT SWAPPING

In this section, we will compare the visibilities of the input state to that of the swapped state, in the case of entanglement swapping between pairs of states in a chain configuration. See Fig. 2 for the case of two entanglement swapping of three pairs of states in a chain.

A. Swapping in a chain of two states: "Loss" in the region of violation of local realism

Let us begin by considering the case of entanglement swapping between two pairs. Consider the $2 \otimes 2$ dimensional Werner state [11]

$$\rho = p\,|B_1\rangle\,\langle B_1| + (1-p)\rho^{(2)}_{noise}. \tag{3}$$

(In this paper, we denote the completely depolarised state of n qubits, $I_n/2^n$, as $\rho^{(n)}_{noise}$, where I_n is the identity operator of the Hilbert space of n qubits.) The state ρ is entangled when $p > \frac{1}{3}$, but the state violates local realism only for $p > \frac{1}{\sqrt{2}}$. Let Alice (A) and Bob (B) share the state ρ, and let Claire (C) and Danny (D) also share such a state [12]. Bob and Claire (who are together) make a measurement on their part of the two states, in the Bell basis $\{B_i\}$ given by eqs. (1) (see Fig. 1). We are interested in whether the state of the qubits of Alice and Danny after the swap, violates local realism. After the Bell measurement, if the measurement result is B_1, the state shared by Alice and Danny is a Werner state of the form

$$\xi^{(2)}_{AD} = p^2\,|B_1\rangle\,\langle B_1| + (1-p^2)\rho^{(2)}_{noise}.$$

Since $\xi^{(2)}_{AD}$ is a Werner state, it is entangled for $p > \frac{1}{\sqrt{3}}$, but violates local realism when $p > \left(\frac{1}{2}\right)^{\frac{1}{4}}$. Of course, the same condition is obtained for the other Bell measurement outcomes. Therefore the region in which the final state $\xi^{(2)}_{AD}$ violates Bell inequalities is strictly contained in the region in which the initial state ρ_{AB} has the same property. We see that there is a region of p, namely $p \in (\frac{1}{\sqrt{2}}, \left(\frac{1}{2}\right)^{\frac{1}{4}})$, for which the output state will not be able to show any violation of local realism (but it is still entangled), whereas the input states do violate in that region. Therefore we have a "loss in the region of violation of local realism" after entanglement swapping.

B. Chain of three states: Further "Loss"

Let us next consider swappings between *three* Werner states, each of dimension $2 \otimes 2$. This can be considered in two different scenarios. It can be either in a chain, scenario as considered below, or it can be in a "star" scenario [2], as considered in the section III.

Suppose A and B share the Werner state ρ given by Eq. (3). And so does C and D as well as E and F. If we now consider the "chain" configuration (schematically represented in Fig. 2), where B and C (as well as D and E) come together to perform Bell measurements (in Fig. 2, it has been shown by boxes), the swapped state between A and F is

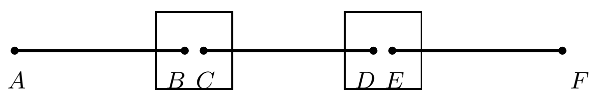

FIG. 2: A chain of two swappings. Boxes represent Bell basis measurements that have been performed by B, C and D, E.

$$p^3 \left|B_1\right\rangle \left\langle B_1\right| + (1-p^3)\rho^{(2)}_{noise}, \tag{4}$$

which is again a Werner state. The swapped state violates local realism for $p > \left(\frac{1}{2}\right)^{\frac{1}{6}}$. Therefore from the perspective of violation of Bell inequalities, we have a further "loss", in the sense described in the preceeding subsection.

C. Chain of N states: "Loss" increases with N

This phenomenon of "loss" becomes more and more pronounced as the number of swappings is increased. Starting with N initial Werner state shared between A_k and B_k ($k = 1, 2, \ldots, N$), the swapped state between A_1 and B_N (after Bell measurements performed by B_1A_2, B_2A_3, ..., $B_{N-1}A_N$) is again the Werner state

$$p^N \left|B_1\right\rangle \left\langle B_1\right| + (1-p^N)\rho^{(2)}_{noise}.$$

Hence the swapped state violates local realism for

$$p > \left(\frac{1}{2}\right)^{\frac{1}{N}}.$$

Therefore in the case of a series of a large number of entanglement swappings, the swapped state can violate local realism only when initial state is almost pure.

