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Saturation of Coffman–Kundu–Wootters inequalities via quantum homogenization

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Abstract
We study how entanglement between an open system and a reservoir is established. The system is considered to be a qubit, while the reservoir is modelled as a collection of qubits. The system and the reservoir qubits interact via a sequence of partial-swap operations. This processes is called *quantum homogenization* since at the output the system as well as all reservoir qubits are in states that are, in a limit sense, equal to the original state of the reservoir qubits. We show that in this process the Coffman–Kundu–Wootters inequalities are saturated. This means that no intrinsic multi-partite entanglement is created.

Keywords: Quantum entanglement, dynamics of open structures

1. Model of homogenization

We start our discussion with a brief description of the so-called *homogenization process* [1]. Let us consider a system, S, represented by a single qubit initially prepared in the unknown state $\varrho_S^{(0)}$ and a reservoir, R, composed of N qubits all prepared in the state ξ, which is arbitrary but the same for all qubits. We will enumerate the qubits of the reservoir and denote the state of the kth qubit as ξ_k. From the definition of the reservoir, it follows that initially $\xi_k = \xi$ for all k, so that the state of the reservoir is described by the density matrix $\xi^{\otimes N}$. Let us assume the following collision-like model: system–reservoir interactions occur in a sequence of qubit–qubit collisions described by the unitary transformation U. Moreover, the system qubit can interact with each of the reservoir qubits once at most. As a result of this arrangement, after n interactions the whole system evolves according to the transformation

$$\varrho_S \otimes \xi^{\otimes N} \mapsto U_n \cdots U_1[\varrho_S \otimes \xi^{\otimes N}]U_1^\dagger \cdots U_n^\dagger, \qquad (1)$$

where $U_j = U \otimes (\bigotimes_{k\neq j} \mathbf{1}_k)$ and $\mathbf{1}_k$ is the identity operator on the kth reservoir qubit.

The homogenization process is motivated by the process of thermalization [2] which describes the evolution that leads the system to equalize its temperature with the temperature of the reservoir. The homogenization does not use the concept of temperature (as one of the characteristics of the state), but instead of this, it works with particular quantum states. The homogenization is a process such that when it terminates, all particles are in states that are very similar to the original state of the reservoir particles. Formally, the homogenization conditions are

$$\forall N \geqslant N_\delta \qquad D(\varrho_S^{(N)}, \xi) \leqslant \delta, \qquad (2)$$

$$\forall k,\, 1 \leqslant k \leqslant N \qquad D(\xi_k', \xi) \leqslant \delta, \qquad (3)$$

where $D(\cdot,\cdot)$ denotes some distance (e.g. a trace–norm distance) between two states. In order to fulfil both conditions (2) and (3) we have to find an appropriate unitary transformation U. In our paper [1] we have shown that the *partial-swap operation*

$$P(\eta) = \cos\eta\mathbf{1} + \mathrm{i}\sin\eta S, \qquad (4)$$

serves the purpose (S stands for the swap operation defined by the relation $S|\psi\rangle\otimes|\phi\rangle = |\phi\rangle\otimes|\psi\rangle$). In what follows we will use the notation $\sin\eta = s$ and $\cos\eta = c$.

The states of the system qubit and the reservoir qubits are obtained by partial traces. We can recursively apply the partial-swap transformation and after the interaction with the nth reservoir qubit, we have

$$\varrho_S^{(n)} = c^2\varrho_S^{(n-1)} + s^2\xi + \mathrm{i}cs[\xi, \varrho_S^{(n-1)}], \qquad (5)$$

as the expression for the density operator of the system qubit, while the nth reservoir qubit is in the state

$$\xi_n' = s^2 \varrho_S^{(n-1)} + c^2 \xi + \mathrm{i}cs[\varrho_S^{(n-1)}, \xi]. \tag{6}$$

2. Entanglement induced by homogenization

Within the context of our investigation it is very natural to ask about the nature of the entanglement created during the process of homogenization. In this section we will address several questions related to this issue. Let us consider a specific initial state of the system and the reservoir: $|\psi\rangle_0 = |1\rangle$ and $|\xi\rangle_j = |0\rangle$. Note that the partial swap is invariant (see [1]) with respect to local unitary transformations of the form $u \otimes u$. That is, it takes the same form in any basis (of a single qubit) we choose.

