Quantum theory: kinematics, linearity and no-signaling condition

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We show that the linearity of an evolution of Quantum Mechanics follows from the definition of kinematics. The same result is obtained for an arbitrary theory with the state space that includes mixtures of different preparations. Next, we formulate the non-signaling theorem and show that the theorem poses no additional restriction on Quantum Mechanics provided the kinematics is given. We also discuss validity of the postulate for the case of more general theories.

I. INTRODUCTION

It was more than seventy years ago when Quantum Mechanics became widely accepted and established as one of the fundamental theories of Nature. Despite its success there are still several questions which, at least for a certain part of the physical community, have not been answered satisfactory yet. The main reason is that there is much more space between abstract mathematical elements of the theory and real objects prepared and measured in our laboratory. The rigorous mathematical formulation of the theory was given in [1] which can be summarized in a few postulates. There have always been numerous attempts to derive the postulates of the Quantum Mechanics from some other more "fundamental" or at least physically well motivated postulates. As for instance to derive the linearity of the evolution from the no-signaling postulate [2]. Such attempts usually raise some discussion [3,4] and a casual non-expert reader may be confused by the language used by experts. In this paper we will try to clarify the relation between the linearity and no-signaling condition in the Quantum Mechanics as well as present more general results.

To begin with let us briefly summarize the most frequently used representation of quantum objects, i.e. the Hilbert space formulation of quantum theory. In order to avoid certain mathematical complications we will work with finite-dimensional Hilbert spaces. However, our discussion remains valid in the case of infinite ones too. In this framework states and observables are associated with specific linear operators acting on a given Hilbert space $\mathcal{H}$. States are represented by *density matrices* ϱ, i.e. positive hermitian operators with unit trace. Let us denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators defined on the Hilbert space $\mathcal{H}$. Then the subset $\mathcal{S}(\mathcal{H})$

$$\mathcal{S}(\mathcal{H}) = \{\varrho \in \mathcal{B}(\mathcal{H}) : \varrho = \varrho^\dagger, \ \varrho \geq 0, \ \text{Tr}\varrho = 1\} \quad (1.1)$$

forms a set of all possible quantum states.

Operators $O : \mathcal{H} \to \mathcal{H}$ for which $O = O^\dagger$ are associated with *projective measurements*. It turns out that a more general notion of a measurement requires a set of operators $\{F_k\}$ for the representation of a single observable M [5]. Each outcome λ_k of the measurement M is associated with one of these operators and the probability for measuring the corresponding outcome is given by the trace rule

$$p_k = \text{Tr}\left(\varrho F_k\right), \quad (1.2)$$

provided that the system was prepared in the state ϱ. In the case of projective measurements these operators possess the property $F_j = F_j^\dagger = F_j^2$, i.e. they are *projective operators*. For a general measurement M these operators must be positive, i.e. $F_k = F_k^\dagger$ and $F_k \geq 0$, and sum up to the identity operator, i.e. $\sum_k F_k = \mathbb{1}$. Let us denote by $\mathcal{P}(\mathcal{H})$ the set of all positive elements of $\mathcal{B}(\mathcal{H})$. Usually we use the concept of operator measure defined on Borel sets $\mathcal{B}(\mathbf{R})$ of real numbers $\mathbf{R}$ (associated with the outcomes of the measurements). Any mapping $F_M : \mathcal{B}(\mathbf{R}) \to \mathcal{P}(\mathcal{H})$ satisfying the properties of a measure, i.e.

1. $F_M(\mathbf{R}) = \mathbb{1}$

2. $F_M(\cup_k A_k) = \sum_k F_M(A_k)$ where A_k are mutually disjoint Borel sets

represents some quantum measurement M. Let us denote the set of all *positive operator-valued measures* (POVMs) F_M with the symbol $\mathcal{M}(\mathcal{H})$.

