Purification and correlated measurements of bipartite mixed states

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We prove that all purifications of a nonfactorable state (i.e., the state that cannot be expressed in a form $\rho_{AB}=\rho_A\otimes\rho_B$) are *entangled*. We also show that for any bipartite state there exists a pair of measurements that are *correlated* on this state if and only if the state is nonfactorable.

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I. INTRODUCTION

Quantum entanglement is one of the most important ingredients of the paradigm of the quantum theory [1,2]. It plays the central role in quantum teleportation [3], quantum dense coding [4], quantum secret sharing [5], and other quantum information processes [6]. Quantum entanglement can be manipulated using the entanglement swapping [7,8] and it can be concentrated via quantum distillation techniques [6].

It is well known that pure entangled states violate the so-called Bell inequalities [9], which implies that these states have nonlocal properties. This means that pure entangled states cannot be created locally. Moreover, for each bipartite pure state, there exists a pair of correlated measurements [1] if and only if the state is entangled.

In the case of mixed states the situation is more complex. Werner [10] has introduced the following definition of entanglement for mixed states: the bipartite mixed state is entangled if and only if it is inseparable. In addition Werner has shown that any separable state can be created exclusively via local operations and classical communication (and hence it does not have nonlocal properties).

In this paper we will concentrate our attention on correlations in measurements performed on mixed bipartite states. The problem of correlations in measurements of two qubits has been studied by Englert and Metwally [11,12]. Specifically, we will derive the necessary and sufficient condition for the existence of correlated measurements on bipartite mixed states.

In what follows, we will utilize the purification ansatz as proposed by Uhlmann [13] via which an impure state of a given quantum system can be purified with the help of ancillas. Our main motivation to study purification of mixed states is to determine the relation between the entanglement present in purified states and the existence of correlations in measurements performed on original bipartite mixed states. We will also study whether these correlations are related to nonlocality of purified states.

In Sec. II we introduce the notion of factorability and we derive the relation between the factorability of bipartite density matrix ρ and the entanglement of any purification of ρ. In Sec. III we prove that for any bipartite density matrix ρ there exists a pair of measurements that are correlated on ρ if and only if ρ is not factorable.

In order to unify the notation and terminology we define correlations in measurements and present two examples, which clarify the problem we address.

Let ρ_{AB} be a bipartite density matrix while $\rho_A=\mathrm{Tr}_B(\rho_{AB})$ and $\rho_B=\mathrm{Tr}_A(\rho_{AB})$ are reduced density matrices of the subsystems A and B, respectively. Let E and F be measurements on the subsystems A and B, respectively, and $\hat{E}$ and $\hat{F}$ be the corresponding observables. Then the measurements E and F are *correlated* on ρ_{AB} if and only if

$$\mathrm{Tr}(\rho_{AB}\hat{E}\otimes\hat{F})\neq\mathrm{Tr}(\rho_A\hat{E})\mathrm{Tr}(\rho_B\hat{F}). \tag{1.1}$$

Example 1. Let us consider two *correlated* sources A and B emitting spin-$\frac{1}{2}$ particles (qubits) such that with the probability $\frac{1}{2}$ both sources simultaneously produce particles in the state $|0\rangle$ and with the probability $\frac{1}{2}$ both particles are simultaneously in the state $|1\rangle$. Hence, the sources produce states $|00\rangle_{AB}$ or $|11\rangle_{AB}$ and the density matrix describing this source is

$$\rho_{AB}=\frac{1}{2}(|00\rangle_{AB}\langle 00|+|11\rangle_{AB}\langle 11|). \tag{1.2}$$

If we subject such a pair of particles to orthogonal (projective) measurements in the bases $\{|0\rangle_A,|1\rangle_A\}$ and $\{|0\rangle_B,|1\rangle_B\}$, then the results of measurements of the state of particles A and B are the same. This is because the pairs were produced in such a way that they are both in the same state. In this case we can apply the formalism of a microcanonical ensemble since we have a set of pairs (of particles) denoted $p_1,p_2,\ldots$, in pure states $|\phi\rangle_1,|\phi\rangle_2,\ldots$, where each of the states $|\phi\rangle_i$ is either $|00\rangle$ or $|11\rangle$. Results of the measurements in this case are two sequences of random variables $a_1,a_2,\ldots$ and $b_1,b_2,\ldots$. Each pair of random variables a_i,b_i is not correlated. But the ensemble of the pairs (which is described as a statistical mixture) exhibits correlations. As pointed out by Werner [10], these correlations have nothing to do with quantum nonlocality and they are caused by classical correlations of the sources.

