Entangled graphs. II. Classical correlations in multiqubit entangled systems

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Bipartite correlations in multiqubit systems cannot be shared freely. The presence of entanglement or classical correlation on certain pairs of qubits may imply correlations on other pairs. We present a method of characterization of bipartite correlations in multiqubit systems using a concept of entangled graphs which has been introduced in our earlier work [M. Plesch and V. Bužek, Phys. Rev. A **67**, 012322 (2003)]. In entangled graphs, each qubit is represented by a vertex while the entanglement and classical correlations are represented by two types of edges. We prove by construction that any entangled graph with classical correlations can be represented by a *mixed* state of N qubits. However, not all entangled graphs with classical correlations can be represented by a pure state.

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I. INTRODUCTION

The laws of quantum mechanics impose strict bounds on bipartite entanglement in multipartite systems. This issue has been first addressed by Wootters *et al.* [1,2] who have derived bounds on shared bipartite entanglement in multiqubit systems. In particular, in their paper, O'Connors and Wootters [2] have searched for a state of a multiqubit ring with maximal possible entanglement between neighboring qubits. Another version of the same problem has been analyzed by Koashi, Bužek, and Imoto, [3] who have derived an explicit expression for the multiqubit, completely symmetric state (entangled web) in which all possible pairs of qubits are maximally entangled.

A more general approach has been suggested by Dür who has introduced a concept of *entanglement molecules* [4], that is, quantum structures such that each qubit is represented by a point (''atom'') while an entanglement between two qubits is represented by a ''bound.'' Dür has shown that under the condition that the ''strength'' of the bound, i.e., a particular value of the degree of entanglement, is arbitrary (though nonzero), an arbitrary entanglement molecule can be represented by a *mixed* state of a multiqubit system. On the other hand, Dür has considered just the condition of inseparability for a given set of pairs of qubits in the multiqubit system, but he did not impose a strict condition of separability for the remaining pairs of qubits. This issue has been addressed in our earlier paper [5] where we have introduced a concept of *entangled graphs*. In the graph, each qubit is represented by a vertex, and an edge between two vertices denotes entanglement between these two qubits (specifically, the corresponding two-qubit density operator is inseparable). By construction, we have proved that any entangled graph with N vertices and k edges can be associated with a *pure* multiqubit state.

In Refs. [1–5], the main issue has been the distribution of bipartite entanglement in multiqubit systems. On the other hand, it is of importance to understand how entanglement as well as classical correlations is shared in multiqubit systems. In order to illuminate this problem, we generalize the concept of entangled graphs. Specifically, we will consider entangled graphs with classical correlations. In the graph, each qubit is represented by a vertex and vertices can be connected by two types of edges; one type corresponds to entanglement between two specific qubits (the corresponding bipartite density operator is inseparable) while the second type corresponds to classical correlations (the corresponding bipartite density operator is separable but not factorized). The main result of our paper is that for any entangled graph with classical correlations, one can find a mixed state that is represented by this graph. We also prove that not every entangled graph with classical correlations can be represented by a pure state, though we find several categories of entangled graphs that can be associated with pure multiqubit states.

II. ENTANGLED GRAPHS WITH CLASSICAL CORRELATIONS

Let us consider a general state ρ of an N-qubit system S. Density matrices ρ_{ij} of all possible pairs in system S are defined as

$$\rho_{ij} = \mathrm{Tr}_{S\setminus\{i,j\}}(\rho), \tag{2.1}$$

where the trace is performed over the set of qubits $S\setminus\{i,j\}$, which denotes the whole system except two qubits i and j. In general, there exist two basic types of bipartite density matrices. Those fulfilling the separability condition (e.g., see Ref. [6])

$$\rho_{ij} = \sum_n \zeta_i^n \otimes \xi_j^n \tag{2.2}$$

are called separable, i.e., these density operators describe states of two qubits that are not entangled but they are *classically* correlated (providing $n \geqslant 2$). All other states are entangled, i.e., they are not separable.

