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Simulation of exponential decay on simple quantum circuits: a case study

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Abstract
One of potential applications of quantum processors should be a simulation of dynamics of quantum systems. In this paper we analyse whether it is possible to simulate a paradigmatic model of an exponential decay of a two-level system using a programmable quantum gate array. That is, we would like to simulate the decay of a two-level system in such a way that the parameters of the decay itself are encoded in the state of the ancilla that serves as a program for the programmable gate array. We show that it is impossible to simulate the exponential decay precisely with the finite-dimensional programmable gate array. On the other hand, we present a very simple model of a Markovian decay process that can be efficiently simulated on a simple programmable gate array. We compare this Markovian process with the exponential decay.

Keywords: Programmable quantum circuits, decoherence, exponential decay

1. Introduction

The Wigner–Weisskopf decay [1] of a two-level quantum system (a two-level atom) is the typical textbook example of a decoherence process [2]. One may consider a two-level atom as the *system* under consideration, which is interacting with an infinite set of modes of the electromagnetic field in a free space. The field modes are modelled as harmonic oscillators. In order to describe the evolution of the system as a Markovian process, several approximations are made. That is, in the dipole approximation, the rotating wave approximation, the weak-interaction limit (first order of perturbation), and, finally, the Markov approximation, one obtains the master equation for the exponential decay of the excited state to the ground state.

In this study we focus on a physically much simpler situation: the time evolution of a decaying two-level atom can be viewed as a quantum operation or quantum channel from the two-level system's point of view. The two-level system can be regarded as a qubit, and exponential decay is referred to as an *amplitude damping channel* in this context [3, 4]. It is well known that, supplementing the qubit with a single-qubit ancilla and assuming a specific interaction between these two systems, amplitude damping channel can be realized—that is, an exponential decay can be mimicked by this very simple model system. This is called the unitary representation of the quantum operation. However, there is no Hamiltonian that the required interaction can be described by. For each time duration to be simulated, a separate quantum circuit has to be built. This would require infinite resources to model the exponential decay.

This problem may be overcome by the application of a programmable quantum gate array (a quantum processor) [5–9]. In this case, the parameters of the evolution should be encoded into the initial quantum state of the ancillary qubit(s), and the quantum circuit itself is fixed. The ancillary qubit may be either dropped after the operation of the circuit or subjected to a measurement, and the result may be accepted or rejected depending on the measurement result. The latter case is called the probabilistic regime, and it has been shown to provide a good way of implementing quantum operations, though with a finite probability of success [7, 8]. We restrict ourselves to the other, deterministic regime [9], which should always see success. We remark here that there exist several other different approaches for simulating generic quantum operations on simple quantum circuits [10–12].

This paper is devoted to an investigation of a deterministic programmable implementation of amplitude damping, and its comparison with the unitary representation. It is a good example to use to study the role of the information flow between two qubits and the entangling power of the interaction in the decoherence process [13, 14].

The paper is organized as follows. In section 2 we summarize some facts concerning the unitary representation of the amplitude damping channel. Section 3 describes the programmable quantum circuit approach. Section 4 compares the two situations from the point of view of information flow and entanglement. Section 5 summarizes our results. Throughout this paper, we will denote the elements of the computational basis by $|0\rangle$ and $|1\rangle$ for qubits, and $|0\rangle$, $|1\rangle$, and $|2\rangle$ for qutrits. In the case of qubits, $|0\rangle$ will be referred to as the ground state, while $|1\rangle$ will be referred to as the excited state, as in the case of two-level atoms.

2. Unitary representation of amplitude damping

Suppose that we have a qubit in an initial state

$$\varrho_{\mathrm{in}} = \begin{pmatrix} \varrho_{00} & \varrho_{01} \\ \varrho_{10} & \varrho_{11} \end{pmatrix}. \tag{1}$$

Amplitude damping is a quantum operation, $\mathcal{E}_{\gamma,t}$, that acts on the input state as

$$\varrho_{\mathrm{out}}(t) = \mathcal{E}_{\gamma,t}[\varrho_{\mathrm{in}}] = \begin{pmatrix} \varrho_{00} + (1 - \mathrm{e}^{-\gamma t})\varrho_{11} & \sqrt{\mathrm{e}^{-\gamma t}}\varrho_{01} \\ \sqrt{\mathrm{e}^{-\gamma t}}\varrho_{10} & \mathrm{e}^{-\gamma t}\varrho_{11} \end{pmatrix}. \tag{2}$$

Here the subscript t stands for the time, and γ is a constant scaling time, and describes the speed of the decay. In fact, the calculation of this decay parameter γ from first principles (i.e. from the wavefunctions of the atomic states and the interaction Hamiltonian) was the main concern of the original Wigner–Weisskopf paper [1]. Throughout the process described by equation (2), the excited state decays into the ground state exponentially.

