Quantum homogenization for continuous variables: Realization with linear optical elements

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Recently Ziman *et al.* [Phys. Rev. A **65**, 042105 (2002)] have introduced a concept of a *universal* quantum homogenizer which is a quantum machine that takes as input a given (system) qubit initially in an arbitrary state ρ and a set of N reservoir qubits initially prepared in the state ξ. The homogenizer realizes, in the limit sense, the transformation such that at the output each qubit is in an arbitrarily small neighborhood of the state ξ irrespective of the initial states of the system and the reservoir qubits. In this paper we generalize the concept of quantum homogenization for qudits, that is, for d-dimensional quantum systems. We prove that the partial-swap operation induces a contractive map with the fixed point which is the original state of the reservoir. We propose an optical realization of the quantum homogenization for Gaussian states. We prove that an incoming state of a photon field is homogenized in an array of beam splitters. Using Simon's criterion, we study entanglement between outgoing beams from beam splitters. We derive an inseparability condition for a pair of output beams as a function of the degree of squeezing in input beams.

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I. INTRODUCTION

One of the most complex phenomena in quantum theory is dynamics of open systems [1]. In general, one can assume an interaction between the open system denoted as S with the environment R. This environment is a quantum system with the Hilbert space of an arbitrary dimension. The whole $S+R$ system evolves unitarily and the question of irreversibility of dynamics of open systems is then a great issue. How does irreversible dynamics of the system S emerge from a unitary evolution of the $S+R$ system? For instance, when a system interacts with a reservoir which is in thermal equilibrium then after some time the system is thermalized—it relaxes towards the thermal equilibrium. This implies that the information about the original state of the system is (irreversibly) "lost" and its new state is determined exclusively by the parameters (temperature) of the reservoir. If the reservoir is composed of a large number N of physical objects of the same physical origin as the system itself, then the thermalization process can be understood as homogenization: out of N objects (the reservoir) prepared in the same thermal state and a single system in an arbitrary state, we obtain $N+1$ objects in the same thermal state. This intuitive picture is based on certain assumptions about the interaction between the system and the reservoir, about the physical nature of the reservoir itself and the concept of the thermal equilibrium. Such a model is very important for the understanding of many processes in quantum physics as well as the fundamental problem of the irreversibility [1,2]. For this reason it is important to analyze rigorously the process of information transfer in this simple model which has been first analyzed in a recent work [3] for qubits. In this paper we present a rigorous analysis of the above picture within the framework of quantum information theory for d-dimensional quantum systems—qudits.

Specifically, we will consider a system S represented by a single qudit initially prepared in an unknown state $\varrho_S^{(0)}$, and a reservoir R composed of N qudits all prepared in the state ξ, which is arbitrary but same for all reservoir qudits. We will enumerate the qudits of the reservoir and denote the state of the kth qudit as ξ_k [4]. From the definition of the reservoir it follows that initially $\xi_k=\xi$ for all k, so the state of the reservoir is described by the density matrix $\xi^{\otimes N}$.

Let U be a unitary operator representing the interaction between a system qudit and one of the reservoir qudits. In addition, let us assume that at each time step the system qudit interacts with just a single qudit from the reservoir (see Fig. 1). Moreover, the system qudit can interact with each of the reservoir qudits at most once. After the interaction with the first reservoir qudit the system is changed according to the following rule (which is a completely positive—CP—map),

$$\varrho_S^{(1)}=\mathrm{Tr}_1[U\varrho_S^{(0)}\otimes\xi_1U^\dagger]. \tag{1.1}$$

Let us repeat the interaction N times, that is, via a sequence of interactions the system qudit interacts with N reservoir qudits all prepared in the state ξ. The final state of the system is then described by the density operator

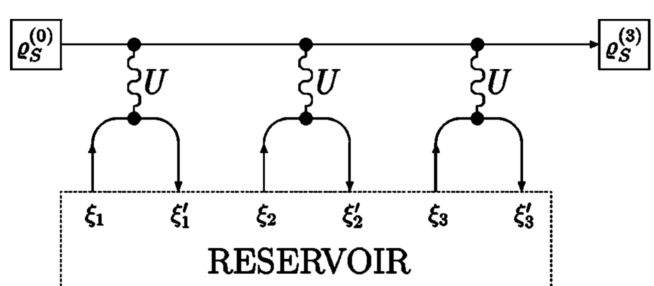

FIG. 1. A simple collisionlike model of homogenization with just three reservoir qudits involved.

$$\varrho_S^{(N)} = \mathrm{Tr}_R[U_N \cdots U_1(\varrho_S^{(0)} \otimes \xi^{\otimes N})U_1^\dagger \cdots U_N^\dagger], \tag{1.2}$$

where $U_k := U \otimes (\otimes_{j\neq k} \mathbb{1}_j)$ describes the interaction between the kth qudit of the reservoir and the system qudit. This model of homogenization is very similar to the *collision model* since the system becomes homogenized via a sequence of individual interactions with the reservoir qudits. The interactions are assumed to be localized in time (i.e., they act like elastic collisions) [5].