In what follows, our task will be to use the concept of entangled graphs [5] to characterize bipartite correlations in multiqubit systems. First we note that when no entanglement between two qubits is present, two classes of bipartite density operators can be identified. These are (1) separable den-

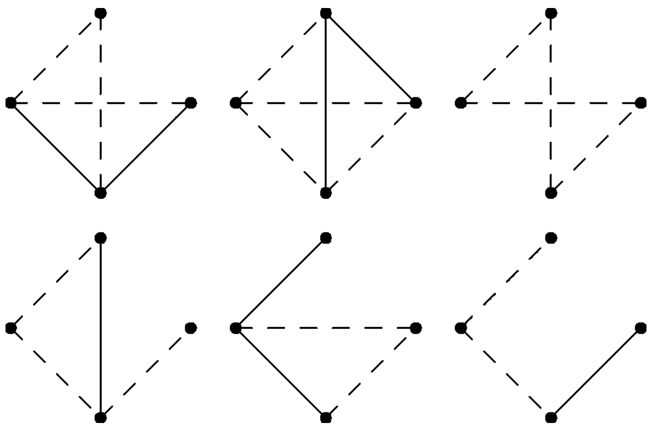

FIG. 1. Some examples of entangled graphs with classical correlation corresponding to states of four qubits. Solid edges are associated with entangled bipartite states, while dashed edges describe correlated but not entangled (i.e., separable but not factorized) bipartite states.

sity operators of form (2.2) and (2) density operators that are given by tensor products of single-particle density operators (i.e., the corresponding two-qubit density operator is factorized [7,8]). Correspondingly, we will divide a set of separable density matrices (2.2) into two categories; if the sum in Eq. (2.2) has at least two terms $(n>1)$, the corresponding density operators describe classically correlated bipartite states. On the other hand, if there is only one term present in the right-hand side of Eq. (2.2) then

$$\rho_{ij}=\rho_i\otimes\rho_j, \tag{2.3}$$

i.e., the bipartite density operator is equal to the tensor product of two single-qubit density operators and the corresponding two qubits are not correlated at all. The single-qubit density operators in Eq. (2.3) are obtained by the standard trace rule

$$\rho_i=\mathrm{Tr}_{S\setminus\{i\}}(\rho), \tag{2.4}$$

where $S\setminus\{i\}$ denotes the set of all qubits except the ith one. If condition (2.3) is fulfilled, then the corresponding two qubits are not correlated at all.

We remind ourselves that in the case of *entangled graphs* [5] (where only entangled and separable matrices were considered), an edge between two vertices has represented entanglement, whereas no edge has simply meant no entanglement. In what follows, we will consider three types of bipartite density operators; (1) nonseparable density operators that describe entangled pairs of qubits, (2) separable density operators that describe classically correlated pairs of qubits, and (3) factorable density operators that describe states of independent (uncorrelated) qubits. Therefore, we will need two types of edges—one that corresponds to the entanglement between two qubits, while the second corresponds to separable qubits that cannot be described by factorized density operators.

With these two types of edges, we can introduce the concept of an entangled graph with classical correlations (some examples of such graphs are presented in Fig. 1):

(1) A system of N qubits is represented by a graph with N vertices.

(2) Vertices in the graph can be connected by two types of edges.

(3) Entanglement edge between two vertices (solid line) denotes nonzero entanglement between relevant qubits; presence of entanglement implies also the presence of classical correlation.

(4) Correlation edge between two vertices (dashed line) denotes classical (and only classical) correlation between relevant qubits that are described by separable but not factorable density operator [see Eq. (2.2)].

(5) No edge between two vertices (no line) denotes no correlation between relevant qubits and the corresponding density operator is given by the tensor product of single-qubit density operators [see Eq. (2.3)].

By definition, for a given multipartite state (pure or mixed) it is always possible to construct a corresponding graph. We simply calculate all bipartite density operators and test for the presence of entanglement, as well as for the condition (2.3) associated with the absence of classical correlation. However, the inverse question is much more attractive: Given an *entangled graph with classical correlations*, is it possible to construct a state, which would be represented by this graph? This question implicitly contains another important issue: Does entanglement and classical correlations between specific pairs of qubits imply entanglement and/or classical correlation on other pairs of qubits in multiqubits systems?