Note that if we consider a chain of N Werner states with different visibilities, i.e. if

$$p_k \left|B_1\right\rangle \left\langle B_1\right| + (1-p_k)\rho^{(2)}_{noise}$$

is shared between A_k and B_k ($k = 1, 2, \ldots, N$), then the swapped state between A_1 and B_N is the Werner state

$$p_1p_2\ldots p_N \left|B_1\right\rangle \left\langle B_1\right| + (1-p_1p_2\ldots p_N)\rho^{(2)}_{noise}.$$

Therefore, again we have that the region of violation of local realism of the swapped state is strictly smaller than the region of violation of the parent states in the $(p_1, p_2, \ldots, p_N)$-space. The former is vanishing when for sufficiently large N.

III. A STAR CONFIGURATION ENTANGLEMENT SWAPPING

In this section, we consider entanglement swapping in a different configuration, than that was considered in section II. We assume a multiparty situation in which initially disjoint subsets of parties share entangled states. In the next stage, single representatives of each subset of parties meet together and perform a GHZ-state measurement. The result of the measurement is sent to the remaining parties. This procedure results in an entangled state shared by them. We shall call this type of entanglement swapping as entanglement swapping in a "star configuration".

A. A star swapping between three states

Consider entanglement swapping in a "star" configuration for three parent states (as is schematically represented in Fig. 3). Suppose that pairs AB, CD and EF, each share the same Werner state $\rho = p\left|B_1\right\rangle \left\langle B_1\right| + (1 - p)\rho^{(2)}_{noise}$. A, C, and E come together and perform a measurement in their $2 \otimes 2 \otimes 2$ dimensional Hilbert space, in the GHZ basis as given in Eq. (2). After the measurement, if G_1 clicks, then B, D, F share the state

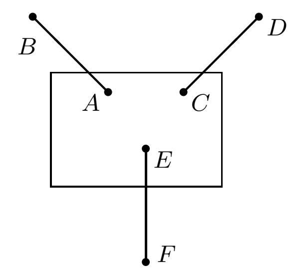

FIG. 3: A star configuration swapping. A GHZ basis measurement is performed on A, C, and E. This is represented by a box.

$$\begin{aligned} \xi^{(3)}_{BDF} &= p^3 \left|G_1\right\rangle \left\langle G_1\right| + (1-p^2)\rho^{(3)}_{noise} \\ &+ \tfrac{1}{2}p^2(1-p)(\left|000\right\rangle \left\langle 000\right| + \left|111\right\rangle \left\langle 111\right|). \end{aligned} \tag{5}$$

Other measurement results give the same state upto local unitary transformations. Note that the swapped state now is not a mixture of white noise and $\left|G_1\right\rangle \left\langle G_1\right|$ only and this is so whenever $p \neq 1$.

We will now use the Mermin-Klyshko (MK) inequalities to study the violation of local realism by the swapped state (see Appendix).

Let us first calculate $\mathrm{tr}\left(B_3\xi^{(3)}_{BDF}\right)$. (See Eq. (A1).)

Suppose that the observables are chosen from the $x{-}y{-}$

plane [13]. That is, we choose

$$\begin{aligned} \sigma_{a_j} &= |+,\phi_j\rangle\langle+,\phi_j| - |-,\phi_j\rangle\langle-,\phi_j| \\ \sigma_{a_j'} &= \left|+,\phi_j'\right\rangle\left\langle+,\phi_j'\right| - \left|-,\phi_j'\right\rangle\left\langle-,\phi_j'\right|, \end{aligned} \tag{6}$$

where

$$\begin{aligned} |\pm,\phi_j\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle \pm e^{i\phi_j}|1\rangle) \\ \left|\pm,\phi_j'\right\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle \pm e^{i\phi_j'}|1\rangle). \end{aligned}$$

The only term in the state given by (5), that will contribute to the expression $\operatorname{tr}\left(B_3\xi^{(3)}_{BDF}\right)$, is

$$\frac{p^3}{2}(|000\rangle\langle111| + |111\rangle\langle000|).$$

(This observation would help us in the more general cases that we consider in the succeeding subsections.)