With the given initial conditions, we easily find the state vector describing the whole system after n interactions:

$$|\Psi_n\rangle = c^n |1\rangle_0 \otimes |0\rangle^{\otimes N} + \sum_{l=1}^{n} |1\rangle_l \otimes |0\rangle^{\otimes N_{\bar{l}}} [\mathrm{i}sc^{l-1}(c + \mathrm{i}s)^{N-l}]. \tag{7}$$

We recall that N is the total number of reservoir qubits, and that the state $|0\rangle^{\otimes N_{\bar{l}}}$ denotes all qubits except the qubit l in the state $|0\rangle$. If the initial state of the system qubit is described by the state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ then the input state of the whole system $|\psi\rangle \otimes |0^{\otimes N}\rangle$ evolves after n interactions into the state

$$|\Omega_n\rangle = \alpha|0^{\otimes(N+1)}\rangle + \beta|\Psi_n\rangle. \tag{8}$$

In what follows we illuminate the issue of quantum entanglement which is created in the process of quantum homogenization. We will utilize the concept of the bi-partite entanglement measure—the concurrence. The concurrence $C_{jk}^{(n)}$ between the jth and kth qubit after n interactions is defined [4] by the following formula:

$$C_{jk}^{(n)} = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{9}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ are eigenvalues of the matrix $R_{jk}^{(n)} := \varrho_{jk}^{(n)} \sigma_y \otimes \sigma_y [\varrho_{jk}^{(n)}]^* \sigma_y \otimes \sigma_y$ and $[\varrho_{jk}^{(n)}]^*$ is a matrix conjugated with respect to $\varrho_{jk}^{(n)}$. In what follows we will always assume that $j < k$, because $C_{jk}^{(n)} = C_{kj}^{(n)}$. In the case of *pure* multi-qubit states one can define a measure of the entanglement between a single qubit and the rest of the system [3] with the help of the determinant of the density operator of the specific qubit under consideration. The entanglement measure, called the *tangle*, between the jth qubit and the rest of the system is given by the expression [3]

$$\tau_j^{(n)} \equiv [C_{j,\bar{j}}]^2 := 4 \det \varrho_j^{(n)}, \tag{10}$$

where $\tau_j^{(n)}$ is the tangle, which is equal (by definition) to the square of the corresponding concurrence.

If we determine the two-qubit and the single-qubit states during the process of homogenization and then apply definitions of the concurrence and the tangle, we will obtain our main results:

$$\tau_{jk}^{(n)} = [C_{jk}^{(n)}]^2 = \begin{cases} 0 & \text{for } n < k \leqslant N \\ 4|\beta|^4 s^4 c^{2(j+k-2)} & \text{for } k \leqslant n \leqslant N; \end{cases} \tag{11}$$

$$\tau_{0k}^{(n)} = [C_{0k}^{(n)}]^2 = \begin{cases} 0 & \text{for } n < k \leqslant N \\ 4|\beta|^4 s^2 c^{2(n+k-1)} & \text{for } k \leqslant n \leqslant N; \end{cases} \tag{12}$$

$$\tau_j^{(n)} = [C_{j,\bar{j}}^{(n)}]^2 = \begin{cases} 0 & \text{for } n < j \leqslant N \\ 4|\beta|^4 s^2 c^{2(j-1)}(1 - s^2 c^{2(j-1)}) & \text{for } j \leqslant n \leqslant N; \end{cases} \tag{13}$$

$$\tau_0^{(n)} = [C_{0,\bar{0}}^{(n)}]^2 = 4|\beta|^4 c^{2n}(1 - c^{2n}). \tag{14}$$

These results show that the system qubit acts as a 'mediator' of a bi-partite entanglement between the reservoir qubits which have never interacted directly. It is obvious that later the two reservoir qubits interact with the system qubit, the smaller is the degree of their mutual entanglement. Nevertheless, this value is constant and does not depend on the subsequent evolution of the system qubit (i.e. it does not depend on the number of interactions n). On the other hand the entanglement between the system qubit and the jth reservoir qubit (for all j) monotonically decreases with the number of interaction steps.