A graph corresponding to N qubits is completely specified by two sets of nonordered pairs of vertices $\{i,j\}$. The first set S^E corresponds to entangled pairs; $\{i,j\}\in S^E \Leftrightarrow \{i,j\}$ are entangled. The second set S^C describes correlated pairs; $\{i,j\}\in S^C \Leftrightarrow \{i,j\}$ that are correlated. Each pair $\{i,j\}\notin S^C$ is completely uncorrelated, i.e., it is in a product state. It is worth remembering that $S^E\subset S^C$, i.e., each entangled pair is also classically correlated. We can also define a specific subset of S^C, the set of classically (and only classically) correlated pairs $S^{CC}=S^C/S^E$. We define also a vector (of the length N) $\vec{m}$, whose components m_i denote the number of qubits, which are uncorrelated with the ith qubit; e.g., m_i is the number of pairs $\{i,j\}\notin S^C$ with a fixed i. Let us denote $M=\frac{1}{2}\sum_{i=1}^{N}m_i$ as the total number of uncorrelated pairs. The inequalities

$$0\leqslant m_i\leqslant N-1,$$

$$0\leqslant M\leqslant \frac{N(N-1)}{2} \tag{2.5}$$

exhibit simple attributes of the system, that no particle can be uncorrelated with more than $(N-1)$ particles and that the maximum number of pairs of qubits in the system is equal to $N(N-1)/2$.

III. MIXED STATES

A mixed state of a quantum mechanical system is always determined by a larger number of parameters than a pure state of the same system. For instance, a pure state of a qubit is represented by a point on a Poincare sphere, that is, each pure state is determined by two parameters. On the other hand, a mixed state (a convex combination of pure states) is represented by a point inside a Poincare sphere and is determined by three parameters. In general, number of parameters, which are needed for a specification of a mixed state, is much larger than the number of parameters needed for specification of a pure state [9]. One of the consequences of this property of mixed states is that it is much easier to fulfill constraints imposed by the graph structure on mixed states with more "free parameters," than on pure states.

In what follows we present a mixed state, which is defined by the sets S^C and S^E. Then, we will prove that the bipartite density operators have all the desired properties, thus this state is represented by the graph specified by the sets S^E and S^C.

The mixed state of N qubits given by the expression

$$\begin{aligned}\rho=\frac{1}{2(N-1)^2}\Bigg\{&\left[N^2-3N+\frac{1}{2}M+2\right]|0\dots0\rangle\langle0\dots0|\\&+\sum_{i=1}^{N}\left[(N-1)-\frac{1}{2}m_i\right]|0\dots01_i0\dots0\rangle\langle0\dots01_i0\dots0|\\&+\sum_{\{i,j}\in S^E\;|0\dots01_i0\dots0\rangle\langle0\dots01_j0\dots0|\\&+\sum_{\{i,j}\in S^E\;|0\dots01_j0\dots0\rangle\langle0\dots01_i0\dots0|\\&+\sum_{\{i,j\}\notin S^C}\frac{1}{2}|0\dots01_i0\dots01_j0\dots0\rangle\langle0\dots01_i0\dots01_j0\dots0|\Bigg\}\end{aligned} \tag{3.1}$$

is characterized by a graph, specified by the number of vertices N and the sets S^E and S^C.

The density operator (3.1) is represented by a convex sum of pure states $|0\dots0\dots0\rangle$, $|0\dots01_i0\dots0\rangle$, $|0\dots01_i0\dots01_j0\dots0\rangle$, and $1/\sqrt{2}(|0\dots01_i0\dots0\rangle+|0\dots01_j0\dots0\rangle)$, so it describes a mixed state of N qubits.

In what follows, we show that for $\{i,j\}\in S^E$, the qubits i and j are entangled. In this case, the reduced (bipartite) density operator, obtained from Eq. (3.1) by tracing over relevant qubits, has the form

$$\rho_{ij}^E=\frac{1}{2(N-1)^2}\begin{pmatrix}2N^2-6N+4&0&0&0\\0&N-1&1&0\\0&1&N-1&0\\0&0&0&0\end{pmatrix}. \tag{3.2}$$

One could use the Peres-Horodecki criterion [6] to determine whether the density matrix (3.2) describes an entangled state of two qubits. Instead of this, we can calculate the concurrence [10] of this state which would allow us to determine the strength of the entanglement. For the density matrix (3.2) the concurrence reads $1/(N-1)^2$, thus it is larger than zero which means that the pair of qubits i and j is indeed entangled.

