Quantum Information and Computation, Vol. 2, Special (2002) 530-539

ENTANGLED GRAPHS

Martin PLESCH[1] and Vladimír BUŽEK[1,2]

[1] *Research Center for Quantum Information, Slovak Academy of Sciences*
Dúbravská cesta 9, Bratislava, Slovakia
and
[2] *Department of Mathematical Physics, National University of Ireland*
Maynooth, Co. Kildare, Ireland

Received October 8, 2002
Revised December 9, 2002

We study how bi-partite quantum entanglement (measured in terms of a concurrence) can be shared in multi-qubit systems. We introduce a concept of the *entangled graph* such that each qubit of a multi-partite system is associated with a vertex while a bi-partite entanglement between two specific qubits is represented by an edge. We prove that any entangled graph can be associated with a *pure* state of a multi-qubit system. We also derive bounds on the concurrence for some weighted entangled graphs (the weight corresponds to the value of concurrence associated with the given edge).

Keywords: entanglement

Communicated by: R. Cleve

1. Introduction

In general an N-partite system can exhibit various types of multi-partite correlations, ranging from bi-partite entanglement to intrinsic multi-partite correlations of the GHZ nature. Correlations associated with the system specify its state (certainly, this specification is not necessarily unique). Ideally we would like to know the whole hierarchy of quantum correlations in the multi-partite system. Presently we are able to determine and quantify bi-partite quantum correlations. Unfortunately, for existence of intrinsic N-qubit correlations we even do not have sufficient and necessary conditions (see Refs. [1, 2]). Nevertheless, as suggested by Coffman, Kundu and Wootters [2] it is very instructive to understand how a bi-partite entanglement is "distributed" in N-qubit system. The inequalities conjectured by Coffman, Kundu and Wootters (the so-called CKW inequalities) open new possibilities how to understand the complex problem of bounds on shared entanglement. The CKW inequalities utilize the measure of entanglement called the concurrence as introduced by Wootters et al. [3]. This measure is defined as follows: Let us assume a two-qubit system prepared in the state described by the density operator ρ. Using this operator one can evaluate the spin-flipped operator defined as

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{1}$$

where σ_y is the Pauli matrix and a star denotes a complex conjugation. Now we define the matrix

$$R = \rho\tilde{\rho}\,, \tag{2}$$

and label its (non-negative) eigenvalues, in decreasing order $\lambda_1, \lambda_2, \lambda_3$ and λ_4. The concurrence is then defined as

$$C = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}\,. \tag{3}$$

This function serves as an indicator whether the two-qubit system is separable (in this case $C = 0$), while for $C > 0$ it measures the amount of bipartite entanglement between two qubits with a number between 0 and 1. Larger the value of C stronger the entanglement between two qubits is.

Unfortunately, no simple measures of entanglement are known for multi-qubit systems. Nevertheless it is still of importance to understand how a bi-partite entanglement is distributed in the N-qubit system. In this paper we will utilize a concept of the entangled graph to illuminate some aspects of the problem. The entangled graph corresponds to a state of N qubits such that each qubit is represented by a vertex while a bi-partite entanglement between two specific qubits is represented by an edge.

Using the concurrence we can easily associate an entangled graph with every N-partite state. On the other hand there is no one to one correspondence between graphs and states. For instance all separable states with N qubits have the same graph – N vertices and no edges. Also all GHZ-like states for $N > 2$ would have the same graph. The question we are going to address can be formulated as follows: Is it possible to construct at least one *pure* state for a given graph?

The main difficulty in answering this question lies in the fact that quantum entanglement cannot be shared freely among many particles. For instance, having four qubits, we are able to prepare a state with two e-bits (two Bell pairs, as an example), but not more. This means that the structure of quantum mechanics imposes strict bounds on bi-partite entanglement in multi-partite systems. This issue has been first addressed by Wootters et al. [2, 4] who have derived important bounds on shared bi-partite entanglement in multi-qubit systems. In fact, one can solve a variational problem to answer a question: What is a pure multi-partite state with specific constraints on bi-partite entanglement? O'Connors and Wootters [4] have studied what is the state of a multi-qubit ring with maximal possible entanglement between neighboring qubits. Another version of the same problem has been analyzed by Koashi et al. [5] who have derived an explicit expression for the multi-qubit completely symmetric state (entangled web) in which all possible pairs of qubits are maximally entangled. In his recent work Dür [6] has introduced a concept of *entanglement molecules*. He has shown that an arbitrary entanglement molecule can be represented by a *mixed* state of a multi-qubit system. On the other hand in his work the problem of pure multi-partite states with specific entangled pairs of qubits has not been discussed thoroughly. Specifically, Dür has considered just the condition of inseparability for given set of pairs, but he did not impose a strict condition of separability for the remaining pairs of qubits.