II. KINEMATICS

When we face the problem of building some new physical theory our first step is to introduce basic objects representing our physical world - the kinematics. This can be accomplished by defining two sets $\mathcal{S}$ and $\mathcal{M}$. The first one represents the states and the second one is associated with the measurements. Each element $M \in \mathcal{M}$ induces a probabilistic measure P_M, where

$$P_M(A, \rho) \quad (2.1)$$

is the probability that the outcome of the measurement M is from the Borel set $A \in \mathcal{B}(\mathbf{R})$ provided that the system was prepared in the state $\rho \in \mathcal{S}$. Here we have

restricted ourselves to the $\mathcal{B}(\mathbf{R})$ as all measurement outcomes are always mapped onto the set of real numbers $\mathbf{R}$. An outcome of a measurement is always a real number like position or number of counts etc.. This probabilistic rule is the only thing we can prove by performing our experiments. Moreover, from the mathematical point of view, this simple rule gives us limitations on both sets in a way that their mutual compatibility is guaranteed. For instance, if the set of states $\mathcal{S}$ is given then for any pair of states the set of all measurements $\mathcal{M}$ must provide a possibility for us to distinguish the two states. These two sets are usually called *kinematics* of the theory. Now we need to associate mathematical objects with the abstract elements of these sets, i.e. find some mathematical realization. In quantum theory there is a very convenient choice using the concept of the Hilbert space. We would like to stress here that this may not be the only choice, but at the same time it cannot be done in arbitrary space. The space should be rich enough and possess all features of the theory, like for example interference, uncertainty relations, etc.

Therefore, let us associate a Hilbert space $\mathcal{H}$ with a given quantum system. The unit elements of $\mathcal{H}$ (denoted by Dirac's ket symbol $|\psi\rangle$), i.e. any vector $|\psi\rangle \in \mathcal{H}$, for which $\langle\psi|\psi\rangle = 1$, is an element of $\mathcal{S}$ and represents a state of the quantum system. It is well known [5] that the set of quantum mechanical states is much larger than the set composed from vector states only. One of the ways how to introduce density matrices (mathematical objects representing generalized quantum states) is by mixing up different pure states (their preparations) together. This is how the density matrices were discovered. "Mixing" can be mathematically described by *convex combinations* not of the vectors, but rather as convex combinations of operators representing the vector states, i.e. of *projectors*. More generally, mixtures can be viewed just like probability distributions defined on the set of unit vectors without any reference to operators. Let us denote such set of distributions by $\mathcal{D}(\mathcal{H})$.

Note 1: From the operational point of view states correspond to our preparation procedures. There are many preparations that lead us to the same state. Moreover, mixing different preparations is a preparation again. The question is whether such mixtures prepare some new states, or not. In Quantum Mechanics some of these distributions are equivalent and are represented by a single density matrix. This fact is due to the given set of observables, that does not allow us to distinguish among different preparations of the same density operator. $\diamond$

The second way how to obtain a density operator as an object representing a state uses the notion of composite quantum system with the Hilbert space given by tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, where symbols A, B denote two different physical systems. If the whole system is described by a vector state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ then there is no guarantee that the subsystems are described by vector states as well. After performing partial trace operation (discarding the second system) we obtain that the resulting

operators possess all features of density operators. Such confirmation gives us new possibilities how to prepare density matrices. We shall call these matrices *reduced density operators* and the set of all density matrices will be denoted as $\mathcal{S}(\mathcal{H})$ in accordance with Eq. (1.1). (The resemblance in our notation between $\mathcal{S}$ and $\mathcal{S}(\mathcal{H})$ is not accidental and will be revealed later.).

Quite natural question arises. What does happen, if we mix together two preparations of two density matrices? Like in the case with vector states, we can associate mixtures of density matrices with the probability distributions defined on the set of states $\mathcal{S}(\mathcal{H})$. Let us denote by $\pi = \{p_j, \varrho_j\}$ such (discrete) *probability measure on the set of density operators* $\mathcal{S}(\mathcal{H})$ and let $\mathcal{K}(\mathcal{H})$ be the set of all such distributions π of density matrices. As a result we obtain new set of quantum states $\mathcal{K}(\mathcal{H})$, which from the point of view of a mathematical description contains the former set $\mathcal{D}(\mathcal{H})$. The members of $\mathcal{S}(\mathcal{H})$, reduced density operators now form extremal points of $\mathcal{K}(\mathcal{H})$. They are not created by mixing different preparations, but by discarding the second system. Next we should find a tool that enables us to differentiate among these states. If we find such tool (like it was find in the case of generalizing vector states) then we can build a new theory with new set of states. We shall call any such theory "Extended Quantum Mechanics (EQM)" according to Ref. [6] where an interesting solution to this problem has been presented.