For every $\{i,j\}\in S^{CC}=S^C/S^E$, the qubits i and j have to be correlated, but not entangled. We can calculate the corresponding reduced density operator

$$\rho_{ij}^C=\frac{1}{2(N-1)^2}\begin{pmatrix}2N^2-6N+4&0&0&0\\0&N-1&0&0\\0&0&N-1&0\\0&0&0&0\end{pmatrix}. \tag{3.3}$$

This matrix is diagonal and the partial transposition would not change it at all. This proves that the corresponding bipartite state is not entangled.

From Eq. (3.3), we can find density operators ρ_i and ρ_j of individual qubits:

$$\rho_i=\rho_j=\frac{1}{2(N-1)^2}\begin{pmatrix}2N^2-5N+3&0\\0&N-1\end{pmatrix}. \tag{3.4}$$

In order to test the presence of classical correlations we will utilize the condition (2.3). The tensor product of two states (3.4) corresponds to uncorrelated (factorized) two-qubit density operator

$$\rho_i\otimes\rho_j=\frac{1}{2(N-1)^2}\begin{pmatrix} 2N^2-6N+\frac{9}{2} & 0 & 0 & 0\\ 0 & N-\frac{3}{2} & 0 & 0\\ 0 & 0 & N-\frac{3}{2} & 0\\ 0 & 0 & 0 & \frac{1}{2}\end{pmatrix} \tag{3.5}$$

and we immediately see that $\rho_i\otimes\rho_j\neq\rho_{ij}^C$, i.e., the pair of qubits in the state (3.3) is correlated.

For the rest of the pairs $\{i,j\}\notin S^C$, the reduced density operator ρ_S can be found to be given by Eq. (3.5), which means that these qubits are not correlated at all since $\rho_{ij}^S=\rho_i\otimes\rho_j$.

Herewith, we have proved that the state (3.1) is represented by a graph specified by the two sets S^E and S^C. The state (3.1) exhibits also some other interesting properties. For instance, concurrencies for all pairs $\{i,j\}\in S^E$ have the same value. This is a natural consequence of the fact that all the density matrices of entangled pairs are identical. From Eq. (3.4), we also see that all single-qubit density operators are identical and that they depend only on the total number of qubits N. This means that the information about the graph itself is encoded only in the correlations; there is no way to extract any information about the specification of the graph only via single-qubit measurements.

IV. PURE STATES

The problem of a construction of *pure* states corresponding to a specific graph is (as expected) more complicated than for mixtures. As mentioned above, the number of "free" parameters in this case is smaller and one cannot "control" off-diagonal matrix elements in the same way as in the case of mixed states [11]. Therefore, we start our discussion with the simplest case of three qubits and we examine thoroughly all possible graphs. Then we formulate a theorem about the existence and nonexistence of some classes of graphs.

A. Three-qubit graphs

In the case of three qubits, there are ten possible entangled graphs with classical correlation. We present these graphs in Fig. 2. We know that for six of these graphs, there exist pure states. For example,

$$a\rightarrow|000\rangle,$$

$$b\rightarrow\frac{1}{\sqrt{2}}|0\rangle(|00\rangle+|11\rangle),$$

$$g\rightarrow\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle),$$

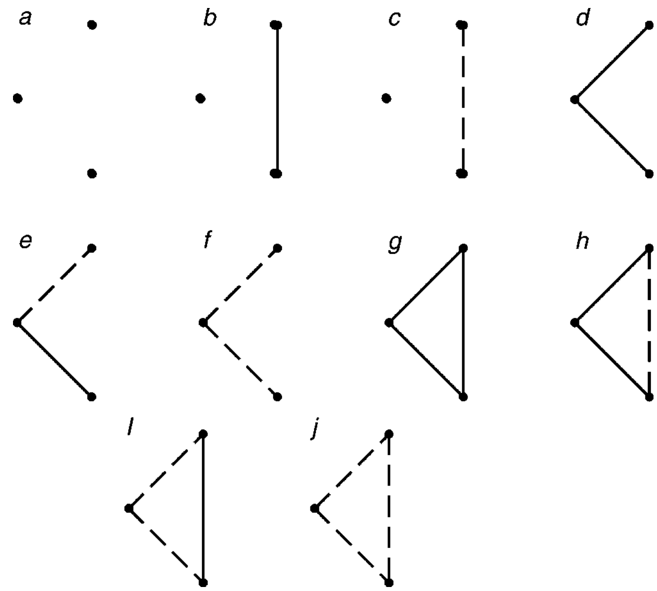

FIG. 2. Ten possible entangled graphs with classical correlations for three qubits. Graphs *c*, *d*, *e*, and *f* do not have representatives among pure states.