In the present paper we will show that any entangled graph with N vertices and k edges can be associated with a *pure* multi-qubit state. We prove this result constructively, by showing the explicit expression of corresponding pure states. We show that any entangled graph of

N qubits can be represented by a pure state from a subspace of the whole 2^N-dimensional Hilbert space of N qubits. The dimension of this subspace is at most quadratic in the number of qubits. Moreover, we will address also the issue of weighted entangled graphs. That is, we will associate with each edge a given value of the concurrence and we will study whether specific weighted graphs can exist.

The paper is organized as follows. In Sec. 2 we prove by construction that an arbitrary entangled graph can be represented by a pure state of N qubits. Section 3 is devoted to the problem of weighted entangled graphs, while in Sec. 4 we summarize our results.

2. Construction of entangled graphs

Let us first consider entangled graphs associated with *mixed* N-qubit states. These graphs consist of N vertices. Let the parameter k denote the number of edges in the graph, with the condition

$$0 \leq k \leq \frac{N(N-1)}{2}. \tag{4}$$

Once k is specified let us define a set S with k members. These will be pairs of qubits between which we expect entanglement; thus for every $i < j$

$$\begin{array}{lclcl} \{i,j\} & \in & S & \Longleftrightarrow & C(i,j) > 0 \\ \{i,j\} & \notin & S & \Longleftrightarrow & C(i,j) = 0. \end{array} \tag{5}$$

A state of the form

$$|\Psi\rangle_{ij} = |\Psi^+\rangle_{ij}|0...0\rangle_{\overline{ij}} \tag{6}$$

exhibits entanglement between qubits i and j and nowhere else. In Eq. (6) the vector $|\Psi^+\rangle_{ij} = (|01\rangle + |10\rangle)/\sqrt{2}$ represents the maximally entangled Bell state between qubits of two qubits i and j. The rest of $N-2$ qubits are assumed to be in the product state $|0...0\rangle_{\overline{ij}}$. Dür in Ref. [6] has proposed a *mixed* state of N qubits, which corresponds to a graph defined by the set S in the form

$$\rho = \frac{1}{k} \sum_{\{i,j\}\in S} |\Psi\rangle_{ij}\langle\Psi|_{ij}. \tag{7}$$

It is much more complex task to find a *pure* state of N qubits corresponding to a specific graph. We will solve this problem below.

2.1. *Pure states*

We start our analysis with entangled graphs that exhibit specific symmetries. Certainly the two most symmetric graphs are those representing separable states [no edges - see Fig.1 a] and those representing W-states, with all vertices connected by edges. A representative of a pure completely separable state is described by the vector $|\Psi\rangle = |0...0\rangle$. The W-state $|W\rangle_N = 1/\sqrt{N}|N-1,1\rangle$, is a maximally symmetric state with one qubit in state $|1\rangle$ and $N-1$ qubits in state $|0\rangle$ (see Refs. [5, 6]). This state maximizes the bi-partite concurrence - its value is given by the expression $C = 2/\sqrt{N}$. We see that the most symmetric entangled graphs do correspond to specific pure multi-qubit states.

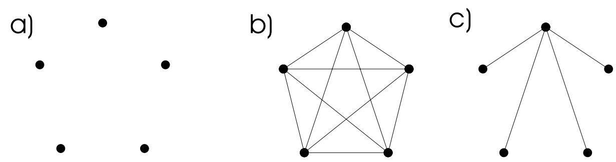

Fig. 1. Examples of entangled graphs associated with states of 5 qubits: **a)** separable states, or any other states with no bipartite entanglement; **b)** W-type states; **c)** star-shaped states

Let us now consider graphs with a lower symmetry. For instance, a star-shaped graph (Fig.1. c). In this case, the given qubit is (equally) entangled with all other qubits in the system, that in turn are not entangled with any other qubit.