Our aim is to investigate possible CP maps induced by the transformation (1.2) and describe the process of homogenization. Homogenization means that due to the interaction U, the states of the qudits in reservoir change only little while after N interactions the system's state become close to the initial state of the reservoir qudits. Formally,

$$\forall \quad k, \quad 1 \leqslant k \leqslant N; D(\xi_k', \xi) \leqslant \delta; \tag{1.3}$$

$$\forall \quad N \geqslant N_\delta; D(\varrho_S^{(N)}, \xi) \leqslant \delta, \tag{1.4}$$

where $D(.,.)$ denotes some distance (e.g., a trace norm) between the states, $\delta > 0$ is a small parameter which is chosen *a priori* to determine the degree of the homogeneity and $\xi_k' := \mathrm{Tr}_S[U\varrho_S^{(k-1)} \otimes \xi U^\dagger]$ is the state of the kth reservoir qudit after the interaction with the system qudit.

We note that homogenization is closely related to *thermalization* [6]. There are, however, two main differences: in thermalization, (i) the state ξ of the reservoir qudits is not completely unknown, but is a thermal state, that is, a state diagonal in a *given* basis (interpreted as the basis of the eigenstates of a single-qudit Hamiltonian); and (ii) the number of qudits in the reservoir is considered to be infinite for any practical purpose.

Our paper is organized as follows: in Sec. II we show that quantum homogenization can be realized with the help of a *partial-swap* operation. In Appendix A we show that the partial swap for qudits generates a contractive CP map on the system qudit with the fixed point being the initial state of the reservoir. This ensures the required convergence of the homogenization process. In Sec. III we address a feasible optical realization (via a sequence of beam splitters) of the homogenization map for continuous variables. In Sec. IV we study the dynamics of the input signal light field homogenized by an array of beam splitters, while Sec. V is devoted to the problem of entanglement between the modes involved in the homogenization process.

II. PARTIAL-SWAP OPERATION

Let us start with the definition of the so-called *swap* operation S acting on the Hilbert space of two qudits which is given by the relation [7]

$$S|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle. \tag{2.1}$$

With the transformation

$$S \quad \varrho^{(0)} \otimes \xi \quad S^\dagger = \xi \otimes \varrho^{(0)}, \tag{2.2}$$

after just a single interaction, the state of the system S is equal to the state ξ of the reservoir qudit; and the interacting qudit from the reservoir is left in the initial state of system. This means that condition (1.4) is fulfilled, while condition (1.3) is not.

In order to fulfill both conditions (1.4) and (1.3) we have to find some unitary transformation which is "close" to the identity on the reservoir qudit, while it performs a *partial*-swap operation, so that the system qudit at the output is closer to the reservoir state ξ than before the interaction. The swap operator is Hermitian, and therefore we can define the unitary *partial-swap* operation

$$P(\eta) = (\cos\eta)\mathbb{1} + i(\sin\eta)S \tag{2.3}$$

that serves our purposes. In what follows we denote $\sin\eta = s$ and $\cos\eta = c$.

In the process of homogenization, the system qudit interacts sequentially with one of the N qudits of the reservoir through the transformation $P(\eta)$. The states of the system qudit and of the reservoir qudit are obtained as partial traces. Specifically, after the first interaction the system qudit is in the state described by the density operator

$$\varrho_S^{(1)} = c^2 \varrho_S^{(0)} + s^2 \xi + ics[\xi, \varrho_S^{(0)}], \tag{2.4}$$

while the first reservoir qudit is now in the state

$$\xi_1' = s^2 \varrho_S^{(0)} + c^2 \xi + ics[\varrho_S^{(0)}, \xi]. \tag{2.5}$$

We can recursively apply the partial-swap transformation and after the interaction with the nth reservoir qudit, we have

$$\varrho_S^{(n)} = c^2 \varrho_S^{(n-1)} + s^2 \xi + ics[\xi, \varrho_S^{(n-1)}], \tag{2.6}$$

as the expression for the density operator of the system qudit, while the nth reservoir qudit is in the state

$$\xi_n' = s^2 \varrho_S^{(n-1)} + c^2 \xi + ics[\varrho_S^{(n-1)}, \xi]. \tag{2.7}$$

In the Appendix A we show that $\varrho_S^{(N)}$ monotonically converges to ξ for all parameters $\eta \neq 0$. This means, in particular, that condition (1.4) does not impose any constraint on η. To show this convergence, we utilized the *Banach theorem* [8] that concerns the fixed point of a contractive transformation. Specifically, let $\mathcal{S}$ be a metric space with a distance function $D(.,.)$. The transformation $\mathcal{T}$ is called *contractive* if it fulfills the inequality $D(\mathcal{T}[\varrho], \mathcal{T}[\xi]) \leqslant kD(\varrho, \xi)$ with $0 \leqslant k < 1$ for all $\varrho, \xi \in \mathcal{S}$. The fixed point of the transformation $\mathcal{T}$ is an element of $\mathcal{S}$ for which $\mathcal{T}[\xi] = \xi$. The Banach theorem states that a contractive map has a unique fixed point [9], and that the iteration of the map converges to it, i.e., $\mathcal{T}^N[\varrho] \to \xi$ for each $\varrho \in \mathcal{S}$.