$$h\rightarrow\frac{1}{2}(|000\rangle+|100\rangle+|110\rangle+|111\rangle),$$

$$i\rightarrow\frac{1}{\sqrt{3}}(|000\rangle+|011\rangle+|111\rangle),$$

$$j\rightarrow\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle).$$

For the other four graphs, it is not possible to find any pure state, which would be represented by them. These four graphs exhibit one common property: they all include at least one vertex, which is connected with just one of the other two vertices. We will show that this property of a graph immediately leads to nonexistence of a pure state associated with this graph. The only possible exception is a graph of form *b* in Fig. 2, when there is an isolated pure entangled two-qubit state [as, for example, a Bell pair $(1/\sqrt{2})(|00\rangle+|11\rangle)$], which is not connected with the rest of the system at all.

B. Multiqubit graphs

For more than three qubits, we have a very large number of possible graphs. Typically, the number of different graphs grows as $\exp(N^2)$ and already for $N=4$ it gives a number greater than 100. Necessarily, one needs to categorize these graphs in order to study the problem. Therefore, let us divide these graphs into two basic categories

(1) Disconnected graphs. These are the graphs whose vertices can be divided into (at least) two groups (each group containing at least one vertex), which are connected neither by an entanglement edge nor by a correlation edge.

(2) Connected graphs. In these graphs, every pair of vertices is connected directly or via other vertices.

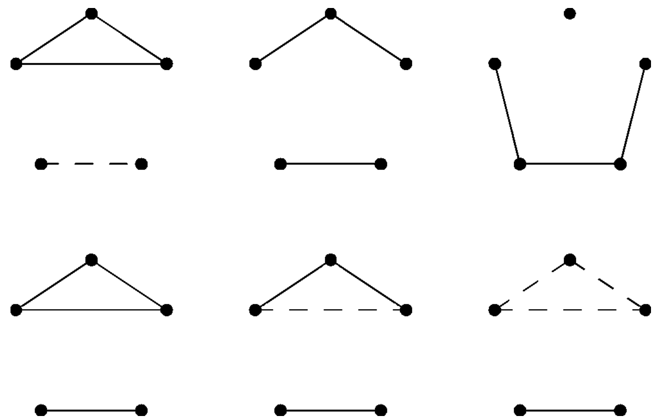

FIG. 3. Six examples of disconnected entangled graphs with classical correlations for five qubits. In the first row, there are examples of graphs that cannot be represented by pure states of five qubits. In the second row, there are examples of graphs that can be represented by pure states.

Let us first consider disconnected graphs. For a large N this group of graphs is much smaller than the second one. For disconnected graphs, the question of existence of pure states can be easily reduced to a problem of graphs with smaller number of vertices.

We can divide all vertices in a disconnected graph into two groups, which are not connected by any edge. We denote the two subsystems as A and B, respectively. As the two subsystems are not correlated, we can write

$$\rho_{AB}=\rho_A\otimes\rho_B\,, \tag{4.1}$$

where ρ_{AB} is the density operator of the whole system while ρ_A and ρ_B are the density operators of the two subsystems. According to our assumption, the whole system is in a pure state, i.e., ρ_{AB} is pure. Consequently, the two subsystems have to be in pure states as well. Thus,

$$|\Psi\rangle_{AB}=|\psi\rangle_A\otimes|\psi\rangle_B \tag{4.2}$$

and we can state that the whole state $|\Psi\rangle_{AB}$ exists, if and only if $|\psi\rangle_A$ and $|\psi\rangle_B$ do exist. One could follow the same argument if there are more disconnections in the graph. Therefore, every disconnected graph can be represented by a pure state if and only if every separated subset of vertices (separated subgraph) can be represented by a pure state. In Fig. 3, we present examples of disconnected graphs that cannot be represented by pure states (first row) and that can be represented by pure states (second row).