Dür [6] has proposed an explicit expression for a pure state associated with this type of entangled graph:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|1\rangle|0...0\rangle + \frac{1}{\sqrt{2}}|0\rangle|N-2,1\rangle\,. \tag{8}$$

In fact, this state maximizes the concurrence between the first and any other qubit. But, the remaining qubits are still mutually entangled. So the state (8) is represented by the graph *b)* (all vertices are connected) rather than graph *c)*. In our analysis we require more stringent constraints than in Ref. [6], where only the conditions on the presence of entanglement between specific qubits have been imposed. We require the conditions (5), that is the presence and absence of bi-partite entanglement for given pairs of qubits in the system.

We find that a pure state which indeed is represented by the star-shaped graph (see Fig. 1 c) is given by the expression

$$|\Psi\rangle = \alpha|W\rangle_N + \beta|0\rangle|1...1\rangle \tag{9}$$

with the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. For $N > 4$, the reduced two-qubit density operator for the first and any other qubit in the system reads

$$\rho_{1i} = \begin{pmatrix} \frac{N-2}{N}|\alpha|^2 & 0 & 0 & 0 \\ 0 & |\alpha|^2\frac{1}{N} + |\beta|^2 & |\alpha|^2\frac{1}{N} & 0 \\ 0 & |\alpha|^2\frac{1}{N} & |\alpha|^2\frac{1}{N} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

One of the eigenvalues of the partially transformed matrix obtained from the matrix (10)

$$\lambda = |\alpha|^2\,\frac{n-2-\sqrt{n^2+8-4n}}{n} \tag{11}$$

is negative for every $\alpha > 0$. Consequently, using the Peres-Horodecki [7] criterion we see that the first qubit is indeed entangled with any other qubit in the system for any non-trivial value of α. Now we have to show, that all other qubits in the system are not mutually entangled (i.e. all pairs of qubits $\{i, j\}$, where $1 < i < j < N$ are separable). The reduced density operator describing a state of two qubits i and j reads

$$\rho_{ij} = \begin{pmatrix} \frac{N-2}{N}|\alpha|^2 & 0 & 0 & 0 \\ 0 & |\alpha|^2 \frac{1}{N} & |\alpha|^2 \frac{1}{N} & 0 \\ 0 & |\alpha|^2 \frac{1}{N} & |\alpha|^2 \frac{1}{N} & 0 \\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix} . \tag{12}$$

The smallest eigenvalue of the partially transposed operator is

$$\lambda = \frac{N - 2\,|\alpha|^2 - \sqrt{\delta}}{2\,N} , \tag{13}$$

where

$$\delta = N^2 - 4\,|\alpha|^2\,(N-1)\,N + 4\,|\alpha|^4\,(2 + (N-2)\,N) . \tag{14}$$

We see that for all α such that

$$|\alpha| \leq \frac{\sqrt{N^2 - 2N}}{N-1} , \tag{15}$$

the smallest eigenvalue λ is non-negative. Consequently, the corresponding density operator is separable. Thus we have found a family of states, that correspond to the desired graph. For the special case of $N = 4$, the reduced operators have a form different from (10) and (12) and also the final condition is more complicated. However, it is quite easy to find one example of the state of four qubits corresponding to the star-shaped graph. The state vector reads:

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{5}} (|0111\rangle + |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \\ &= \frac{2}{\sqrt{5}}|W\rangle_4 + \frac{1}{\sqrt{5}}|0111\rangle. \end{aligned} \tag{16}$$

Above we have analyzed the most symmetric entangled graphs. In what follows we propose a general algorithm how to construct a pure state for an arbitrary graph. Let us consider a pure state of N ($N > 4$) qubits described by the vector

$$|\Psi\rangle = \alpha|0..0\rangle + \beta|1..1\rangle + \sum_{\{i,j\}\in S} \frac{\gamma}{\sqrt{k}}|1\rangle_i|1\rangle_j|0...0\rangle_{\overline{ij}} \tag{17}$$

with the normalization condition $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. In what follows we will show that for a certain range of parameters this state matches a graph given by the condition (5).