III. HOMOGENIZATION OF LIGHT FIELDS

In the preceding section we have presented a simple model of an open system interacting with reservoir particles. We have shown that a partial-swap operation induces a contractive map on a system qudit, with the initial state of reservoir qudits as the fixed point. This model can serve for a detailed analysis of an information transfer and the problem of reversibility from the point of view of quantum information theory. As shown in Ref. [3], the process of homogenization can be reversed and the original state of the system $\varrho_S^{(0)}$ and the reservoir ξ can be recovered only when the $N+1$ qubits of the output state interact, via the inverse of the original partial-swap operation, in the "correct" order. This *classical* information about the sequence of interaction is vital for reversibility of the quantum process under consideration.

Our results are valid not only for qudits but also for continuous variables. That is, the model can be used for a description of an interaction of optical fields in appropriate settings. In particular, the partial-swap operation can be realized with the help of a beam splitter, so that the whole process of the homogenization can be represented as a transformation of a signal mode via a sequence of interactions with reservoir modes on highly transitive beam splitters. In an idler port of each of the beam splitter we launch a mode, playing the role of the reservoir "particle." All reservoir modes are initially in the same state ξ. It can be shown that at the output of the sequence of the beam splitters, the signal mode is in the state which (in the limit sense $N\to\infty$) is the state of the reservoir modes, provided that the reservoir modes are initially prepared in a Gaussian state with zero mean amplitude (e.g., a thermal state, a squeezed vacuum state, and a squeezed thermal state).

Beam splitters for partial swap

Two input fields are mixed at a beam splitter to give two output fields. We can model a simple interaction of two photons by the use of a beam splitter—a linear optical device. The input states described by bosonic operators $\hat{a}$ and $\hat{b}$ are mixed at the beam splitter. The output field annihilation operators are given by $\hat{c}=\hat{B}\hat{a}\hat{B}^\dagger$ and $\hat{d}=\hat{B}\hat{b}\hat{B}^\dagger$, where the beam-splitter operator $\hat{B}$ is (see Ref. [10])

$$\hat{B}=\exp\left[\frac{\theta}{2}(\hat{a}^\dagger\hat{b}e^{i\phi}-\hat{a}\hat{b}^\dagger e^{-i\phi})\right], \tag{3.1}$$

with the transmitivity and reflexivity of the beam splitter given by expressions $t=\cos\theta/2$ and $r=\sin\theta/2$, respectively.

Our task is to examine how a state of a photon changes after many weak interactions with photons from reservoir. We assume the interactions are weak, thus the transmitivity of the beam splitters will be approaching unity. In this case, the signal photon is left almost undisturbed by each interaction. Nevertheless, with the increasing number of interactions, the input signal is slowly transformed under the influence of the reservoir modes. On the other hand the interacting reservoir states are only slightly changed. We will show that this process of *quantum homogenization* [3] can be realized in an array of beam splitters. A beam splitter is not the general partial-swap operation, but it realizes the homogenization for the reservoir modes prepared in Gaussian states with the zero mean amplitude (see below).

The beam splitter is also one of few experimentally accessible devices, which may act as an entangler [11]. In the array of beam splitters the original signal is homogenized and is simultaneously correlated with reservoir modes. During the process of homogenization, the information that has been originally encoded in the state of the signal mode is gradually transferred into the correlations between interacting modes. Depending on the character of quantum-statistical properties of the incoming modes, these correlations might have purely quantum nature. That is, at the output the light modes are entangled. In what follows we will use it Simon's criterion [12] to determine whether the modes at the output are indeed entangled.

IV. DYNAMICS OF HOMOGENIZED LIGHT FIELD

In this section we turn our attention to a particular optical realization of quantum homogenization with the help of a beam-splitter array. We note that the beam-splitter transformation, in general, does not realize a partial-swap operation. It is easy to show that in general the beam splitter does not obey the conditions for homogenization: Let us assume that the two inputs of a beam splitter are in a Fock state $|1\rangle$. In this case the two output modes of the 50:50 beam splitter are in the state $(|2,0\rangle+|0,2\rangle)/\sqrt{2}$. On the other hand we expect the output of the partial swap in this case to be $|1,1\rangle$.

However, we will show the beam-splitter array realizes quantum homogenization with reservoir modes prepared in Gaussian states with zero displacement. Quantum homogenization is a process, in which an initial quantum state ζ is changed into the reservoir state η by many small sequential interactions with reservoir states initially prepared in the same state η. Modeling of reservoirs by beam splitters has been previously studied in Ref. [13].