For connected graphs, we have not been able to find any simple algorithm to determine the existence of a pure state that would represent a given graph. However, we can formulate theorems about specific classes of graphs, which exhibit some special properties.

It is obvious that every graph containing only one vertex has a representation among pure states (any pure state of a qubit). Also, for a graph containing two vertices, which are connected by an entanglement edge, one is able to find a pure state [for example, a Bell-state $(1/\sqrt{2})(|00\rangle+|11\rangle)$]. On the other hand, there is no pure state that would correspond to a two-vertex graph with a correlation edge This can be seen from the fact that such a state would have to be written in a form

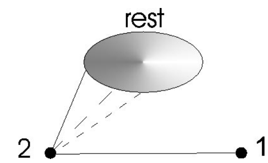

FIG. 4. Schematic visualization of the structure of a graph with an open edge. The vertex 1 is connected with the rest of the system via a single edge.

$$|\psi\rangle_{AB}\langle\psi|=\sum_{i=1}^{k}\lambda_i(\rho_A^i\otimes\rho_B^i), \tag{4.3}$$

with $k>1$, $\lambda_i>0$, and $\rho_A^i\neq\rho_A^j$, what is clearly not possible.

For more qubits, we can determine one type of graph that cannot be represented by pure states: these are the graphs with the so-called *open edges*. If a vertex in a connected multivertex graph is connected with the rest of the graph with just a single edge (correlation or entanglement), then we will call it as an *open edge* (since it is not a part of any closed chain of edges). Now we can formulate a theorem for graphs with open edges:

A connected graph with N vertices $(N>2)$ containing at least one open edge can never be represented by a pure state of N qubits.

To prove this theorem, let us denote the qubit, which is connected only via one edge (the open edge) with the rest of the system, as the first qubit. The qubit mediating this connection (the other end of the open edge) will be denoted as the second one. The structure of the remaining part of the graph is not important for our consideration and we simply denote it as the ''rest'' (see Fig. 4).

Since we consider the whole N-qubit system to be in a pure state, the state of the first qubit has to be mixed (otherwise it could not be correlated or entangled with other parts of the system). The density operator of the first qubit then can be written in a form

$$\rho_1=a|\psi\rangle\langle\psi|+(1-a)|\psi^{\perp}\rangle\langle\psi^{\perp}|, \tag{4.4}$$

where $0<a<1$ and the two states $|\psi\rangle$ and $|\psi^{\perp}\rangle$ are mutually orthogonal, i.e., $\langle\psi|\psi^{\perp}\rangle=0$.

The part of the graph denoted as the rest with $N-2$ vertices is also in a mixed state with the corresponding density operator ρ_{rest} that can be written in a form

$$\rho_{rest}=A|\Psi\rangle\langle\Psi|+(1-A)\rho_{\Psi}^{\perp}\,, \tag{4.5}$$

where $0<A<1$ and $\rho_{\Psi}^{\perp}$ is a density operator of $N-2$ qubits which is orthogonal to the state $|\Psi\rangle$, i.e., $\langle\Psi|\rho_{\Psi}^{\perp}|\Psi\rangle=0$. Because we assume that the part of the graph (corresponding to $N-2$ qubits) that we denote as rest is not correlated with the vertex 1 at all, we can express the joint density operator $\rho_{1\oplus rest}$ of the first qubit and the rest as the tensor product of two density operators ρ_1 and ρ_{rest}, i.e.,

$$\rho_{1\oplus rest}=\rho_1\otimes\rho_{rest}. \tag{4.6}$$

On the other hand, by the definition of our task, the whole graph corresponding to N qubits has to be in a pure state $|\Xi\rangle_{1\oplus 2\oplus rest}$. Correspondingly, the qubit (vertex) number 2 has to purify simultaneously both density operators ρ_1 and ρ_{rest} in such a way that $\rho_1=\mathrm{Tr}_{2,rest}(|\Xi\rangle\langle\Xi|)$ and $\rho_{rest}=\mathrm{Tr}_{2,1}(|\Xi\rangle\langle\Xi|)$, while $\rho_{1\oplus rest}=\mathrm{Tr}_2(|\Xi\rangle\langle\Xi|)$.