For the GHZ state $|G_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, one has

$$\max \operatorname{tr}(B_3 |G_1\rangle\langle G_1|) = 2,$$

and this maximal violation of local realism by the GHZ state is reached in the $x-y$-plane. The contribution to $\operatorname{tr}(B_3|G_1\rangle\langle G_1|)$ is only from the term $\frac{1}{2}(|000\rangle\langle111| + |111\rangle\langle000|)$. Therefore the maximal value reached by $\operatorname{tr}\left(B_3\xi^{(3)}_{BDF}\right)$, for any choice of ϕ_j and ϕ_j' by the parties, is

$$\max \operatorname{tr}\left(B_3\xi^{(3)}_{BDF}\right) = 2p^3. \tag{7}$$

Consequently, the state $\xi^{(3)}_{BDF}$ violates a MK inequality for $\max \operatorname{tr}(B_3\xi^{(3)}_{BDF}) > 1$, i.e. for

$$p > \left(\frac{1}{2}\right)^{\frac{1}{3}} \simeq .7937.$$

Our initial Werner state ρ violates Bell inequalities when

$$p > \frac{1}{\sqrt{2}} \simeq .7071.$$

One should compare this with the case of entanglement swapping between *two* Werner states, where the swapped state gives violation for

$$p > \left(\frac{1}{2}\right)^{\frac{1}{4}} \simeq .8409.$$

In considering violation of local realism by the state $\xi^{(3)}_{BDF}$, we have used only the Mermin-Klyshko inequalities. However in this case, one can also consider the WWWZB inequalities [14, 15, 16], which are a necessary and sufficient condition for the violation of local realism by the N-qubit correlations of an arbitrary state of N qubits, when there are two settings at each site.

Let us first define the correlation tensor for N-qubit states. An N-qubit state ρ can always be written down as

$$\frac{1}{2^N}\sum_{x_1,\ldots,x_N=0,x,y,z} T_{x_1\ldots x_N}\sigma^{(1)}_{x_1}\otimes\ldots\otimes\sigma^{(N)}_{x_N}, \tag{8}$$

where $\sigma^{(k)}_0$ is the identity operator and the $\sigma^{(k)}_{x_i}$'s ($x_i = x,y,z$) are the Pauli operators of the k-th qubit. The coefficients

$$T_{x_1\ldots x_N} = \operatorname{tr}(\rho\sigma^{(1)}_{x_1}\otimes\ldots\otimes\sigma^{(N)}_{x_N}), \quad (x_i = x,y,z) \tag{9}$$

are elements of the N-qubit correlation tensor $\hat{T}$ and they fully define the N-qubit correlation functions of the state ρ.

Consider now the state $\xi^{(3)}_{BDF}$, obtained via entanglement swapping, as given in Eq. (5). One can check that the three-qubit correlation tensor $\hat{T}$ of this state, contains only those terms which are also present for the GHZ state G_1. Precisely, the correlation tensor $\hat{T}$ of $\xi^{(3)}_{BDF}$, is given by

$$\begin{aligned} \hat{T}_{\xi^{(3)}_{BDF}} = \; & p^3(\vec{x}_1\otimes\vec{x}_1\otimes\vec{x}_1 - \vec{x}_1\otimes\vec{x}_2\otimes\vec{x}_2 \\ & -\vec{x}_2\otimes\vec{x}_1\otimes\vec{x}_2 - \vec{x}_2\otimes\vec{x}_2\otimes\vec{x}_1), \end{aligned}$$

whereas the correlation tensor of the GHZ state G_1 is just

$$\begin{aligned} \hat{T}_{G_1} = \; & \vec{x}_1\otimes\vec{x}_1\otimes\vec{x}_1 - \vec{x}_1\otimes\vec{x}_2\otimes\vec{x}_2 \\ & -\vec{x}_2\otimes\vec{x}_1\otimes\vec{x}_2 - \vec{x}_2\otimes\vec{x}_2\otimes\vec{x}_1, \end{aligned}$$

where $\vec{x}_1 = \vec{x}$ and $\vec{x}_2 = \vec{y}$. Hence, when the quantum correlation function is computed by inserting $\hat{T}_{\xi^{(3)}_{BDF}}$ into the generalised Bell inequality of WWWZB, one gets the value which is by factor $2p^3$ greater than the one allowed by local realism. This is because for the GHZ state, the value is by factor 2 greater than the one allowed by local realism. This maximal value ($2p^3$) is attained in the $x-y$ plane, and was already obtained (in Eq. (7)) for the state $\xi^{(3)}_{BDF}$, when we considered the MK inequalities. Therefore the state $\xi^{(3)}_{BDF}$ violates local realism for $p > (1/2)^{1/3}$. Moreover, from our considerations of the WWWZB inequalities, we have that for lower values of the parameter p, the three-qubit correlations of $\xi^{(3)}_{BDF}$ have a local realistic model for two measurement settings at each site.

B. Other forms of the star configuration of swapping

In the preceeding subsection, we have shown that the "star configuration" leads to stronger resistance to noise admixture than with Bell measurements in the "chain configuration" (discussed in section II). The parent states that we considered (in the preceeding subsection) were bipartite states. Let us now consider the case of entanglement swapping with measurements in a GHZ basis, when the parent states are multipartite states.