Example 2. Now let us consider a source that repeatedly produces three spin-$\frac{1}{2}$ particles A,B,C in the Greenberger-Horne-Zeilinger state [6]

$$|\phi\rangle_{ABC}=\frac{1}{\sqrt{2}}(|000\rangle_{ABC}+|111\rangle_{ABC}). \tag{1.3}$$

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Obviously, the reduced density matrix ρ_{AB} of particles A and B is the same as in the previous example described by Eq. (1.2). This means that measurements in the bases $\{|0\rangle_A,|1\rangle_A\}$ and $\{|0\rangle_B,|1\rangle_B\}$ yield the same results as in the previous example. Nevertheless, in order to describe the present situation we have to employ the macrocanonical formalism. Results of the measurements are two random variables, which are correlated, and this correlation is caused by a quantum nonlocality, which follows from the fact that measurements $\{|0\rangle_A,|1\rangle_A\}$ and $\{|0\rangle_B,|1\rangle_B\}$ performed on $|\phi\rangle_{ABC}$ are correlated due to the present quantum entanglement.

In order to appreciate the relevance of these two examples, we note that in spite of the fact that the measurements performed on the two systems generate the same experimental results their interpretation might be totally different. We conclude that although the properties of separable states can be explained locally (i.e., without employing entanglement), the actual physical reason behind these ''classical'' correlations can be related to the quantum nonlocality in preparation of the system.

II. PURIFICATIONS AND FACTORABILITY

We start this section with the definitions of factorability, separability, and purification of density matrices.

The density matrix ρ_{AB} is *factorable*, if it can be written in the form

$$\rho_{AB}=\rho_A\otimes\rho_B . \tag{2.1}$$

The density matrix ρ_{AB} is *separable*, if it can be written in the form

$$\rho_{AB}=\sum_i p_i\rho_A^{(i)}\otimes\rho_B^{(i)} . \tag{2.2}$$

Let us consider a bipartite system AB in the state described by a density matrix ρ_{AB}. Let AB be a subsystem of some larger system ABC_1C_2, which is in a pure state $|\psi\rangle$. Obviously, there is a whole class of states $|\psi\rangle$, which represent *purifications* of the density matrix ρ_{AB}, i.e., which fulfill the condition

$$\mathrm{Tr}_{C_1C_2}(|\psi\rangle\langle\psi|)=\rho_{AB} . \tag{2.3}$$

It is important to note that the purification of a given state ρ_{AB} is not unique. First, from the Schmidt decomposition [1] it follows that we can choose auxiliary systems C_1 and C_2 of an arbitrary dimension such that $\dim(C_1C_2)\geqslant\dim(AB)$. Second, the purification is not unique even when we fix dimensions of Hilbert spaces C_1 and C_2 because if $|\psi\rangle_{ABC_1C_2}$ is a *purification* of ρ_{AB}, then $U_{C_1C_2}|\psi\rangle_{ABC_1C_2}$ is also a purification of ρ_{AB} for any unitary operator $U_{C_1C_2}$ acting on C_1C_2.

Theorem 1. Let ρ_{AB} be a nonfactorable density matrix. Then any purification $|\psi\rangle_{ABC_1C_2}$ of ρ_{AB} is entangled in a sense that it cannot be written in the factorized form

$$|\psi\rangle_{ABC_1C_2}=|\psi_1\rangle_{AC_1}\otimes|\psi_2\rangle_{BC_2}. \tag{2.4}$$

Conversely, if all purifications of ρ_{AB} are entangled, then ρ_{AB} is nonfactorable.

In order to prove this theorem let us suppose that there is a purification $|\psi\rangle_{ABC_1C_2}=|\psi_1\rangle_{AC_1}\otimes|\psi_2\rangle_{BC_2}$ of ρ_{AB}. From the definition of purification it holds that

$$\mathrm{Tr}_{C_1C_2}(|\psi\rangle_{ABC_1C_2}\langle\psi|)=\rho_{AB} . \tag{2.5}$$

However, from the definition of the partial trace we have

$$\begin{aligned}\mathrm{Tr}_{C_1C_2}(&|\psi_1\rangle_{AC_1}|\psi_2\rangle_{BC_2\,AC_1}\langle\psi_1|_{BC_2}\langle\psi_2|)\\ &=\mathrm{Tr}_{C_1}(|\psi_1\rangle_{AC_1}\langle\psi_1|)\otimes\mathrm{Tr}_{C_2}(|\psi_2\rangle_{BC_2}\langle\psi_2|)\\ &=\rho'_A\otimes\rho'_B ,\end{aligned} \tag{2.6}$$

which is in a contradiction with the fact, that ρ_{AB} is a nonfactorable density matrix.