We can describe a single-mode photon field prepared in a state $|\Psi\rangle$ by its Wigner function $W_{|\Psi\rangle}(\xi)$, which is a quasiprobability function in phase space. This function is a Fourier transform of the Weyl characteristic function $C^{(w)}_{|\Psi\rangle}(\eta)$,

$$W_{|\Psi\rangle}(\xi)=\frac{1}{\pi}\int C^{(w)}_{|\Psi\rangle}(\eta)\exp(\xi\eta^*-\xi^*\eta)\, d^2\eta.$$

The characteristic function $C^{(w)}_{|\Psi\rangle}(\eta)$ of a system described by the density operator $\hat{\rho}$ is defined as

$$C^{(w)}_{|\Psi\rangle}(\eta)\equiv\mathrm{Tr}[\hat{\rho}\hat{D}(\eta)],$$

where $\hat{D}(\eta)\equiv\exp[\eta\hat{a}^\dagger-\eta^*\hat{a}]$ is the displacement operator. We will choose the characteristic function notation, mainly because the computations will be relatively simple in this form. When the two input states are represented by the Weyl characteristic function $C^{(w)}_a(\zeta)C^{(w)}_b(\eta)$, the Weyl characteristic function of the two-mode output field after a beam-splitter operation reads [11]

$$C^{(w)}_{out}(\zeta,\eta)=C^{(w)}_a(t\zeta+re^{i\phi}\eta)C^{(w)}_b(t\eta-re^{i\phi}\zeta). \tag{4.1}$$

Let us consider only the output of the signal mode (c mode as labeled in Fig. 2) and its evolution. That is, this time we are not interested in what happens to the disturbed reservoir modes. It is convenient to consider only the one-mode

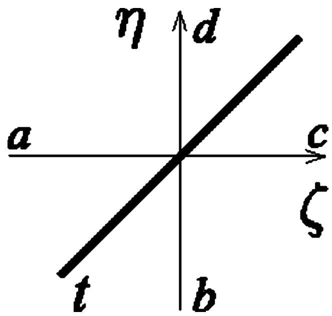

FIG. 2. Schematic description of a beam splitter.

characteristic function of the main output after each beam-splitter interaction. The characteristic function of the output signal from a beam splitter takes a very simple form [we are choosing $\phi=0$ in Eq. (3.1), so that the transmitivity and reflexivity are real]

$$C_c(\zeta)=[C_a(t\zeta+r\eta)C_b(t\eta-r\zeta)]_{\eta=0}$$
$$=C_a(t\zeta)C_b(-r\zeta). \tag{4.2}$$

Let us now assume that the incoming photon is in a Gaussian state with the Weyl characteristic function

$$C_a(\zeta)=\exp[iA_0\zeta_r-2iB_0\zeta_i-\tfrac{1}{2}C_0\zeta_r^2-\tfrac{1}{2}D_0\zeta_i^2], \tag{4.3}$$

and that the reservoir modes are prepared in Gaussian states with zero displacement (squeezed thermal states) with the Weyl characteristic function

$$C_b(\eta)=\exp[-\tfrac{1}{2}E\eta_r^2-\tfrac{1}{2}F\eta_i^2]. \tag{4.4}$$

The output from a single beam-splitter operation is also a Gaussian state. Therefore it keeps the form of C_a, while the coefficients A, B, C, and D change.

$$C_c^{(1)}(\zeta)=\exp[2i(At)\zeta_r-2i(Bt)\zeta_i]\times\exp[-\tfrac{1}{2}(Ct^2+Er^2)\zeta_r^2$$
$$-\tfrac{1}{2}(Dt^2+Fr^2)\zeta_i^2]. \tag{4.5}$$

After a given number k of beam-splitter operations (see Fig. 3), $C_c^{(k)}$ has still the Gaussian form (4.5). When we perform a geometrical sum, we obtain the following results for the coefficients in the characteristic function of the output after the kth beam-splitter interaction:

$$\begin{aligned} A_k&=t^kA_0,\\ C_k&=t^{2k}C_0+(1-t^{2k})E,\\ B_k&=t^kB_0,\\ D_k&=t^{2k}D_0+(1-t^{2k})F. \end{aligned} \tag{4.6}$$

Now we can take a limit $k\to\infty$, considering the transmitivity $0<t<1$ (there is no interaction between the input states for the extreme values of transmitivity, i.e., $t=1$ and 0). It is now obvious that the beam-splitter array homogenizes the incoming state and changes it towards the reservoir state, so that $\lim_{k\to\infty} C_c^{(k)}=\exp[-\tfrac{1}{2}E\eta_r^2-\tfrac{1}{2}F\eta_i^2]$. Both displacements

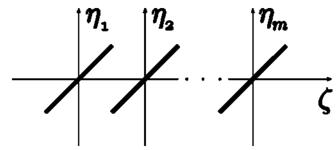

FIG. 3. Schematic description of an array of beam splitters.

A_k and B_k are approaching zero in this process and coefficients C_k and D_k are gaining the properties of the reservoir state (they are approaching E and F, respectively).

From above we can derive two important conclusions:

(1) If the reservoir modes are initially prepared in Gaussian states with zero displacement, then the state of the reservoir is invariant under the action of a sequence of beam-splitter transformations. This means that this state is the fixed point of the corresponding CP map and the reservoir is "stable." This can be easily verified when we consider in Eq. (4.3) for the characteristic function of the signal mode, the expression equal to the characteristic function of the reservoir mode, i.e., $A=B=0$, $C=E$, and $D=F$. In this case expression (4.4) for the characteristic function of the output of the signal is the same as the characteristic function of the input state of the reservoir. The beam-splitter operation is indeed the partial-swap operation restricted on the class of Gaussian states with zero displacement.