The set of states $\mathcal{K}(\mathcal{H})$ (as well as $\mathcal{D}(\mathcal{H})$) is endowed with a convex structure. That is, any mixture of two probability distributions (elements of $\mathcal{K}(\mathcal{H})$) is again a probability distribution defined on $\mathcal{S}(\mathcal{H})$. Instead of expressing a mixing of preparations by $\pi = \{p_j, \varrho_j\}$ we use the natural convex structure and write any element $\pi \in \mathcal{K}(\mathcal{H})$ as $\pi = \sum_j p_j \varrho_j$, where ϱ_j is now a point in the state space $\mathcal{K}(\mathcal{H})$ representing the state $\{p_j = 1, \varrho_j\}$. In the formulation of the kinematics of the Quantum Mechanics any mixture of density matrices, i.e an element of $\mathcal{K}(\mathcal{H})$, is associated with a certain element of the set $\mathcal{S}(\mathcal{H})$. That is instead of probability distributions defined on the set $\mathcal{S}(\mathcal{H})$ we can speak only about the set $\mathcal{S}(\mathcal{H})$ with its natural convex structure.

Note 2: Only the convex structure of the state space $\mathcal{S}(\mathcal{H})$ enables us to identify (and compare) different probability distributions (mixings of preparations) defined on the space $\mathcal{S}(\mathcal{H})$ with the elements of $\mathcal{S}(\mathcal{H})$. (Let us note the vector states do not have such property; they are just extremal points of $\mathcal{S}(\mathcal{H})$). $\diamond$

For the time being we postpone the definition of the set of all possible measurements $\mathcal{M}$ for different types of state spaces. The rationale being that in Quantum Mechanics every observable is related to the generator of a one-parametric semi-group that is a dynamical evolution. The same can be done even in the case of a more general theory see for instance Ref. [6].

III. DYNAMICS

Under quantum dynamics two concepts can be understood. Firstly, some rule how quantum system evolves with time, and secondly some set of transformations of quantum states (different from measurement) without any explicit reference to time. Of course, any state transformation takes some time, but we will not consider its time duration here. This second notion determines the whole possible set of objects in which the first rule (with time) can draw a line to record the time dependency. In each time the state is transformed according to a map that belongs to all allowable set of state transformations. Let us now formalize these ideas. In general the allowed dynamical maps form a set

$$\mathcal{E}(\mathcal{S}) = \{\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}_\Lambda \subset \mathcal{S}(\mathcal{H})\} \qquad (3.1)$$

This set contains any transformation Λ of the quantum states. There are yet no restrictions such as the shape of the target set $\mathcal{S}_\Lambda$, or the linearity of Λ. In what follows by dynamics of quantum theory we will understand the evolution without any reference to time. It means we will investigate the general properties, what the dynamics should satisfy.

Now we discuss the issue of linearity [7] of Quantum Mechanics. To obtain the linearity of the evolution Λ one has to consider the following "mixing procedure". Let us assume that there is no possibility how to distinguish between two different preparations (decompositions) of any density operator by performing all possible measurements $\mathcal{M}(\mathcal{H})$ [8]. A preparator might have this information, but in quantum theory (with the state space $\mathcal{S}(\mathcal{H})$) we believe it is useless. Let us allow nonlinear evolution Λ and let the preparator prepare single particle (in state $\psi_j \in \mathcal{S}(\mathcal{H})$) and evolve it according to this evolution ($\psi_j \mapsto \psi'_j = \Lambda[\psi_j]$). Such procedure is experimentally acceptable. Let us consider two different preparations of states ψ_j and ϕ_k with probabilities p_j, q_k such that

$$\sum_j p_j \psi_j = \sum_k q_k \phi_k$$

where the equality means that the density operators are the same. Applying the (non-linear) evolution Λ the outgoing states need not represent the same density matrix, i.e.

$$\sum_j p_j \Lambda[\psi_j] \neq \sum_k q_k \Lambda[\phi_k]$$

It means that the preparator is able to differentiate between the two mixtures but then he must be able to differentiate between them from the beginning [9]. Thus we should include observables that enables him to do so. In order to preserve the kinematics (the set $\mathcal{S}(\mathcal{H})$ and the set $\mathcal{M}(\mathcal{H})$) the non-linear evolution must be forbidden. Otherwise, the above procedure enables us to distinguish preparations resulting in the same density operator ϱ.