Consider therefore the state

$$\rho_3 = F \left|G_1\right\rangle \left\langle G_1\right| + (1-F)\rho^{(3)}_{noise}$$

where $\left|G_1\right\rangle = \frac{1}{\sqrt{2}}(\left|000\right\rangle + \left|111\right\rangle)$. This state violates local realism for

$$F > \frac{1}{2}.$$

Let two such states be shared between A, B, C and D, E, F, with A and D placed together (Fig. 4). A and D

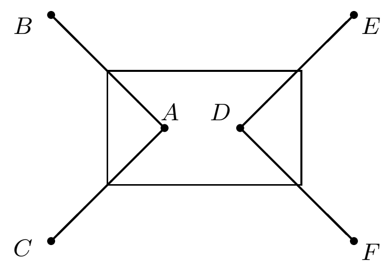

FIG. 4: A star configuration swapping. A Bell measurement is performed on A and D which is denoted by a box.

make a Bell measurement on their parts of the two states. After the measurement, the resulting state violates the MK inequalities in the $x-y$-plane, for

$$F > \left(2^{\frac{3}{2}}\right)^{-\frac{1}{2}} \simeq .5946.$$

Note here that we do not need the explicit form of the state. The terms that contribute to the violation of MK inequality in the $x-y$-plane are $\left|0\ldots0\right\rangle\left\langle1\ldots1\right|$ and $\left|1\ldots1\right\rangle\left\langle0\ldots0\right|$.

With three ρ_3's, and a swapping in the 3-qubit GHZ basis (given in Eq. (2)) on the 3 qubits (one from each of the ρ_3's) (see Fig. 5), the MK inequality is violated in the $x-y$-plane for

$$F > \left(2^{\frac{5}{2}}\right)^{-\frac{1}{3}} \simeq .5612.$$

Thus the following picture is emerging: entanglement swapping involving GHZ measurements is less fragile (to violation of local realism) than Bell measurements, with repect to the noise admixtures in the initial states.

C. The general star configuration entanglement swapping

We will now generalize the entanglement swapping process in the star configuration. Consider the following M-qubit state:

$$\rho_M = V \left|GHZ_M\right\rangle \left\langle GHZ_M\right| + (1-V)\rho^{(M)}_{noise} \qquad (10)$$

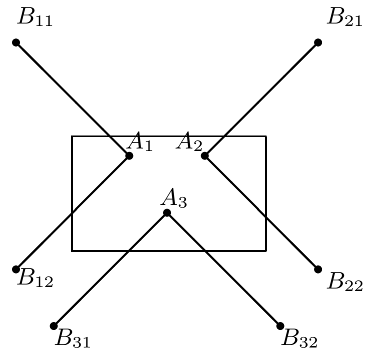

FIG. 5: A star configuration swapping. A_1, B_{11}, B_{12} and A_2, B_{21}, B_{22} and A_3, B_{31}, B_{32} share noisy GHZ states. All B's are at distant locations but A's are in the same lab. A GHZ basis measurement is performed by A_1, A_2, and A_3, as depicted in the figure by a box.

where $\left|GHZ_M\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle^{\otimes M} + \left|1\right\rangle^{\otimes M}\right)$. Take N copies of ρ_M. The i-th copy ($i = 1, 2, \ldots, N$) is shared between A_i and $B_{i1}, B_{i2}, \ldots, B_{i(M-1)}$. We suppose that all A_is are at the same location of the observer called Alice. (The schematic diagram in Fig. 5 is drawn when both N and M are three.) She makes a measurement in the N-qubit GHZ basis. (See Eq. (2) for the three qubit GHZ basis.) As in the previous cases, we take the observables in the $x-y$-plane, i.e., the ones given by Eq. (6), in the MK inequality. Here also we do not need the explicit form of the state. The terms that contribute to the violation of MK inequality in the $x-y$-plane are $\left|0\ldots0\right\rangle\left\langle1\ldots1\right|$ and $\left|1\ldots1\right\rangle\left\langle0\ldots0\right|$. Therefore we obtain that the resulting $N(M-1)$-qubit state violates this inequality for

$$V > V_N^{(M)} \equiv \left(2^{\frac{N(M-1)-1}{2}}\right)^{-\frac{1}{N}}. \qquad (11)$$

This expression is easily obtained once we remember our observation for the derivation of Eq. (7) [17].