In order to prove the second implication we will prove the following: If ρ_{AB} is factorable, then there exists a purification of ρ_{AB}, which is not entangled. In fact, we will prove a stronger statement by restricting the dimension of the purification. Let $\dim(\mathcal{H}_{AB})=n$. Then there exists a purification of ρ_{AB} of dimension n^2, which is not entangled. It is well known, that there exist purifications $|\phi_1\rangle_{AC_1}$ of ρ_A and $|\phi_2\rangle_{BC_2}$ of ρ_B such that $\dim(\mathcal{H}_A)=\dim(\mathcal{H}_{C_1})$ and $\dim(\mathcal{H}_B)=\dim(\mathcal{H}_{C_2})$. Then $|\psi\rangle_{ABC_1C_2}=|\phi_1\rangle_{AC_1}\otimes|\phi_2\rangle_{BC_2}$ is a purification of ρ_{AB} of the desired dimension, which is not entangled.

Theorem 1 can be easily generalized for n-partite systems in the following way: Let $\rho_{A_1,\dots,A_n}$ is a density matrix, which is not factorable in the sense that it cannot be written as $\rho_{A_1,\dots,A_n}=\rho_{A_1}\otimes\cdots\otimes\rho_{A_n}$. Then any purification $|\psi\rangle_{A_1,\dots,A_nC_1,\dots,C_n}$ of $\rho_{A_1,\dots,A_n}$ is entangled in the sense that it cannot be written in the form

$$|\psi\rangle_{A_1,\dots,A_nC_1,\dots,C_n}=|\psi_1\rangle_{A_1C_1}\otimes\cdots\otimes|\psi_n\rangle_{A_nC_n}. \tag{2.7}$$

Conversely, if each purification of $\rho_{A_1,\dots,A_n}$ is entangled, then $\rho_{A_1,\dots,A_n}$ is not factorable.

From above it follows that if ρ_{AB} is a nonfactorable state and $\rho_{ABC_1C_2}$ an arbitrary mixed state such that $\mathrm{Tr}_{C_1C_2}(\rho_{ABC_1C_2})=\rho_{AB}$, then $\rho_{ABC_1C_2}$ is not factorable in the sense that it cannot be written as $\rho_{ABC_1C_2}=\rho_{AC_1}\otimes\rho_{BC_2}$. This follows from Theorem 1, because each purification of $\rho_{ABC_1C_2}$ is also a purification of ρ_{AB} and thus it is entangled.

It is also straightforward to show that a factorable density matrix ρ_{AB} has both entangled and unentangled purifications.

Specifically, from the factorability we have $\rho_{AB}=\rho_A\otimes\rho_B$. Let $|\psi\rangle_{AC_1}$ is a purification of ρ_A and $|\phi\rangle_{BC_2}$ is a purification of ρ_B. Then $|\psi\rangle_{AC_1}\otimes|\phi\rangle_{BC_2}$ is a purification of ρ_{AB}. Let $|\omega\rangle=U_{C_1C_2}(|\psi\rangle_{AC_1}\otimes|\phi\rangle_{BC_2})$, where $U_{C_1C_2}$ is a unitary operator acting on C_1C_2. Clearly $|\omega\rangle$ is a purification of ρ_{AB} for any $U_{C_1C_2}$ and there is a $U_{C_1C_2}$ such that $|\omega\rangle$ is entangled.

We conclude the present section with the following observation: If a system AB, which is a part of a larger system ABC_1C_2, is in a nonfactorable state ρ_{AB}, then it must be a part of a larger system which is *entangled*. In other words, when we have a nonfactorable system, then any larger system (in a pure state) containing this system is entangled. Moreover, for each purification $|\psi\rangle$ of ρ_{AB}, no unitary $U_{C_1C_2}$ operation can be found such that $U_{C_1C_2}|\psi\rangle$ is unentangled. Hence, the nonfactorability of $|\psi\rangle$ is not caused by the correlation between C_1 and C_2. The most interesting fact is that all previous statements hold regardless if ρ_{AB} is separable or not.