Let us apply the above consideration for the Extended Quantum Mechanics with the larger set of states $\mathcal{K}(\mathcal{H})$, specifically onto the elements of $\mathcal{K}(\mathcal{H})$. If we assume that this set represents all the possible quantum states, then the evolution defined as a mapping $\Lambda : \mathcal{K}(\mathcal{H}) \to \mathcal{K}_\Lambda \subset \mathcal{K}(\mathcal{H})$ must be "linear", (i.e. affine). To show this it is enough to repeat the previous discussion, only instead of $\mathcal{S}(\mathcal{H})$ consider the set $\mathcal{K}(\mathcal{H})$. As a result we obtain that the evolution Λ is linear in the following sense

$$\Lambda[\sum_j p_j \pi_j] = \sum_j p_j \Lambda[\pi_j]$$

where $\pi_j \in \mathcal{K}(\mathcal{H})$ and $p_j \geq 0$, $\sum_j p_j = 1$. The linearity implies that the evolution is completely determined by its action on extremal states (Dirac distributions on $\mathcal{S}(\mathcal{H})$) associated with the members of the set $\mathcal{S}(\mathcal{H})$. Let us remind that $\mathcal{S}(\mathcal{H})$ is not a subset of $\mathcal{K}(\mathcal{H})$, but they are different sets with different elements. Therefore, the question of the linearity of Λ on the set $\mathcal{S}(\mathcal{H})$ is ill defined, because in general, the map Λ can transform extremal states of $\mathcal{K}(\mathcal{H})$ into mixtures in $\mathcal{K}(\mathcal{H})$. But if we assume that the evolution Λ maps extremal points into extremal points, the definition (restriction) of $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is possible and the linearity can be discussed.

Note 3: This result is nothing else but our comprehension of the notion probability. Probabilities arise in our description naturally due to the preparations of mixtures. Notice, that the set $\mathcal{K}(\mathcal{H})$ is a set of all probability distributions defined on $\mathcal{S}(\mathcal{H})$. It means that any element $\pi \in \mathcal{K}(\mathcal{H})$ can be written as $\{p_j, \varrho_j\}$ where $\varrho_j \in \mathcal{S}(\mathcal{H})$ and $\{p_j\}$ is the probability distribution. Here, the elements π are understood as statistical ansambles (mixings of preparations) - the state ϱ_j is prepared with the probability p_j. Consequently, it must hold that

$$\Lambda\left[\{p_j, \varrho_j\}\right] = \{p_j, \Lambda[\varrho_j]\} ,$$

since each participant from the ansamble, a system prepared in one of the states ϱ_j, evolves independently on the other participants. $\diamond$

So far we have considered only particular cases of Quantum Mechanics and Extended Quantum Mechanics, but above arguments can be used for any physical theory with specified kinematics. Therefore, we can formulate the following theorem

Theorem *Consider a set of states $\mathcal{S}$ and a set of measurements $\mathcal{M}$ (compatible with $\mathcal{S}$), i.e. the kinematics of the theory is given. If the set $\mathcal{S}$ is endowed with the "convex structure" then evolution must be linear.*

In fact, any non-linear evolution leads either to a contradiction or to a new kinematics. Let us suppose that the space $\mathcal{S}$ is endowed with the convex structure and an evolution Λ is non-linear. The convex structure of the space $\mathcal{S}$ is a consequence of the possibility of mixing

preparations. In other words for a set of elements $\pi_j \in \mathcal{S}$ the mixture $\{p_j, \pi_j\}$ is an element $\pi = \sum_j p_j \pi_j$ of the space $\mathcal{S}$ (see the end of the section Kinematics). The non-linearity of the evolution Λ implies that there exist at least one pair of sets of states $\{\varrho_j\} \in \mathcal{S}$ and $\{\xi_i\} \in \mathcal{S}$ where $\sum_j p_j \varrho_j = \sum_i q_i \xi_i$ such that