Firstly, we show that a pair of qubits i and j such that $\{i,j\} \in S$ is indeed entangled. The corresponding reduced density operator reads

$$\rho_{ij} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 \frac{k-n_i-n_j+1}{k} & 0 & 0 & \frac{\alpha\gamma^*}{\sqrt{k}} \\ 0 & |\gamma|^2 \frac{n_i-1}{k} & |\gamma|^2 \frac{n_{ij}}{k} & 0 \\ 0 & |\gamma|^2 \frac{n_{ij}}{k} & |\gamma|^2 \frac{n_j-1}{k} & 0 \\ \frac{\alpha^*\gamma}{\sqrt{k}} & 0 & 0 & |\beta|^2 + \frac{|\gamma|^2}{k} \end{pmatrix} , \tag{18}$$

where n_i is the number of connections originating from the ith vertex (the number of qubits we wish to have entangled with the ith one) and n_{ij} is the number of vertices, that are connected directly with the ith and jth vertex. The following inequalities for these variables hold:

$$\begin{aligned} 1 &\leq n_i \leq k\,; \\ 0 &\leq n_{ij} < \frac{k}{2}\,; \\ 2 &\leq n_i + n_j \leq k+1\,; \\ 1 &\leq n_i n_j \leq \frac{(k+1)^2}{4}\,. \end{aligned} \tag{19}$$

One of the eigenvalues of the density operator obtained by the partial transposition of the operator (18) reads

$$\lambda = \frac{|\gamma|}{2k}\left(|\gamma|\,(n_i+n_j-2) - \sqrt{4|\alpha|^2 k + |\gamma|^2 (n_i - n_j)^2}\right). \tag{20}$$

In the non-trivial case of $|\gamma| > 0$ we need only to show that

$$|\gamma|^2\,(n_i+n_j-2)^2 < 4|\alpha|^2 k + |\gamma|^2 (n_i-n_j)^2. \tag{21}$$

If we use the inequalities (19), we find the following constraints

$$\begin{aligned} |\gamma|^2\,(n_i+n_j-2)^2 < |\gamma|^2 k^2 \leq 4|\alpha|^2 k\,, \\ 4|\alpha|^2 k \leq 4|\alpha|^2 k + |\gamma|^2 (n_i-n_j)^2, \end{aligned} \tag{22}$$

from which it follows that if the condition

$$0 < |\gamma|^2 k \leq 4|\alpha|^2 \tag{23}$$

is fulfilled then a specific pair qubits described by the density operator (18) is entangled.

Till now we have proved that a specific pair of qubits in multi-partite system is entangled. In order, to show that the corresponding state vector indeed is associated with a desired entangled graph we have to show that all other pairs of qubits are separable. Density operators for pairs of qubits $\{i,j\} \notin S$ are given by the expression:

$$\rho_{ij} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 \frac{k-n_i-n_j}{k} & 0 & 0 & 0 \\ 0 & |\gamma|^2 \frac{n_i}{k} & |\gamma|^2 \frac{n_{ij}}{k} & 0 \\ 0 & |\gamma|^2 \frac{n_{ij}}{k} & |\gamma|^2 \frac{n_j}{k} & 0 \\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix}$$

with the involved parameters satisfying the set of inequalities

$$\begin{aligned} 0 &\leq n_i \leq k\,; \\ 0 &\leq n_{ij} \leq \frac{k}{2}\,; \\ 0 &\leq n_i + n_j \leq k\,; \\ 0 &\leq n_i n_j \leq \frac{k^2}{4}\,. \end{aligned} \tag{24}$$

Instead of checking that all the eigenvalues of the corresponding partially transposed operator are non-negative, we will show that under certain conditions the concurrence of the state (24) will be zero. The eigenvalues of the operator R given by Eq. (2) are

$$\begin{aligned} \lambda_1 &= \lambda_2 = |\alpha\beta|^2 + |\gamma|^2 \frac{k - n_i - n_j}{k}\,; \\ \lambda_{3,4} &= |\gamma|^4 \left(\frac{n_{ij} \pm \sqrt{n_i n_j}}{k}\right)^2 ; \qquad \lambda_4 \geq \lambda_3\,, \end{aligned} \tag{25}$$

and, according to the definition of the concurrence (3), it is enough to show that $\lambda_1 \geq \lambda_4$ (since then λ_1 is the maximal eigenvalue and already $\sqrt{\lambda_1} - \sqrt{\lambda_2} = 0$ and so the concurrence vanishes). That is, we require that

$$|\alpha\beta|^2 + |\gamma|^2 \frac{k - n_i - n_j}{k} \geq |\gamma|^4 \left(\frac{n_{ij} + \sqrt{n_i n_j}}{k}\right)^2 . \tag{26}$$

When we use the inequalities (24) we obtain the final condition

$$|\alpha\beta|^2 \geq |\gamma|^4 > 0\,, \tag{27}$$

which guarantees that the state (24) is separable.