3. Conclusion: saturation of the CKW inequality

Let us now formulate the Coffman–Kundu–Wootters (CKW) conjecture [3]:

Coffman–Kundu–Wootters conjecture. *Consider the system composed of N qubits in a pure state. Then for each qubit j the following inequality holds:*

$$\Delta_j = C_{j,\bar{j}}^2 - \sum_{k,k\neq j} C_{jk}^2 \geqslant 0. \tag{15}$$

We will say that the state *saturates* CKW inequalities, if $\Delta_j = 0$ for all values of j. Using our results from the previous section we can verify the validity of the CKW conjecture (providing the whole system is in a pure state). Using the expressions (11)–(14) we find that the CKW inequality is saturated during the whole process of homogenization, i.e.

$$\Delta_j(n) = [C_{j,\bar{j}}^{(n)}]^2 - \sum_{k,k\neq j} [C_{jk}^{(n)}]^2 = 0. \tag{16}$$

We have proved that for quantum states of the form (8) that are generated in the process of quantum homogenization the CKW conjecture is valid. Moreover, it turns out that the quantum homogenization process leads to states that saturate the CKW inequality. This result is interesting also from the point of view of the creation of multi-partite entanglement. The difference Δ_j can be used to indicate the presence of the so-called *intrinsic* multi-partite entanglement. The vanishing value of Δ_j indicates that quantum correlations that are created in the homogenization process are just of bi-partite origin. No multi-partite entanglement is established in this process.

It is not difficult to show that any pure state $|\Psi\rangle$

$$|\Psi\rangle = \alpha_0 |0\rangle^{\otimes N} + \sum_{j=1}^{N} \alpha_j |0^{\otimes(N-1)}\rangle_{\bar{j}} \otimes |1\rangle \tag{17}$$

of an N-partite system saturates the CKW inequalities.

Firstly, we note that for a pair of qubits described by the density matrix

$$\varrho_{jk} = \begin{pmatrix} a_{jk} & d_{jk} & e_{jk} & 0 \\ d^*_{jk} & b_{jk} & f_{jk} & 0 \\ e^*_{jk} & f^*_{jk} & c_{jk} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (18)$$

the concurrence can be found in an analytical form

$$C^2_{jk} = 4 f_{jk} f^*_{jk}, \quad (19)$$

where $f_{jk} = \langle 01|\varrho_{jk}|10\rangle$. By inspection we find out that in the N-partite system prepared in the state (17) all bi-partite density operators have the form (18) with a matrix element f_{jk} given by the expression $f_{jk} = \alpha_j \alpha^*_k$. Consequently, the square of the concurrence $C^2_{jk} = 4|\alpha_j|^2|\alpha_k|^2$.

The state of a single qubit in the N-partite system (17) is described by the matrix

$$\varrho_j = \begin{pmatrix} |\alpha_0|^2 + \sum_{k\neq j} |\alpha_k|^2 & \alpha_0\alpha^*_j \\ \alpha_j\alpha^*_0 & |\alpha_j|^2 \end{pmatrix}, \quad (20)$$

and the corresponding tangle between the jth qubit and the rest of the system is

$$\tau_j = 4 \det \varrho_j = 4|\alpha_j|^2 \sum_{k\neq j} |\alpha_k|^2 = \sum_{k\neq j} C^2_{jk}. \quad (21)$$

Therefore we can conclude that the CKW inequalities are saturated, i.e. $\Delta_j = \tau_j - \sum_{k\neq j} C^2_{jk} = 0$ for all j.

Finally, we note that N-qubit states that are created in the process of quantum homogenization do belong to the class of states given by equation (17). Moreover, it is easy to see that N-partite pure states of the form

$$|\Psi\rangle = |\Psi_{n_1}\rangle \otimes \cdots \otimes |\Psi_{n_K}\rangle \otimes |\Phi\rangle \quad (22)$$

saturate the CKW inequality. In the above expression, the state $|\Psi_{n_j}\rangle$ describing n_j qubits is of the form given by equation (17), while $n_1 + \cdots + n_K = M$ and $|\Phi\rangle$ is a factorized state of the remaining $N - M$ qubits. The question regarding the most general state that saturates the CKW inequalities is still open.

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