$$\sum_j p_j \Lambda(\varrho_j) \neq \sum_i q_i \Lambda(\xi_i)$$

The two states $\sum_j p_j \varrho_j$ and $\sum_i q_i \xi_i$ represent two mixing preparations, so that $\Lambda(\sum_j p_j \varrho_j) = \sum_j p_j \Lambda(\varrho_j)$ and $\Lambda(\sum_i q_i \xi_i) = \sum_i q_i \Lambda(\xi_i)$. What's more, they represent the same point and thus transform into a single point $\Lambda(\sum_j p_j \varrho_j) = \Lambda(\sum_i q_i \xi_i)$ which is in contradiction with the assumption that the map Λ is non-linear as

$$\sum_j p_j \Lambda(\varrho_j) = \Lambda(\sum_j p_j \varrho_j) = \Lambda(\sum_i q_i \xi_i) = \sum_i q_i \Lambda(\xi_i) \ .$$

In the case when $\mathcal{S}$ is not convex (like for instance $\mathcal{S} = \{$unit vectors in $\mathcal{H}\}$), then we can create a new set of states $(\mathcal{D}(\mathcal{H}))$, which must either be compatible with the set of observables $\mathcal{M}$, or we also need to change the set of measurements $\mathcal{M}$ in order to preserve the mutual compatibility. In the case of Quantum Mechanics we find out solution, where $\mathcal{S} = \mathcal{S}(\mathcal{H})$ and $\mathcal{M} = \mathcal{M}(\mathcal{H})$. In particular, we can represent pure states $|\psi\rangle$ like one-dimensional projections P_ψ and define probability distributions on these projections. The linear structure of the space of linear operators $\mathcal{B}(\mathcal{H})$ enables us to associate these distributions $\{p_j, P_{\psi_j}\}$ with the linear operators $\sum_j p_j P_{\psi_j} \in \mathcal{B}(\mathcal{H})$. Let us remind that such assignment is a map "from-many to-one". As a result of the identification of all distributions represented by the same operator we obtain the set of density operators $\mathcal{S}(\mathcal{H})$ which is compatible with the set of observables $\mathcal{M}(\mathcal{H})$, i.e. all POVMs.

Example 1: Next we will use the theorem to show in what sense the evolution in Classical Mechanics is linear, too. *The phase space Ω plays role analogical to the* Hilbert space $\mathcal{H}$ (in the case of Quantum Mechanics), or space $\mathcal{S}(\mathcal{H})$ (in the case of Extended Quantum Mechanics). It means that elements of the phase space Ω are extremal points (denoted by $\delta_{\vec{\omega}}$) of the set of all classical states $\mathcal{P}(\Omega)$, i.e. *probability distributions* on Ω. The same arguments as before will lead us to the linearity (on $\mathcal{P}(\Omega)$) in the following sense. Any (discrete) probability distribution $\pi(\vec{\omega}) = \sum_k \pi_k \delta_{\vec{\omega}_k} \leftrightarrow \{\pi_k, \vec{\omega}_k\}$ must evolve with $\Lambda : \mathcal{P}(\Omega) \to \mathcal{P}_\Lambda \subset \mathcal{P}(\Omega)$ according to the rule

$$\pi(\vec{\omega}) = \sum_k \pi_k \delta_{\vec{\omega}_k} \mapsto \Lambda[\pi(\vec{\omega})] = \sum_k \pi_k \Lambda[\delta_{\vec{\omega}_k}]$$

Again, in the sense of $\mathcal{P}(\Omega)$ the evolution Λ is linear, but the transformation of the points in Ω (i.e. $\Lambda : \Omega \to \Omega$) need not be linear. (Unlike the Quantum Mechanics, if one starts with the phase space Ω then the new set of classical states equals to the set of all probability distributions $\mathcal{P}(\Omega)$.) $\diamond$