D. There is no loss in the asymptotic regime

We remember that our parent state ρ_M, as given in (10), violates local realism for

$$V > \left(\frac{1}{\sqrt{2}}\right)^{M-1}.$$

Note that $V_N^{(M)}$ (as given by Eq. (11)) is monotonically decreasing with respect to N. A plot of the critical visibility $V_N^{(M)}$ for $M = 2$, that is the visibility obtained when the swapping in a star configuration is performed on N number of copies of two qubit Werner states, is given in Fig. 6. It clearly shows the monotonic decrease of the critical visibility in N.

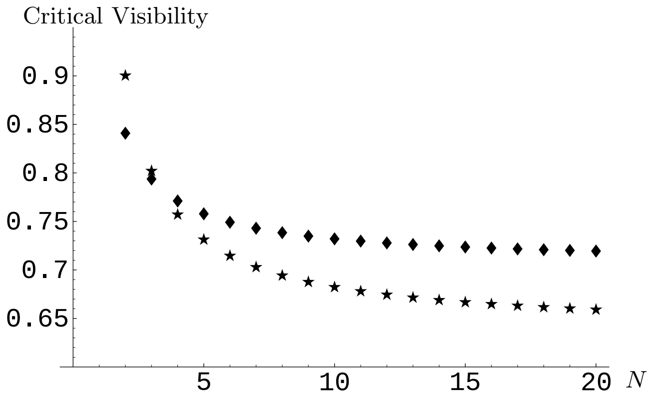

FIG. 6: (a) Plot of the critical visibilities $V_N^{(M)}$ for $M = 2$ (the stars). This is the critical visibility required (in the parent state) for violation of local realism by the swapped state, in the case when one representative from each of N Werner states (Eq. (3)) come together to perform an entanglement swapping in the N-qubit GHZ basis (see Fig. 7). The violation of local realism in the swapped state is considered by using the MK inequalities. (b) The critical visibility V_N^f (the diamonds) is the that for violation of local realism by considering functional Bell inequalities in the same N-qubit swapped state. The figure shows their relative monotonic decrease in N.

Thus the system is surprisingly robust to noise admixture, with respect to violation of local realism in the following sense. The amount of (white) noise that the parent state can afford so that the state after entanglement swapping still violates local realism, increases monotonically as we consider swapping between higher number of parties, in a star configuration. Moreover, one has

$$V_N^{(M)} \to \left(\frac{1}{\sqrt{2}}\right)^{M-1} \text{ as } N \to \infty.$$

This shows that the amount of noise that the parent state can afford, so that the state obtained after entanglement swapping violates the MK inequality, in the asymptotic limit of arbitrarily large number of subsystems (in the way considered above, that is in the star situation), *coincides* with the amount of noise that can be afforded by the parent state itself to violate local realism. The loss in the region of violation of local realism is more and more recovered as we consider entanglement swapping between higher and higher number of parties and ultimately in the asymptotic limit, there is *no* loss in the region of violation of local realism.

In this general situation, the state obtained after performing the entanglement swapping, is an incoherent mixture of some product states and a "weakened" GHZ state (i.e., a GHZ state admixed with white noise). The product states contribute only to the $T_{z\dots z}$ component of the the correlation tensor $\hat{T}$ (cf. Eq. (9)) of the state obtained after entanglement swapping. Here we have considered violation of the MK inequalities (by the state obtained after swapping) only in the $x-y$-plane. This is because the gradual reduction of loss of the region of violation of local realism after entanglement swapping, and disappearance of this loss asymptotically, is already obtained in this plane. However we do not rule out a faster reduction of loss if all the WWWZB inequalities are considered.

From the perspective of the recent works indicating that Bell inequality violation is a signature of "useful entanglement" [18, 19, 20, 21], our result here can be also viewed as showing (in a particular case) that in an entanglement swapping process, this useful entanglement is lost, but this loss may be asymptotically vanishing. Below in sections III E and III F, we will show that useful entanglement can even be "gained" in an entanglement swapping process, and this gain can be possible even without going into the asymptotic regime. Here by "gain", we mean a situation in which the swapped state violates local realism, even when the parent state does not violate. That is, there exists values of the visibility V, for which the parent states do not, while the swapped state does violate Bell inequalities after performing the swap. We will perform the swapping in a star configuration.

E. Star entanglement swapping in the light of functional Bell inequality

The Bell inequalities we have considered upto now are the ones in which there is only a finite number of (in fact, two) settings per local site. However there are Bell inequalities in which one may consider even a continuous range of settings of the local apparatus, as described in Appendix A 2.