One can check that there are many states which fulfill the conditions (23) and (27). In particular, let us assume the state (17) with

$$\alpha = \frac{k}{\sqrt{k^2 + 2k + 4}}\,; \qquad \beta = \frac{2\alpha}{k}\,; \qquad \gamma = \alpha\sqrt{\frac{2}{k}}\,. \tag{28}$$

This state indeed corresponds to the desired graph. This proves that one can associate with an arbitrary entangled graph a pure state. Moreover, by construction we have proved that in general this state is a superposition of at most N^2 vectors from the 2^N-dimensional Hilbert space of N qubits.

3. Weighted graphs

A natural generalization of the scheme of entangled graphs is to introduce weights to the edges of the graphs. The weight is equal to the corresponding value of the concurrence. The graph itself is defined by the number of qubits N and by the concurrencies between all pairs of qubits C_{ij}. Again, one can ask, for which graphs there exists a mixed, or even a pure state.

For mixed-state case, Dür in his work [6] has suggested states for which the total concurrence in the system, irrespective on the number of qubits or a particular configuration, was limited by 1. This approach is working very efficiently for weekly entangled systems, but does not answer the question, how strong the entanglement can be. It guarantees the existence (for mixed states) of all the graphs, where $\sum_{i,j} C_{ij} \leq 1$, but does not solve the existence question for all other graphs.

In this section, we will try to answer the question of existence of weighted entangled graphs for *pure* states for two simple examples. This will be the so-called entangled web state, where all qubits are entangled with the same strength. And, the second one, the so called star-shaped graph, where one specific qubit is entangled with all the others as strong as possible.

However, firstly we have to recall the basic property, that all weighted graphs have to fulfill. From earlier results of Wootters et al. [2] it follows that only such states can exist, that do not break the CKW inequalities:

$$\vee\, j, \quad \sum_i C_{ij}^2 \leq 1. \tag{29}$$

Wootters et al. [2] have conjectured that all N-qubit states have to fulfill the inequalities of the form (29). Except for the case $N = 3$ these inequalities have not been proven yet, though numerical studies do strongly suggest that the CKW inequalities do hold. Therefore in this paper we assume their validity.

This already helps us to rule out a significant number of possible weighted graphs. However, these inequalities are not tight, and not all the graphs, that do not break (29), are realizable by pure or even mixed states. The most simple example is the entangled web, studied by Koashi et. all. in [5] who addressed the question, how strong the mutual concurrence in a system of N equally entangled qubits can be. Koashi et al. have proved that the best possible choice (or, rather one of the best possibilities) for maximizing the concurrence is the W-state

$$\begin{aligned} |W\rangle_N &= |N-1;1\rangle \\ &= \frac{1}{\sqrt{N}}\left(|0..01\rangle + |0..010\rangle + ... + |10..0\rangle\right), \end{aligned} \tag{30}$$

that is a fully symmetric state with one qubit in state $|1\rangle$ and all the others in the state $|0\rangle$. The concurrence between any two qubits in such a system is

$$C_{web} = \frac{2}{N}, \tag{31}$$

which is significantly lower than the bound given by CKW (29), i.e.

$$C_{CKW} = \frac{1}{\sqrt{N-1}}. \tag{32}$$

Using this result, we can formulate the theorem of existence of fully symmetric weighted entangled graphs. If the required concurrence in such a graph is greater than $\frac{2}{N}$, there does not exist a state represented by that graph. In other cases, we search for a state, where the concurrence between any two qubits will reach a certain given value $C \leq C_{web}$. The searched state has the form

$$|\Psi\rangle_{web} = \sqrt{\frac{CN}{2}}|W\rangle_N + \sqrt{1-\frac{CN}{2}}|0...0\rangle. \tag{33}$$

Since the state is fully symmetric, all the two-particles density matrices are mutually equal. The calculation of concurrence according to (3) is rather simple and gives us the expected value C.

A more complex example is the star-shaped graph. There are two possible views on this problem. The first one is, that we want to maximize the concurrence between the first and the rest of the qubits in the system, but we do not care about the value of the concurrence between other pairs in the system. Then the problem is easy to solve, because the state

$$|\Psi\rangle_{star} = \sqrt{\frac{1}{2}}|1\rangle|0...0\rangle + \sqrt{\frac{1}{2}}|0\rangle|W\rangle_{N-1} \tag{34}$$

reaches the boundary given by the CKW inequalities,[a] since the concurrence between first and any other qubit is just

$$C_{1i} = \frac{1}{\sqrt{N-1}}. \tag{35}$$

However, the concurrence between any other pair in the system does not vanish and it takes the value

$$C_{ij} = \frac{1}{N-1}, \tag{36}$$

so this is rather a form of the entangled web with some of the edges stronger, than the star-shaped graph.