(2) Due to the fact that coherent states form an overcomplete basis, an arbitrary input state of the signal mode can be decomposed into coherent states (the so-called P representation). Taking into account the linear superposition principle and the above result, we find that an arbitrary state of the signal mode is properly homogenized on the array of beam splitters providing reservoir modes initially prepared in Gaussian states with zero displacement.

In Fig. 4 we show an example of homogenization of a

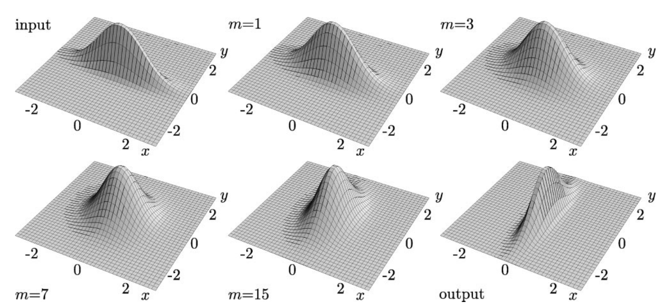

FIG. 4. Wigner functions $W(\xi)$ of the signal mode after m beam-splitter interactions with the reservoir modes. We denote $x=\mathrm{Re}(\xi)$; $y=\mathrm{Im}(\xi)$ and we use physical units such that the phase-space coordinates x and y are dimensionless. We consider the input state of the signal mode to be a squeezed vacuum. This mode is homogenized in the array of beam splitters with transmitivity $t=0.95$. The reservoir modes are prepared in the same squeezed vacuum state except that the orientation of squeezing is rotated by 90 deg. (The complementary quadrature is squeezed.) We see that in the process of homogenization the signal mode is gradually transformed into the original state of the reservoir. The degree of squeezing of the input signal and the reservoir modes is determined by the parameter $q=\ln 2$ (for explanation see Sec. V).

squeezed Gaussian state. The reservoir states are squeezed in a direction perpendicular to the direction of squeezing of the signal state. We see that at the output the signal mode is transformed into the state of the reservoir mode which nicely illustrates the homogenization process as discussed above.

V. ENTANGLEMENT IN THE BEAM-SPLITTER ARRAY

In this section we will show how quantum correlation—entanglement between pairs of output beams arises via the beam-splitter interactions of squeezed Gaussian states. All interactions under consideration are unitary, so the whole process is reversible. If we knew the exact sequence of reservoir photons with which our signal photon had interacted, we could reverse the process by running it "backwards" so that the initial state of the signal mode is completely recovered [3].

The beam splitter can serve as an entangler [11]. It mixes the input states and can produce inseparable output. When two coherent states are incident on a beam splitter, the output is given by

$$|\Psi^{(out)}\rangle = \hat{B}\hat{D}_\alpha(\alpha)\hat{D}_\beta(\beta)|0,0\rangle = \hat{D}_\alpha(t\alpha + r\beta)\hat{D}_\beta(t\beta - r\alpha)|0,0\rangle. \tag{5.1}$$

Yet this output is clearly not entangled. Kim *et al.* [11] found that simply displacing the input fields does not increase entanglement of the output fields, because the impact of the displacement of the input fields can always be canceled by local unitary operations on the output fields. In order to generate entanglement via a beam splitter which is a linear optical device we need to have inputs exhibiting purely quantum-statistical features such as sub-Poissonian photon statistics or quadrature squeezing [11]. Let us consider that the two inputs (i.e., the signal and the reservoir modes) are initially prepared in squeezed states. Both states are squeezed by the same amount, but the squeezing direction is not the same. These single-mode states are generated by the squeezing operator $\hat{S}(\zeta) = \exp[(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2})/2]$. It has been shown earlier (see, e.g., Ref. [11] and references therein) that in the case the output modes of a beam splitter can be entangled. The degree of entanglement depends on specific direction of squeezing of two inputs [14]. Moreover, if the inputs are thermal squeezed states (i.e., the squeezing operator does not act on the vacuum state but on a thermal state) then the creation of entanglement depends on the amount of the thermal noise in the inputs.

In the case of continuous variables it is not a trivial task to determine whether two modes that are in a mixed state are entangled. That is, the general inseparability condition is not known. On the other hand, for Gaussian states the inseparability condition has been derived recently [12,15]. In particular, Simon [12] has derived the inseparability condition that is simple to use since it is directly related to properties of a two-mode characteristic function. Simon's criterion is a generalization of the Peres-Horodecki partial-transposition criterion [16] into infinite-dimensional systems, where it has a geometrical interpretation as a mirror reflection in the Wigner space.

When testing the separability of a bipartite (two-mode) Gaussian state, it is convenient to express the characteristic function $C(\alpha,\beta)$ as follows:

$$C(\alpha,\beta) = \exp(-\tfrac{1}{2}\xi M \xi^T), \quad \xi = (\alpha_r, \alpha_i, \beta_r, \beta_i),$$

where M is a symmetric 4×4 matrix. As linear displacements do not affect separability of two modes, we have neglected them in the analysis of entanglement. After some local unitary operations, any two-mode Gaussian state can be expressed with only six nonzero coefficients,

$$M = \begin{pmatrix} a & 0 & f & 0 \\ 0 & b & 0 & g \\ f & 0 & c & 0 \\ 0 & g & 0 & d \end{pmatrix}. \tag{5.2}$$

The Simon separability criterion then reads

$$S = (ac - f^2)(bd - g^2) - ab - cd - 2|fg| + 1 \geqslant 0. \tag{5.3}$$

If the function S is non-negative then the bipartite state is separable. Otherwise it is entangled.