However, this is impossible even if we assume that the density operator $\rho_\Psi^\perp$ in Eq. (4.5) is a projector (i.e., $\rho_\Psi^\perp=|\Psi^\perp\rangle\langle\Psi^\perp|$) since even in this case the density operator $\rho_{1\oplus rest}$ is equal to a statistical mixture of four mutually orthogonal states:

$$\begin{aligned}\rho_{1\oplus rest}=\rho_1\otimes\rho_{rest}=&aA|\psi\Psi\rangle\langle\psi\Psi|+a(1-A)\\&\times|\psi\Psi^\perp\rangle\langle\psi\Psi^\perp|+(1-a)A|\psi^\perp\Psi\rangle\langle\psi^\perp\Psi|+(1-a)\\&\times(1-A)|\psi^\perp\Psi^\perp\rangle\langle\psi^\perp\Psi^\perp|.\end{aligned} \tag{4.7}$$

As discussed earlier in Sec. III in order to purify the state (4.7), we would need a four-dimensional ancilla [9], which obviously is not available in our considerations since the vertex 2 is just a qubit with a two-dimensional Hilbert space. This proves the Theorem 2.

C. Other classes of graphs

1. Entangled webs

Let as consider graphs with *all* pairs of vertices connected with an edge (either correlation or entanglement). These types of graphs can be represented by pure states of the form

$$|\Xi\rangle=\alpha|0,\dots,0\rangle+\beta|1,\dots,1\rangle+\sum_{\{i,j\}\in S^E}\frac{\gamma}{\sqrt{k}}|1\rangle_i|1\rangle_j|0,\dots,0\rangle_{S\setminus\{i,j\}} \tag{4.8}$$

with the normalization condition $|\alpha|^2+|\beta|^2+|\gamma|^2=1$, and $\alpha,\beta,\gamma>0$. The pure state (4.8) describes a graph such that pairs of vertices in the set S^E are entangled while all other pairs of vertices are correlated [5].

2. Connected graphs with no open edges

Through numerical simulations we have searched for pure states corresponding to graphs of four qubits. We have found a pure state for every connected graph with no open edges. Specifically, taking into account general properties of graphs described above, we have found only 20 graphs that remain ambiguous. With the help of our simulation, we have been able to find at least one pure state as a representative of each graph.

Unfortunately, we have not been able to generalize this result for connected graphs with no open edges for more than four vertices.

V. CONCLUSIONS

In order to understand how correlations and entanglement are shared among qubits in multiqubit systems, we have introduced a concept of entangled graphs with classical correlations. Every qubit is represented by a vertex, and correlations between two qubits are represented by edges. Two types of edges stand for two possible types of (nonzero) correlations: the entanglement edge corresponds to entanglement between a specific pair of qubits (vertices), while the correlation edge denotes classical correlation. No edge between two qubits means that the corresponding bipartite density operator is the tensor product of single-qubit density operators.

We have shown that any graph with N vertices can be represented by a mixed state of N qubits. On the other hand, only some graphs can be represented by pure states. In particular, we have shown that connected graphs with N vertices that contain an open edge can never be represented by a pure state. Interestingly enough, in the case of three- and four-vertex graphs, we have been able to find pure states for all other graphs (i.e., connected graphs with no open edges).

ACKNOWLEDGMENTS

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[10] The concurrence is a measure of bipartite entanglement between two qubits that has been introduced by Wootters [see W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998)]. To calculate it, we define a so-called spin-flipped operator $\tilde{\rho}=(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$, where σ_y is the Pauli matrix and the star means complex conjugation. Now we define the matrix $R=\rho\tilde{\rho}$ and label its (non-negative) eigenvalues, in decreasing order, λ_1, λ_2, λ_3 and λ_4. The concurrence is then defined as $C=\max\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\}$ and characterizes the amount of bipartite entanglement between two qubits with a number between 0 and 1.

[11] The off-diagonal elements are the biggest "problem" for classical correlation. The reason is that if one type of measurements (let us say, σ_x) is not correlated, the other type (such as σ_z) can be.