IV. MEASUREMENTS

As has already been mentioned above the kinematics of the theory can be viewed as a set of states $\mathcal{S}$ and a set of measurements $\mathcal{M}$ where the two sets have to be mutually "compatible". Our notion of "measurements" correspond to "measurable quantities", or "observables". Hence, they do not contain any description of a dynamics of the corresponding *physical process of measurement*. In the case of Quantum Mechanics the set $\mathcal{S}$ is the set of all density operators $\mathcal{S}(\mathcal{H})$, while $\mathcal{M}$ is the set of all positive operator valued measures $\mathcal{M}(\mathcal{H})$. And these two sets are "compatible". The set $\mathcal{M}$ contains enough elements so that we are able to differentiate between any two elements of the set $\mathcal{S}$. Moreover, if we take any representation of a given state (The set $\mathcal{S}$ has a certain structure; see above) then no measurement can differentiate between any such representations. On the other hand if we take the set $\mathcal{S}$ and ask what are all possible measurements i.e. all possible probability measures defined on the set $\mathcal{S}$ then we find out that the sought set is $\mathcal{M}$.

In order to retain the larger set of states $\mathcal{K}(\mathcal{H})$ of Extended Quantum Mechanics, but still use only density operators (i.e. elements of $\mathcal{S}(\mathcal{H})$) for our description of states, we must be able to differentiate between two different types of density matrices: *genuine mixtures* and *elementary mixtures* (for more details see section Kinematics or Ref. [6]). The elementary mixtures represent reduced density operators and genuine mixtures are associated with statistical mixtures of these reduced density operators. However, in order to discriminate between an elementary mixture (reduced density operator) ϱ and a genuine mixture $\{\lambda_j, \omega_j\}$ (with $\omega_j \in \mathcal{S}(\mathcal{H})$ being elementary mixtures) associated with the same density operator $\varrho = \sum_j \lambda_j \omega_j$ (decomposition is fixed), we have to introduce an observable M that is non-linear (see Eq. (2.1))

$$P_M(A, \sum_j \lambda_j \omega_j) \neq \sum_j \lambda_j P_M(A, \omega_j) \ , \qquad (4.1)$$

for at least one $A \in \mathcal{B}(\mathbf{R})$. Consequently, if we decide to deal with the set of density operators $\mathcal{S}(\mathcal{H})$, then the set $\mathcal{M}$ has to include non-linear observables. Let us remind the reader that quantum mechanical observables, self adjoint operators, are in Quantum Mechanics identified with the generators of a dynamical evolution. Once we allow non-linear evolution then we have to include observables that are non-linear and vice versa. But let us stress here that the evolution of genuine mixtures (elements of $\mathcal{K}(\mathcal{H})$) is always linear while elementary mixtures (elements of $\mathcal{S}(\mathcal{H})$) can evolve according to non-linear maps.

V. NO-SIGNALING CONDITION

The impossibility to transmit information faster than the propagation of light seems to be an interesting problem in the context of quantum theory. Undoubtedly, in quantum theory there is no dynamical restriction how fast the particles can move. However, no-signaling does not deal with dynamical properties of the theory, but rather kinematic ones, namely with the possibility to use the *projection postulate* of quantum measurements in the information transfer [10]. Naturally, only the theory with the projection postulate (or more generally any postulate of similar type) can be questioned. Therefore, our following discussion will be focused only on two specific theories: Quantum Mechanics and Extended Quantum Mechanics (for definition see sections Kinematics and Measurements).

Let us consider a projective measurement M represented by the set of projective operators $\{F_k\}$ (for more details see Introduction and Ref. [5]) and let a quantum system be prepared in a state ϱ. The projection postulate states:

After performing a projective measurement M (in non-demolition experiments) resulting in the observation of the value λ_k the system is described by the following state

$$\varrho_k = \frac{F_k \varrho F_k}{\mathrm{Tr}\varrho F_k} \ . \tag{5.1}$$

The postulate is an independent postulate of Quantum Mechanics and has been introduced on account of the following: When we repeat the same measurement (on the same object) right after the first one then the results of the two consecutive measurements are always the same. After many repetitions of the same measurement on systems all prepared in the state ϱ (i.e. following the same preparation process) the final ensemble will be described by a mixture of states ϱ_k associated with different outcomes, i.e. $\varrho_f = \{p_k, \varrho_k\} = \sum_k p_k \varrho_k$ with $p_k = \mathrm{Tr}\varrho F_k$. As a result we have that measurements (without postselection) prepare systems in mixtures. That is, for anybody who does not have access to observed values, the outcome of the measurement is described by this mixture ϱ_f. Moreover, another measurement $\widetilde{M}$ (with corresponding operators $\{\widetilde{F}_\alpha\}$) can result in a different mixture $\{\widetilde{p}_\alpha, \widetilde{\varrho}_\alpha\}$.