III. CORRELATIONS IN MEASUREMENTS

Theorem 2. Let ρ_{AB} be a nonfactorable density matrix. Then there exists a pair of orthogonal measurements represented by observables E and F (measured on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively), which are correlated on ρ_{AB}.

Proof. In order to prove the theorem we will use the negated implication. That is, let ρ_{AB} be a density matrix such that any two orthogonal measurements E and F performed on ρ_{AB} are uncorrelated. Then $\rho_{AB}=\rho_A\otimes\rho_B$ is factorable.

A result of a measurement can be represented as a random variable. Therefore, the results of the measurement are uncorrelated iff the corresponding random variables are uncorrelated, i.e., the covariance $C(E,F)$ fulfills the condition $C(E,F)=0$. Let $\rho_A=\mathrm{Tr}_B(\rho_{AB})$ and $\rho_B=\mathrm{Tr}_A(\rho_{AB})$, then the covariance of uncorrelated measurements fulfills the condition

$$C(E,F)=\langle E\otimes F\rangle_{\rho_{AB}}-\langle E\rangle_{\rho_A}\langle F\rangle_{\rho_B}=0, \tag{3.1}$$

from which it follows that

$$\mathrm{Tr}(E\otimes F\rho_{AB})=\mathrm{Tr}(E\rho_A)\mathrm{Tr}(F\rho_B)=\mathrm{Tr}(E\otimes F\rho_A\otimes\rho_B). \tag{3.2}$$

We want to show that this identity implies $\rho_{AB}=\rho_A\otimes\rho_B$ and hence that ρ_{AB} is factorable.

The condition (3.2) holds for any two Hermitian operators E and F. Let us choose some fixed basis $\{|\phi_i\rangle_{AB}\}_i$ on $\mathcal{H}_A\otimes\mathcal{H}_B$. We will show that

$$(\rho_{AB})_{ij}=(\rho_A\otimes\rho_B)_{ij}\quad\forall\quad i,j. \tag{3.3}$$

Because Eq. (3.2) holds for any two Hermitian operators E_i and F_i it also holds that

$$\sum_i \alpha_i\mathrm{Tr}(E_i\otimes F_i\rho_{AB})=\sum_i \alpha_i\mathrm{Tr}(E_i\otimes F_i\rho_A\otimes\rho_B) \tag{3.4}$$

for any $\alpha_i\in\mathbb{C}$ and hence

$$\mathrm{Tr}\left(\sum_i \alpha_iE_i\otimes F_i\rho_{AB}\right)=\mathrm{Tr}\left(\sum_i \alpha_iE_i\otimes F_i\rho_A\otimes\rho_B\right). \tag{3.5}$$

To prove Eq. (3.3) it is enough to show that

$$\mathrm{Tr}(A\rho_{AB})=\mathrm{Tr}(A\rho_A\otimes\rho_B) \tag{3.6}$$

for any matrix A such that

$$A_{ij}=1\quad\text{for fixed}\quad i,j\quad\text{and}\quad A_{xy}=0\quad\text{otherwise}. \tag{3.7}$$

However, an arbitrary matrix on $\mathcal{H}_A\otimes\mathcal{H}_B$ can be expressed as

$$\sum_i \alpha_iE_i\otimes F_i, \tag{3.8}$$

where E_i and F_i are Hermitian matrices and α_i is an arbitrary complex number. This completes the proof.

The remaining part of this problem is trivial. When $\rho_{AB}=\rho_A\otimes\rho_B$ (ρ_{AB} is factorable), then the systems A and B are not correlated which follows from Theorem 2.

IV. CONCLUSION

We proved that any purification of a nonfactorable state is always *entangled*. This means that any system that contains a nonfactorable subsystem is also nonfactorable. Moreover, we described purifications of factorable states and we proved that for any bipartite density matrix ρ there exists a pair of measurements, which are correlated on ρ if and only if ρ is nonfactorable. Taking into account the fact that any purification of a nonfactorable state is entangled, we conclude that these correlations have their origin in quantum nonlocality.

This can be interpreted as an alternative approach to Werner's explanation of the origin of correlations in measurements on separable (but nonfactorable) states. Our approach supplements the original work of Werner [10]. Specifically, we showed that correlations on bipartite mixed state exist if and only if the state is nonfactorable. These correlations can be explained locally (see Werner [10]) when the state is separable, or they can be explained via quantum entanglement of purified states (see Sec. II).

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