The unitary representation of this quantum operation is depicted in part (a) of figure 1. Beside our system qubit S, take an ancillary qubit A. Define a set of unitary operators on the two-qubit system of S and A such that

$$\begin{aligned} U(t)|0\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}} &= |0\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}}; \\ U(t)|1\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}} &= \sqrt{\mathrm{e}^{-\gamma t}}|1\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}} + \sqrt{1 - \mathrm{e}^{-\gamma t}}|0\rangle_{\mathrm{S}}|1\rangle_{\mathrm{A}}. \end{aligned} \tag{3}$$

For a given value of γt, the quantum logic network realizing U has to be built. The ancilla should be prepared in the state $|0\rangle$ initially. After operating the circuit, we drop the ancilla, and obtain

$$\varrho_{\mathrm{out}}(t) = \mathrm{tr}_{\mathrm{A}}[U(\gamma t)(\varrho_{\mathrm{in}} \otimes |0\rangle\langle 0|_{\mathrm{A}})U^{\dagger}(\gamma t)]. \tag{4}$$

Note that the information about γt is treated completely classically, as it is directly 'encoded' into the quantum circuit—that is, this information represents a prescription for classical control of quantum dynamics (e.g. turning on and off an external laser field).

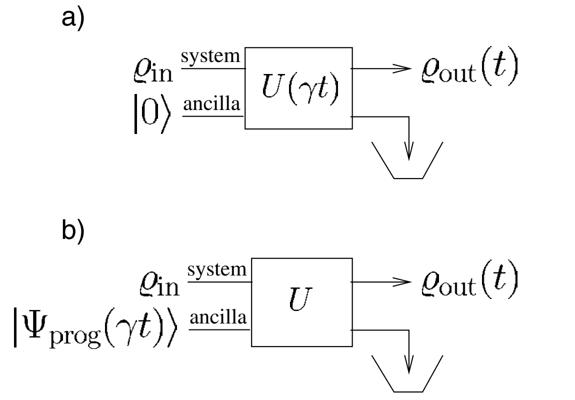

Figure 1. Arrangements under consideration. Panel (a) shows the unitary representation of amplitude damping, while panel (b) shows the programmable quantum network approach that we adopt.

Amplitude damping is a Markovian process: $\varrho_{\mathrm{out}}(t)$ satisfies the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho = \hat{L}\varrho, \qquad \hat{L}\varrho = \sum_{i,j=1}^{3} C_{i,j}^{\mathrm{AD}}([\hat{F}_i\varrho, \hat{F}_j^{\dagger}] + [\hat{F}_i, \varrho\hat{F}_j^{\dagger}]), \tag{5}$$

where $\hat{F}_i = \frac{1}{\sqrt{2}}\hat{\sigma}_i$, $i = 1, \ldots, 3$, are the normalized Pauli matrices. The right-hand side of equation (5) is in the standard Gorini–Kossakowski–Sudarshan (GKS) form [15]; the Hermitian matrix

$$C^{(\mathrm{AD})} = \frac{\gamma}{4}\begin{pmatrix} 1 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

represents the amplitude damping process itself [16]. The time evolution in Markovian processes can always be written in the form of equation (5).