Example 2: Let us consider a pair of two-dimensional quantum systems (qubits) denoted as A and B prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left\{ |0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \right\} \ .$$

To demonstrate the projection postulate we will consider a specific measurement M represented by two operators F_k defined as

$$F_0 = \mathbb{1}_A \otimes |0\rangle_B \langle 0|$$
$$F_1 = \mathbb{1}_A \otimes |1\rangle_B \langle 1| \ .$$

It is easy to see that the measurement M is actually a projective measurement acting on the system B only with two projectors $|0\rangle_B \langle 0|$ and $|1\rangle_B \langle 1|$. After obtaining the first outcome the bipartite system is according to Eq. (5.1) in the state $\varrho_0 = |1\rangle_A \langle 1| \otimes |0\rangle_B \langle 0|$ and when the second outcome is measured the system is in the state $\varrho_1 = |0\rangle_A \langle 0| \otimes |1\rangle_B \langle 1|$. Thus, by measuring the bipartite system we prepare a mixture of states ϱ_0 and ϱ_1 with equal probabilities as $\mathrm{Tr}|\psi\rangle\langle\psi|F_0 = \mathrm{Tr}|\psi\rangle\langle\psi|F_1 = 1/2$. What is more interesting is the fact that by measuring the subsystem B the state of the subsystem A changes as well. For instance, if the eigenvalue λ_0 is measured then the state of the bipartite system is ϱ_0 and the state of the subsystem A is

$$\mathrm{Tr}_B \varrho_0 = |1\rangle_A \langle 1|$$

which is different from its original state

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi| = \frac{1}{2}\mathbb{1}_A$$

$\diamond$

The fact that a measurement performed on the system B can change a description of the state of the system A no matter how far from each other they are is certainly a peculiar property. What is more this change is considered to be instantaneous. Therefore, it is correct to ask, whether such property cannot be used for signaling or transmission of information at speeds larger than the speed of light.

In general, consider a bipartite system where A and B are corresponding parts and let the system be in a state ϱ. Any projective measurement M performed on the system B can be represented with operators $F_k = \mathbb{1}_A \otimes P_k$, where P_k are operators (projectors) acting on the system B only. After measuring M the state of the system A is (according to Eq. (5.1)) in one of the states $\mathrm{Tr}_B \varrho_k$ with probability p_k. Due to the fact that the only thing we can predict are the probabilities of individual outcomes of a given measurement but not which of them is observed in a single event, it follows that by measuring the system B we prepare the system A in the mixture $\varrho_A^M = \{p_k, \mathrm{Tr}_B \varrho_k\}$. (Let us note here that the observer possessing the system A does not know the results of the measurements performed on the system B.) For different measurements M the mixture ϱ_A^M can be different [11]. However, if we express the ϱ_A^M as a density operator and use Eq. (5.1) then it is easy to see that

$$\varrho_A^M = \sum_k p_k \mathrm{Tr}_B \varrho_k = \mathrm{Tr}_B \varrho = \varrho_A \ . \tag{5.2}$$

It means, that the resulting state of the system A is described by the original density operator ϱ_A, only its decomposition is different.

As we have shown, by using different measurements on the first part of a bipartite system we can prepare different realizations (mixtures) of a given density operator of the second part. It means that the only information we can "signal" using this procedure is the information on the particular realization of a given density operator. But the *kinematics* of the Quantum Mechanics is such that two different realizations of a given density operator represent the same state (see the section Kinematics) and there is no measurement that the owner of the second system could use to distinguish the two preparations. In other words, within the standard quantum state space $\mathcal{S}(\mathcal{H})$ two different realization of ϱ_A (two different statistical mixtures or a statistical mixture and the reduced density operator) represent the same point. Therefore, the no-signaling holds and follows from the kinematic properties of the set $\mathcal{S}(\mathcal{H})$.