Let us consider violation of local realism by the swapped state as revealed by a functional Bell inequality. For simplicity, let us consider the parent states to be a two-qubit state, although all our considerations can be generalised to a parent state of higher number of qubits. Suppose therefore that the Werner state, given by Eq. (3), is shared between two parties, A and B. Numerical calculations have indicated that the Werner state violates local realism for $p > \frac{1}{\sqrt{2}}$ even for a high number of settings per observer [22, 23]. We use as a working hypothesis that $p = \frac{1}{\sqrt{2}}$ is indeed the threshold value below which there exist an explicit local realistic model which returns the quantum predictions for the continuous range of settings.

Consider now the "star" configuration described before, where A_1B_1, A_2B_2, $\dots$, A_NB_N share N Werner states, each given by Eq. (3) (see Fig. 7). The A_i's come together and perform a measurement in the N-qubit GHZ basis (see Eq. (2)) as has been discussed previously. We will now consider violation of local realism of the swapped state, by using the functional Bell inequalities.

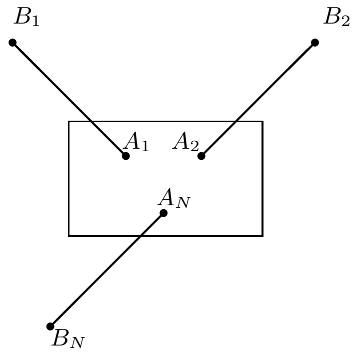

FIG. 7: N Werner states are distributed between A_i and B_i (for $i = 1, 2, \ldots, N$) and a GHZ basis measurement is performed at $A_1, A_2, \ldots, A_N$.

Consider the local observable at the jth location to be

$$\sigma_{a_j}(\phi_j) = |+, \phi_j\rangle \langle +, \phi_j| - |-, \phi_j\rangle \langle -, \phi_j| \,. \tag{12}$$

where

$$|\pm, \phi_j\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \pm e^{i\phi_j} |1\rangle \right). \tag{13}$$

The aggregate ξ_j of local parameters, in the functional Bell inequality, at the jth site is just the single parameter ϕ_j here. One can now easily show that the swapped state violates local realism (by violating the functional Bell inequality with the observables as defined in Eq. (12)) for [24, 25, 38]

$$p > V_N^f \equiv \frac{2}{\pi} (2)^{\frac{1}{N}} \,. \tag{14}$$

V_N^f is plotted in Fig. 6 and compared with the critical visibility $V_N^{(2)}$, for the same swapped state, but when violation of the MK inequalities are considered.

F. A kind of superadditivity for Werner states

There is an important consequence of the relation (14). For $N \geq 7$, the critical visibility V_N^f is strictly less than $\frac{1}{\sqrt{2}}$, which is the critical visibility for the Werner state to violate local realism (on the basis of the multi-settings numerical results in Ref. [22]).

Therefore, in the process of entanglement swapping, there seems to be a kind of superadditivity with respect to violation of local realism. Suppose Alice shares 7 Werner states with 7 Bobs, $B_1, \ldots, B_7$. Each of the states is given by Eq. (3), with a visibility $p \in (\frac{2}{\pi}, \frac{1}{\sqrt{2}})$. Such Werner states do not violate local realism [22]. Now suppose that Alice makes a measurement in the generalised GHZ basis and communicates her result to all the Bobs, over a classical channel. The state created at the Bobs, violates local realism (by violating the functional Bell inequality, as discussed above) for (cf. (14))

$$p > \frac{2}{\pi} (2)^{\frac{1}{7}} \simeq .7029,$$

which is strictly less than $\frac{1}{\sqrt{2}} \simeq .7071$. Yet for such visibilities which are lower than $\frac{1}{\sqrt{2}}$, any *single* pair of particles shared between Alice and any one Bob, will not be able to violate local realism. (Recall that we have assumed that taking more settings at each site does not help to improve the critical visibility of violation of local realism by the Werner state [22].) It is in this sense that we obtain a kind of "superadditivity" in violation of local realism.

For sufficiently large N,

$$V_N^f \to \frac{2}{\pi} \simeq .6366.$$

Let us note here a surprising coincidence. An explicit construction of local hidden variable model for the Werner state exists (till date) for all possible projection measurements by the two parties, for just $p \leq \frac{2}{\pi}$ [26, 27].

It must be stressed that the kind of superadditivity obtained here is not related to a distillation protocol [28]. As distinct from a distillation protocol, we do not consider measurements depending on previous measurements. Also in our case, only the Alices are together while the Bobs can be far apart (collective operations are required on both ends in the usual distillation protocols).