3. Programmed decay

Consider now the arrangement in figure 1(b). Our intention is to encode the scaled time γt into the initial state of the ancilla. That is, the dynamics of the quantum system under consideration is fully controlled by the state of another quantum system (the ancilla). This type of scenario can be utilized in the situation where the state of the control system is totally unknown (e.g. may be an output of another quantum process). Under such circumstances, one approach would be to estimate the state of the ancilla and use it for a classical control as discussed above. On the other hand, one might like to utilize the whole potential of quantum information processing and to use the quantum information of the program system without measuring it. That is, to use it as an input program register to the programmable quantum processor. This approach would remove the need to build different circuits for different values of scaled time. Instead of equation (4), we then have

$$\varrho_{\mathrm{out}}(t) = \mathrm{tr}_{\mathrm{A}}[U(\varrho_{\mathrm{in}} \otimes |\Psi(\gamma t)\rangle\langle\Psi(\gamma t)|_{\mathrm{A}})U^{\dagger}]. \tag{7}$$

Surprisingly, this goal cannot be achieved. That is, the parameter γt in the exponential decay cannot be encoded into a single-qubit program register. This can be shown as follows. There must be a state of the ancilla for which the circuit remains 'inert'. Let us choose the basis state $|0\rangle_{\mathrm{A}}$ for this purpose. Thus for any pure input state $|\Psi\rangle_{\mathrm{S}}$,

$$U|\Psi\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}} = |\Psi\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}}. \tag{8}$$

We can write $|0\rangle_{\mathrm{A}}$ for the right-hand side; if this was not the case, the circuit U could be supplemented by another circuit acting on the ancilla only, to take the output state to $|0\rangle_{\mathrm{A}}$. As the ancilla is dropped, this does not alter the operation of the scheme.

For the $t \to \infty$ limit, we also need an appropriate ancilla state so that the system is certainly in the ground state finally:

$$U|\Psi\rangle_{\mathrm{S}}|\infty\rangle_{\mathrm{A}} = |0\rangle_{\mathrm{S}}|\infty^{\Psi}\rangle_{\mathrm{A}}. \tag{9}$$

This is required since an arbitrarily long time may be considered. Now the ancilla may alter, and its final state may depend on the input state of the system.

The scalar multiplication of equations (8) and (9) yields

$${}_{\mathrm{A}}\langle 0|\infty\rangle_{\mathrm{A}} = {}_{\mathrm{S}}\langle\Psi|0\rangle_{\mathrm{S}}\, {}_{\mathrm{A}}\langle 0|\infty^{\Psi}\rangle_{\mathrm{A}}. \tag{10}$$

Substituting $|\Psi\rangle_{\mathrm{A}} = |1\rangle$, it immediately follows that $|\infty\rangle_{\mathrm{A}} = |1\rangle_{\mathrm{A}}$. From this, by substituting $|\Psi\rangle_{\mathrm{A}} = |0\rangle_{\mathrm{A}}$, we find that $|\infty^{0}\rangle_{\mathrm{A}} = |1\rangle_{\mathrm{A}}$. Thus equation (9) should read

$$U|\Psi\rangle_{\mathrm{S}}|1\rangle_{\mathrm{A}} = |0\rangle_{\mathrm{S}}|1^{\Psi}\rangle_{\mathrm{A}}. \tag{11}$$

A unitary linear operator satisfying equations (8) and (11) may be defined as

$$\begin{aligned} U|0\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}} &= |0\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}}; \\ U|0\rangle_{\mathrm{S}}|1\rangle_{\mathrm{A}} &= |0\rangle_{\mathrm{S}}|1\rangle_{\mathrm{A}}; \\ U|1\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}} &= |1\rangle_{\mathrm{S}}|0\rangle_{\mathrm{A}}; \\ U|1\rangle_{\mathrm{S}}|1\rangle_{\mathrm{A}} &= |0\rangle_{\mathrm{S}}|2\rangle_{\mathrm{A}}, \end{aligned} \tag{12}$$

where we had to assume that the ancilla is at least a *qutrit*. This circuit will unfortunately not produce the amplitude damping of the form of equation (2). We will show that the process realized this way is a combination of amplitude and phase damping. Further enlargement of the ancilla space will not alter this situation fundamentally, as a state orthogonal to $|0\rangle_{\mathrm{A}}$ and $|1\rangle_{\mathrm{A}}$ must appear in the last row of equation (12).

It is interesting to note that the same scheme with a qubit ancilla works for most *unital* operations, which preserve the maximally mixed state (having a density matrix proportional to the identity matrix), e.g., a Pauli channel.