Nevertheless, we can impose the requirement on the separability of other pairs in the system (e.g. $C_{ij} = 0 \quad j, i > 1$) and again search for the strongest possible entanglement. One would expect, that the boundary could be more strict, since we have imposed a new constrain to the system. This is however true only partially.

Let us suppose a state of the form

$$|\Psi\rangle_{star} = \sqrt{\frac{1}{2}}|0...0\rangle + \sqrt{\frac{1}{2}\alpha}|1\rangle|W\rangle_{N-1} + \sqrt{\frac{1-\alpha}{2}}|1...1\rangle \tag{37}$$

with

$$\alpha \leq \frac{(N-1)\left(\sqrt{N^2-4N+5}-1\right)}{(N-2)^2} = \alpha_{\max} < 1. \tag{38}$$

The choice of the constant $\alpha_{\max}$ guarantees, that the concurrence between any pair of qubits in the system, except the pairs connected with the first qubit, will be zero. Concurrence (for $\alpha = \alpha_{\max}$) between the first and other qubits has a rather complicated structure, but as we expect it in a form somehow similar to (35), we can expand it into the Taylor series and see, that it reads

$$C_{star} = \frac{1}{\sqrt{N-1}}\left(1 - \frac{1}{4(N-1)^2} - \frac{1}{4(N-1)^3} + ...\right). \tag{39}$$

So, for large N, one is able to find a rather simple pure state, that has concurrencies between the first and any other qubit nearly as strong as possible, and has no bipartite entanglement within the rest of the system. Even for $N = 4$ the difference between the highest possible concurrence, given by the CKW inequalities $C_{\max} \cong 0.577$ and the achieved concurrence $C \cong 0.555$ is rather small.

When requiring concurrencies smaller than the limiting value, one has to adjust only the parameter α. By decreasing the value of α from $\alpha_{\max}$ to zero the concurrence decreases, since the state (37) approaches the GHZ state of N qubits, where no bipartite entanglement is

[a]Here we note once again, that the CKW inequalities (29) have been conjectured but have not been proved yet. These inequalities are very reasonable, so we consider them to be valid and some of our further considerations do depend on their validity.

present at all. Because the concurrence is always a continuous function of the parameters of the state, we can cover all values of the concurrence between zero and the maximally achievable value. So, finally, we are able to formulate the theorem for star-shaped graphs: Given a star-shaped graph, where the concurrence between the first and any other qubit fulfills the constraint

$$C \leq C_{star}, \tag{40}$$

we are able to find a pure state, which is characterized by this graph. The only open question remains for graphs with $C_{star} < C \leq C_{\max}$, which is, however, only a small interval for big N.

4. Conclusion

We have introduced a concept of the *entangled graphs*: that is an entangled multi-qubit structure such that every qubit is represented by a vertex while entanglement between two qubits is represented as an edge between relevant vertices. We have shown that for every possible graph with non-weighted edges there exists a *pure* state, which represents the graph. Moreover, such state can be constructed as a superposition of a rather small number of states from a subspace of the Hilbert space. The dimension of this subspace grows linearly with the number of entangled pairs (thus, in the worst case, quadratically with the number of particles).

In addition we have analyzed also weighted entangled graphs. In particular we have constructed a specific star-shaped entangled graph that in the limit of large number of qubits saturates the Coffman-Kundu-Wootters inequality.

Acknowledgements

We thank Mário Ziman for many helpful discussions. This work was supported by the IST-FET-QIPC project QUPRODIS. VB would like to acknowledge the SFI E.T.S.Walton award.

References

1. G. Alber, T. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M Rötteler, H. Winfurter, R.F. Werner, A. Zeilinger, Quantum information (Springer Verlag, Berlin, 2001)
2. V. Coffman, J. Kundu, W. K. Wootters, Phys. Rev. A **61**, 052306 (2000)
3. S. Hill, W.K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997); W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998)
4. K.M. O'Connor and W.K. Wootters, Phys. Rev. A **63**, 052302 (2001).
5. M. Koashi, V. Bužek, N. Imoto, Phys. Rev. A **62**, 050302 (2000)
6. W. Dür, Phys. Rev. A **63**, 020303(R) (2001)
7. A. Peres,*Phys. Rev. Lett.* **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).