In Ref. [29], two Werner states shared by A_1B_1 and A_2B_2, respectively, are shown to violate local realism, although the individual states are non-violating. But in Ref. [29], collective tests are required at both ends. That is, both A_1 and A_2, and B_1 and B_2 are required to be together. In our case, although the Alices must be together, the Bobs are separated. Therefore the "superadditivity" reached in this subsection is of a different kind than the one in Ref. [29].

IV. DISCUSSION

We have shown (under a plausible assumption) that if the initial state has a local realistic model, after performing entanglement swapping, the final swapped state can violate local realism. This was obtained by using the initial states as Werner states. We regard this as a kind of superadditivity for Werner states.

However, there can be a general question: Consider a state that is entangled and yet does not violate local realism. Is it possible to show that there exists some entanglement swapping process, after which the swapped state will violate local realism?

The "star" configuration entanglement described in this paper, gives a positive answer to this question for the case of Werner states in certain ranges of the visibility parameter (under a plausible assumption). Can such configurations be obtained in other cases also? Can this be a general method of obtaining violation of local realism?

In this respect, let us consider another interesting example. The state

$$\rho_\lambda = \lambda \left|\psi\right\rangle \left\langle\psi\right| + \frac{1-\lambda}{2}(\left|00\right\rangle \left\langle 00\right| + \left|11\right\rangle \left\langle 11\right|), \qquad (15)$$

where $\left|\psi\right\rangle = a\left|01\right\rangle - b\left|10\right\rangle$ (and $\lambda > \frac{1}{2(1-ab)}$), is entangled whenever $\lambda > 1/(1+2ab)$ [30]. For $\lambda \leq 1/(1+a^2b^2)$, this state does not violate any Bell inequality. However, despite the fact that for $\lambda \in (\frac{1}{(1+2ab)}, \frac{1}{(1+a^2b^2)})$, the state ρ_λ can be modelled with local hidden variable models, it was shown in Ref. [31] that after a suitable local filtering operations, the resulting state violates local realism. Since the required operations are local, one may interpret this result as a sort of "self-superadditivity".

Consider the following entanglement swapping process. Assume that there are four observers Alice, Bob, Claire and Danny. Alice and Bob share a state ρ_λ, and so do the other two. After a Bell measurement performed by Bob and Claire, (whom we are at the same location), if the measurement result is $\left|B_1\right\rangle$, the state at AD, collapses into

$$\begin{aligned} \xi_{AD}^{(\rho_\lambda)} &= \tfrac{1}{A}[\tfrac{\lambda^2a^2b^2}{2}\left|B_1\right\rangle\left\langle B_1\right| \\ &+ \tfrac{(1-\lambda)^2}{8})(\left|00\right\rangle\left\langle 00\right| + \left|11\right\rangle\left\langle 11\right|) \\ &+ \tfrac{\lambda(1-\lambda)}{2}(a^2\left|01\right\rangle\left\langle 01\right| + b^2\left|10\right\rangle\left\langle 10\right|)], \end{aligned} \qquad (16)$$

where $A = \lambda^2a^2b^2 + (1-\lambda^2)/4$. This state violates local realism for

$$\lambda > \frac{1}{\sqrt{1+4(\sqrt{2}-1)a^2b^2}}.$$

Therefore the region of violation of local realism for the swapped state $\xi_{AD}^{(\rho_\lambda)}$ is *strictly greater* than that for the parent states (that is, for ρ_λ). Recall that in Ref. [31], it was shown that one can obtain a nonclassical state, after operating locally on an input classical state. Here we have used a different method to obtain the same end: after performing entanglement swapping on two copies of the input classical state, the swapped state can violate local realism.

Note that the "superadditivity" reported in this section (for the state ρ_λ), as also in section III F (for the Werner states), is of a different kind than in Ref. [31].

Importantly, note here that the superadditivity reported in section III F, is for Werner states. And for Werner states, one cannot reproduce the kind of "self-superadditivity" by using local filtering operations [32], as was done in [31].

From these examples, it seems that it may be a generic feature that an entangled state which satisfies local realism, will violate local realism after a suitable entanglement swapping procedure. If this is true, then this method can be used to detect entanglement in the laboratory. Suppose Alice and Bob who are in a different locations, share some state. They want to find out whether their shared state is entangled or not. One way is to do a Bell experiment and find whether their state violates local realism. If the state violates local realism, then they conclude that their state is entangled. If the state does not violate local realism, they cannot infer anything about the entanglement of the state. However Alice and Bob can apply the method discussed in this paper. They can perform entanglement swapping on some copies of the state in a suitable configuration, and then check whether the resulting state violates local realism. If yes, then they can infer that the input state was entangled.