Let us now examine the process that is implemented by the programmable array with the circuit corresponding to the transformation (12). The ancilla is now a qutrit, in the suitable initial 'program' state

$$|\Psi_{\mathrm{prog}}(\gamma t)\rangle = \sqrt{\mathrm{e}^{-\gamma t}}|0\rangle + \sqrt{1-\mathrm{e}^{-\gamma t}}|1\rangle. \tag{13}$$

Substituting equations (12) and (13) into (7), the result is

$$\varrho_{\mathrm{out}}^{(\mathrm{PAD})}(t) = \begin{pmatrix} \varrho_{00} + (1-\mathrm{e}^{-\gamma t})\varrho_{11} & \mathrm{e}^{-\gamma t}\varrho_{01} \\ \mathrm{e}^{-\gamma t}\varrho_{10} & \mathrm{e}^{-\gamma t}\varrho_{11} \end{pmatrix}. \tag{14}$$

In what follows, we call this process programmed amplitude damping. Comparing with equation (2), we find that the off-diagonal matrix elements of the density matrix decay faster in this process than in the case of amplitude damping: the square root in the off-diagonal matrix element disappears.

The programmed amplitude damping is a Markovian process with the GKS matrix

$$C^{(\mathrm{PAD})} = \frac{\gamma}{4}\begin{pmatrix} 1 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

This may be compared with equation (6). The difference is the appearance of a constant matrix element in the lower right corner that describes a phase damping about the z-axis of the Bloch sphere. This illustrates the convenience of the GKS formalism: a linear combination of GKS matrices describes a process where these two Markovian processes are combined in such a way that another Markovian process is obtained. The combination may be realized by the repeated alternating application of the two processes, with an infinitesimally small time step in each case. In our case, programmed amplitude damping is a combination of amplitude and phase damping channels.

Thus we have found that amplitude damping cannot be implemented on a programmable quantum gate array with finite resources deterministically, but a similar Markovian process can be implemented instead.

4. Information transfer

Let us now investigate the role of information in the processes described above. In order to do so, we calculate the von Neumann entropy of each subsystem involved. Let us assume that the input states of the system as well as the ancilla qubits are pure (they are represented by points on a Bloch sphere). Correspondingly, the total von Neumann entropy of system + ancilla is zero. Given the fact that the evolution is unitary, the total entropy will remain zero during the time evolution, while according to the Araki–Lieb theorem the two subsystems will have the same entropy S:

$$S_{\mathrm{system}} = S_{\mathrm{ancilla}} := S. \tag{16}$$

It is also easy to see that S is independent of the declination ϕ on the Bloch sphere. S is physically interesting, because it characterizes both the mutual information of the two subsystems, and the entangling power of U: for a pure entangled state, the von Neumann entropy of one of the subsystems is a measure of the entanglement.

Figure 2 shows the entropy S for amplitude damping and the programmed amplitude damping processes, as a function of the azimuthal angle θ on the Bloch sphere, and the probability of decay $p = 1 - \exp(-\gamma t)$. The value $p = 0$ corresponds to $t = 0$, while $p = 1$ describes the $t \to \infty$ case.

Figure 2 is self-explanatory, and clearly displays the characteristics of both processes. It should also be noticed that, except for the case when the entropy is zero, the entropy in the programmed amplitude damping is always superior to the entropy of amplitude damping. There is more mutual information in each subsystem in the programmed amplitude damping, the joint state is more entangled, and hence decoherence is faster.

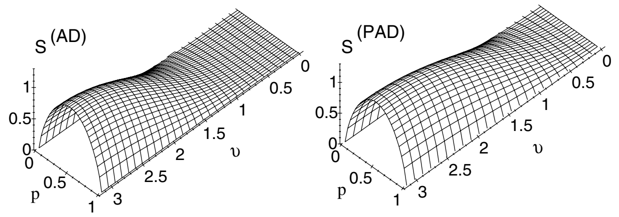

Figure 2. The entropy of one of the subsystems in amplitude damping (AD) and programmed amplitude damping (PAD) processes as a function of azimuthal angle θ of the initial pure state on the Bloch sphere, and decay probability $p = 1 - \exp(-\gamma t)$.

5. Conclusions

We have considered the possibility of quantum programmed deterministic simulation of exponential decay. We have found that exponential decay cannot be simulated this way, but a similar procedure that can be simulated in this way was introduced. Both processes are Markovian; they were compared in the GKS representation. This provides a good example of the convenience of this formalism. The two processes illustrate well the role of information flow and the entangling power of the interaction.

Acknowledgments

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