A. Two-mode output characteristic function

Let us consider that the signal mode is initially described by a Gaussian characteristic function (4.3) with the initial parameters: $A_0 = \alpha_r e^q$, $B_0 = \alpha_i e^{-q}$, $C_0 = (2\bar{n}+1)e^{2q}$, $D_0 = (2\bar{n}+1)e^{-2q}$. The parameter q is related to the degree of quadrature squeezing in the signal mode, while $\bar{n}$ describes a number of thermal photons in the mode. The reservoir is in the squeezed thermal state with the mean number of thermal photons $\bar{m}$ and squeezing parameter P. The reservoir is then represented by its characteristic function in the form of Eq. (4.4) with $E = (2\bar{m}+1)e^{2p}$ and $F = (2\bar{m}+1)e^{-2p}$. Using Eq. (4.1), it is possible to find the $(m+1)$-mode characteristic function of the whole output of the beam-splitter array. However, we do not need all this information. For our purposes it suffices to analyze only two modes—the chosen reservoir mode $\eta^{(k)}$ and the final output mode ζ, or two chosen reservoir modes $\eta^{(k)}$ and $\eta^{(\ell)}$.

Using Eq. (4.1) we find the two-mode output characteristic function from the kth beam splitter

$$C_{out}(\zeta,\eta) = L_k(\eta)\exp[2iA_k(\eta)\zeta_r - 2iB_k(\eta)\zeta_k] \times \exp[-\tfrac{1}{2}C_k\zeta_r^2 - \tfrac{1}{2}D_k\zeta_i^2], \tag{5.4}$$

with A_k, B_k, and L_k being functions of the η mode. The parameters A_k and B_k are related to possible entanglement between the two modes under consideration. The exact values of the coefficients in Eq. (5.4) are found using Eq. (4.6),

$$A_k = t^k A_0 + \frac{rt^{2k-1}}{2k}[E - C_0]\eta_r, \tag{5.5}$$

$$C_k = t^{2k} C_0 + E(1 - t^{2k}),$$

$$B_k = t^k B_0 + \frac{r t^{2k-1}}{2k}[F - D_0]\eta_r,$$

$$D_k = t^{2k} D_0 + F(1 - t^{2k}),$$

with L_k given by the expression

$$L_k = \exp[2iA_{k-1} r\eta_r - 2iB_{k-1} r\eta_i] \times \exp[-\tfrac{1}{2}(C_{k-1} r^2 + t^2 E)\eta_r^2 - \tfrac{1}{2}(D_{k-1} r^2 + t^2 F)\eta_i^2]. \quad (5.6)$$

The terms containing A_0 and B_0 are linear in ζ and do not depend on η. Thus they will not affect entanglement so it is not needed to use them in further computation. We have two ways to continue, depending on which two-mode characteristic function we want to obtain.

1. Characteristic function of the signal and kth reservoir mode

In what follows we present the characteristic function of the final output signal mode and the kth reservoir mode $\eta^{(k)}$. After the kth beam splitter the output interacts with reservoir states on the remaining $m-k$ beam splitters. We obtain the characteristic function in the form

$$C(\zeta, \eta^{(k)}) = \exp[-\tfrac{1}{2}(a\zeta_r^2 + b\zeta_i^2 + c\eta_r^2 + d\eta_i^2)] \times \exp[-\tfrac{1}{2}(f\eta_r\zeta_r + g\eta_i\zeta_i)] \times \exp[-(\text{terms linear in } \zeta \text{ and } \eta)], \quad (5.7)$$

with the coefficients in the M matrix found from Eqs. (5.5) and (4.6),

$$a = t^{2m} C_0 + (1 - t^{2m})E,$$

$$b = t^{2m} D_0 + (1 - t^{2m})F,$$

$$c = r^2 t^{2(k-1)} C_0 + [1 - r^2 t^{2(k-1)}]E,$$

$$d = r^2 t^{2(k-1)} D_0 + [1 - r^2 t^{2(k-1)}]F,$$

$$f = -t^{m+k-1} r[E - C_0],$$

$$g = t^{m+k-1} r[F - D_0]. \quad (5.8)$$

2. Characteristic function of two reservoir modes

Let us consider a characteristic function of two reservoir modes after they interact with the signal. Assume that the modes are labeled as k and ℓ. That is, there are $\ell - k - 1$ beam splitters in between. Let us denote $\xi = \eta^{(\ell)}$ and $\eta = \eta^{(k)}$. The remaining $m - \ell$ beam splitters to the end of the beam-splitter array cannot change the separability of the k,ℓ pair, because they affect only the terms containing ζ in the characteristic function and those are not quadratic in η,ξ. Thus for examination of the entanglement of the k,ℓ pair we only need to consider $C(\eta,\xi) = C(\eta^{(k)}, \eta^{(\ell)})$ at the ℓth beam splitter,

$$C(\eta,\xi) = \exp[-\tfrac{1}{2}(a\zeta_r^2 + b\zeta_i^2 + c\xi_r^2 + d\xi_i^2)] \times \exp[-\tfrac{1}{2}(f\eta_r\xi_r + g\eta_i\xi_i)] \times \exp[-(\text{linear terms})]. \quad (5.9)$$