In the context of the Extended Quantum Mechanics with the state space $\mathcal{K}(\mathcal{H})$ the situation is different. The projection postulate corresponds to the projection onto a Dirac distribution $\delta_\psi = \{1, P_\psi\}$, where P_ψ is a projector associated with the vector state $|\psi\rangle \in \mathcal{H}$. If we apply this postulate onto bi-partite systems, then we are able to prepare two different elements of $\mathcal{K}(\mathcal{H})$ from a spatially distant place in the Universe (see Eq. (5.2) and discussion above). Let us remind the reader that in the theory with the state space $\mathcal{K}(\mathcal{H})$ two different decompositions of a given density operator represent different points in $\mathcal{K}(\mathcal{H})$. Therefore, the situation is different from that in Quantum Mechanics. Due to the experimental possibility to discriminate two decompositions of a given density operator, our information transfer (based on this property) will be as fast as we are able to distinguish the two prepared states (the projection is considered to be instantaneous). We can always place the second system far enough from the first one to violate the second principle of relativity, i.e. we will be able to signal at a speed greater than the speed of light.

In conclusion, in the Extended Quantum Theory with the projection postulate the no-signaling condition does not hold [12]. On the other hand, in Quantum Mechanics the no-signaling holds and follows from the kinematics of Quantum Mechanics. We have also shown (see section Dynamics) that the linearity follows from the kinematic properties too. Let us stress here that this result is independent on the no-signaling condition which, in the particular case of kinematics of Quantum Mechanics, is therefore redundant. Finally, even though the Extended Quantum Mechanics together with the projection postulate is not compatible with the no-signaling condition we cannot exclude all non-linear theories (for instance those not using the projection postulate), inasmuch as we have not considered the most general case.

[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin (1932)
[2] Ch. Simon, N.Gisin, V.Bužek, *Phys. Rev. Lett.* **87** 170405 (2001)
[3] P. Bóna, Commnet on "No Signaling Condition and Quantum Dynamics", quant-ph/0201002.
[4] George Svetlichny, Critique of "No Signaling Condition and Quantum Dynamics", quant-ph/0208049.
[5] see for instance A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer Acadenic Publishers, Dordrecht.
[6] P. Bóna, *Acta Phys. Slovaca* **50**, 1-198 (2000)
[7] The notion of the linearity of an evolution Λ makes sense, only when the set $\mathcal{S}$ is a subset of the linear (vector) space. In that case we shall call a transformation $\Lambda : \mathcal{S} \to \mathcal{S}$ linear, if for all λ_j and all $\omega_j \in \mathcal{S}$ (such that $\sum_j \lambda_j \omega_j \in \mathcal{S}$) holds $\Lambda[\sum_j \lambda_j \omega_j] = \sum_j \lambda_j \Lambda[\omega_j]$. Otherwise the evolution Λ is non-linear.
[8] This assumption determines the state space of quantum theory to be $\mathcal{S}(\mathcal{H})$. For more details see section Kinematics.
[9] In principle, we might consider evolutions where a single state evolves into two or more different states. There is a deep reason why not to allow any evolution of that sort. The way physics has been developed is that given the initial conditions and all interactions, the evolution (not measurement) of the system is deterministic, i.e. ("dynamical") future of the system is completely predicted by its initial state and given dynamics. If this were not true then we would need to change not only our theory but also our way of looking at the world. However, this has nothing to do with the possible distinguishability of different preparations of a density operator.
[10] From this point of view it makes no sense to speak about the no-signaling without the projection postulate as was done in the introduction in [2]. But we also should say that despite the authors claims in the indroduction they later used it in subsequent paragraphs of the cited paper.
[11] This holds for any non-factorizable state $\varrho_{AB} \neq \varrho_A \otimes \varrho_B$ where $\rho_{AB} \in \mathcal{S}(\mathcal{H})$.
[12] To avoid the non-signaling in the variant of quantum theory with larger state space $\mathcal{K}(\mathcal{H})$, perhaps we should reconsider the validity of the projection postulate. We should exclude instantaneous action onto the second system. Probably we will need to introduce some dependency on the mutual distance of the measuring device and the affected system. But we do not want to force any new idea about this reconsideration.