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APPENDIX A: BELL INEQUALITIES

For obtaining violation of local realism by the swapped state, we will consider two different types of multipartite Bell inequalities: multiparticle Mermin-Klyshko inequalities [33, 34, 35, 36, 37]. (subsection A 1) and the functional Bell inequality [38] (subsection A 2).

1. The Mermin-Klyshko inequalities

A Bell operator for the so-called Mermin-Klyshko (MK) inequality for N qubits (shared between observers A_1, A_2, ..., A_N) can be defined recursively as [39]

$$B_k = \frac{1}{2}B_{k-1}\otimes(\sigma_{a_k}+\sigma_{a_k^{'}}) + \frac{1}{2}B_{k-1}^{'}\otimes(\sigma_{a_k}-\sigma_{a_k^{'}}), \quad (A1)$$

with $B_k^{'}$ obtained from B_k by interchanging a_k and $a_k^{'}$, and

$$B_1 = \sigma_{a_1} \quad \text{and} \quad B_1^{'} = \sigma_{a_1^{'}}.$$

The party A_j is allowed to choose between the measurements σ_{a_j} and $\sigma_{a_j^{'}}$. Here $\vec{a_j}$ and $\vec{a_j^{'}}$ are two three-dimensional unit vectors ($j = 1, 2, \ldots, N$), and for example, $\sigma_{a_j} = \vec{\sigma}.\vec{a_j}$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

An N-qubit state η violates MK inequality if

$$\mathrm{tr}\,(B_N\eta) > 1.$$

2. The functional Bell inequalities

To study the violation of local realism of the swapped state, we will (along with the MK inequalities) also consider the functional Bell inequalities [38].

The functional Bell inequalities [38] essentially follow from a simple geometric observation that in any real vector space, if for two vectors h and q one has $\langle h \mid q \rangle < \parallel q \parallel^2$, then this immediately implies that $h \neq q$. In simple words, if the scalar product of two vectors has a lower value than the length of one of them, then the two vectors cannot be equal.

Let ϱ_N be a state shared between N separated parties. Let O_n be an arbitrary observable measured at the nth location ($n = 1, \ldots, N$). The quantum mechanical prediction E_{QM} for the correlation in the state ϱ_N, when these observables are measured, is

$$E_{QM}(\xi_1, \ldots, \xi_N) = \mathrm{tr}(O_1 \ldots O_N \varrho_N), \tag{A2}$$

where ξ_n is the aggregate of the local parameters at the nth site. Our objective is to see whether this prediction can be reproduced in a local hidden variable theory. A local hidden variable correlation in the present scenario must be of the form

$$E_{LHV}(\xi_1, \ldots, \xi_N) = \int d\lambda \rho(\lambda) \Pi_{n=1}^{N} I_n(\xi_n, \lambda), \tag{A3}$$

where $\rho(\lambda)$ is the distribution of the local hidden variables and $I_n(\xi_n, \lambda)$ is the predetermined measurement-result of the observable $O_n(\xi_n)$ corresponding to the hidden variable λ.

Consider now the scalar product

$$\langle E_{QM} \mid E_{LHV} \rangle = \int d\xi_1 \ldots d\xi_N E_{QM}(\xi_1, \ldots, \xi_N) E_{LHV}(\xi_1, \ldots, \xi_N) \tag{A4}$$

and the norm

$$\parallel E_{QM} \parallel^2 = \int d\xi_1 \ldots d\xi_N \left(E_{QM}(\xi_1, \ldots, \xi_N)\right)^2. \tag{A5}$$

If we can prove that a strict inequality holds, namely for all possible E_{LHV}, one has

$$\langle E_{QM} \mid E_{LHV} \rangle \leq B, \tag{A6}$$

with the number $B < \parallel E_{QM} \parallel^2$, we will immediately have $E_{QM} \neq E_{LHV}$, indicating that the correlations in the state ϱ_N are of a different character than in any local realistic theory. We then could say that the state ϱ_N violates the "functional" Bell inequality (A6), as this Bell inequality is expressed in terms of a typical scalar product for square integrable functions. Note that the value of the product depends on a continuous range of parameters (of the measuring apparatuses) at each site.

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$$V > V_N \cos\alpha \sin\alpha.$$

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$$\langle E_{QM}|E_{LHV}\rangle \leq 4^N < \parallel E_{QM} \parallel^2 = \frac{p^N}{2}(2\pi)^N,$$

that is whenever $p > V_N^f \equiv \frac{2}{\pi}2^{\frac{1}{N}}$.

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