The terms in the matrix M are found analogically as in the previous case;

$$a = r^2 t^{2(k-1)} C_0 + [1 - r^2 t^{2(k-1)}]E,$$

$$b = r^2 t^{2(k-1)} D_0 + [1 - r^2 t^{2(k-1)}]F,$$

$$c = r^2 t^{2(\ell-1)} C_0 + [1 - r^2 t^{2(\ell-1)}]E,$$

$$d = r^2 t^{2(\ell-1)} D_0 + [1 - r^2 t^{2(\ell-1)}]F,$$

$$f = r^2 t^{\ell+k-2}[E - C_0],$$

$$g = r^2 t^{\ell+k-2}[F - D_0]. \quad (5.10)$$

B. Simon's criterion

We are now ready to test the separability of the outputs using Simon's criterion (5.3). Let us assume that our input states are squeezed in directions perpendicular to each other (see Fig. 4), which maximizes the possible entanglement [11]. We thus expect that the two output beams from a beam splitter can be inseparable, even if the interaction is weak $(t \to 1)$, assuming the inputs are sufficiently squeezed.

It is interesting to examine the entanglement between the signal and reservoir modes. In particular, it is of importance to understand how the entanglement (if created) deteriorates in the presence of thermal noise in reservoir modes. It is also of great interest to understand whether the signal mode can act as a mediator in entangling reservoir modes which have never interacted directly. In what follows we will illuminate these issues.

1. Inseparability between signal and ith reservoir mode

Let us return to our assumptions. We will consider that the signal mode is initially in the squeezed thermal state described by a characteristic function (4.3) with $C_0 = Ne^{2q}$, $D_0 = Ne^{-2q}$, where $N = (2\overline{n} + 1)$. The reservoir modes are in Gaussian states (4.4) with $E = Me^{2p}$, $F = Me^{-2p}$, where $M = (2\overline{m} + 1)$. For simplicity we will assume a situation when the signal and the reservoir are equally squeezed with the direction of squeezing perpendicular to each other.

Let us use the coefficients and express the Simon's criterion with $x = t^{2m}$, $y = r^2 t^{2k-2}$, and $z = rt^{m+k-1}$. It is useful to express z^2 in terms of x and y as $z^2 = xy$, which later simplifies the expression

$$S=-2xy(1+\xi)[2MN\cosh 4q-N^2-M^2]+(M^2-1)\times[M^2(1-x-y)^2-1+N^2(x+y)^2+2MN(x+y)\times(1-x-y)\cosh 4q]. \tag{5.11}$$

If we collect the corresponding terms, the coefficient in front of cosh 4s is equal to

$$Q=2MN\begin{bmatrix}(x+y)(1-x-y)(M^2-1)\\ -2xy(1+\xi)\end{bmatrix},$$

where ξ is the sign of $(2MN\cosh 4q-N^2-M^2)$ [it resolves the absolute value $|fg|$ in Eq. (5.3)].

If the squeezing parameter q is small, so that $\xi=-1$, S is non-negative. If there are nonsqueezed states on the input of a beam splitter, the output modes *cannot* be entangled. On the other hand S is growing with q until the sign ξ changes, because cosh 4q is increasing and Q is positive. Afterwards, the possible negative sign of S depends on the sign of the coefficient Q in front of cosh 4s. If it changes into a negative number, S will be negative for high enough q. Testing the coefficient on negativity yields a boundary for M and thus for $\overline{m}$,

$$M^2<1+\frac{4xy}{(x+y)(1-x-y)}=K_t,$$

$$\overline{m}<\frac{\sqrt{K_t}-1}{2}, \tag{5.12}$$

otherwise the sign in front of cosh 4q would remain positive and thus S would increase again and remain positive, so the state would be always separable. This holds for nonzero $(1-x-y)$. However, this is true if $0<t<1$ and $m\geqslant 2$, because

$$(1-x-y)=(1-t^{2k-2})+(t^{2k}-t^{2m}).$$

The first term is nonzero (and positive) for $k\geqslant 2$ and the second term is positive for $m>k$. Together it results in the condition $m\geqslant 2$.

The threshold K_t decreases with growing m and also with larger k, thus, as expected, tightening the constraint on the value $\overline{m}$. Because of the inequality $K_t>1$, we can never find entanglement between the considered output beams.

In Fig. 5 we show the violation of the Simon's separability criterion in dependence on the number of total reservoir modes m interacting with the signal and the transmitivity t for fixed values of squeezing q and the mean numbers of thermal photons $\overline{n}$ and $\overline{m}$. We see that the first reservoir mode is still entangled to the signal state even after 200 interactions for high values of transmitivity. We also see that thermal noise (nonzero values of $\overline{m}$) leads to deterioration of entanglement between the involved modes.

It is also interesting to analyze how inseparability of a particular pair of modes ($m=2$, $k=1$) depends on the amount of thermal noise in the reservoir and the signal (Fig. 6). The squeezing and the transmitivity are fixed now. We see

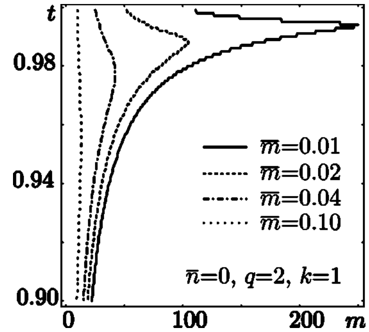

FIG. 5. Inseparability of the signal after m beam splitters and the first reservoir state ($k=1$) in dependence on the transmitivity t of the beam splitters for various values of thermal noise in the reservoir $\overline{m}$. The pair is inseparable in the region to the left of the corresponding line.

that for large overall values of thermal noise, the two output modes under consideration are separable. It is a natural observation that thermal noise leads to deterioration of quantum entanglement.

It is also interesting to consider no thermal noise in the reservoir. If $\overline{m}=0$ (or $M=1$), expression (5.11) is very simple and certainly negative for any x and y, if the squeezing is higher than the threshold given by the change of the sign $\xi=+1$,

$$q>\frac{1}{4}\text{arccosh}\left(1+\frac{2\overline{n}^2}{2\overline{n}+1}\right). \tag{5.13}$$

If this condition is satisfied, then any pair (k,m) of states becomes entangled! This is possible only because there is no additional noise introduced with every beam-splitter interaction. The amount of entanglement is, however, decreasing with increasing m or k.

2. Inseparability of two reservoir modes

The calculation is analogous to the previous case. We put $x=r^2t^{2k-2}$, $y=r^2t^{2\ell-2}$, and $z=r^2t^{\ell+k-2}$. We use the equality $z^2=xy$ here too, and we obtain the same expression for S (5.11) and the constraint on $\overline{m}$ (5.12) in terms of x and y.

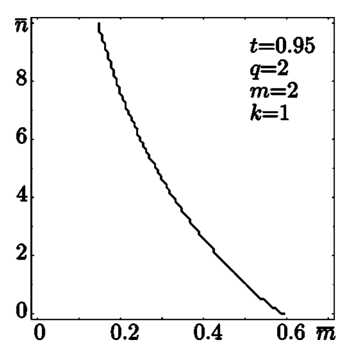

FIG. 6. Inseparability of the ($m=2$, $k=1$) pair in dependence on the amount of thermal noise in the reservoir $\overline{m}$ and the signal state $\overline{n}$. The pair is inseparable to the left of the line.

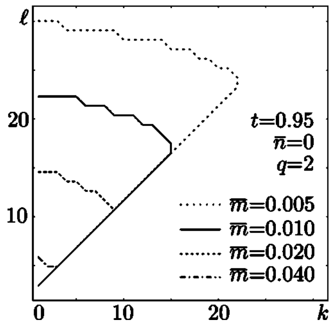

FIG. 7. Inseparability of pairs (k,ℓ) of reservoir states for different values of thermal noise in the reservoir. The pair is inseparable in the region to the left of the corresponding line.

Let us check the sign of the expression $(1-x-y)$ in Eq. (5.12) in this case,

$$(1-x-y)=(1-t^{2k-2})+(t^{2k}-t^{2\ell-2})+t^{2\ell}.$$

The first two terms are non-negative $(k\geqslant 0,\ \ell\geqslant k)$, the last term is always positive, so the whole expression is nonzero.

In Fig. 7 we see the region of inseparability of the k,ℓ-reservoir state pairs. Transmitivity, squeezing, and mean numbers of particles $\overline{n},\overline{m}$ are fixed. The inseparability of the reservoir pairs is much more sensitive to the thermal noise in the reservoir. It is though obvious, because the reservoir pairs have become entangled through a mediator that gains more and more noise with each interaction. The signal state soon loses the ability to entangle more distant pairs of reservoir modes, but still becomes entangled with a single-reservoir state (compare Fig. 5).

However, if there is no thermal noise in the reservoir, the situation is very much the same as in the case of a signal-reservoir pair. If the squeezing is high enough (5.13), every pair of reservoir photons becomes entangled.

VI. CONCLUSIONS

In this paper we have presented a model of the *universal* quantum homogenizer for qudits that is realized via a sequence of partial-swap operations between the system (signal) qudit and the set of reservoir qudits. The universality of the device means that the process can be realized for arbitrary initial states of the system as well as the reservoir qudits. We have shown that our results are valid not only for qudits but also for continuous variables (e.g., qudits are replaced by modes of an electromagnetic field). We have shown that a sequence of partial-swap operations induces a contractive map with the fixed point being the initial state of the reservoir qudits. In this scenario the original qudit at the end of the homogenization process is in the same state as the reservoir qudits. Since the whole process of homogenization is governed by unitary transformations then it is a legitimate question to ask: Where is the original information encoded in the initial state of the system qudit? It turns out that this quantum information is transferred into correlations between interacting qudits. It is interesting to note that the quantum information that is transferred (redistributed) into the correlations between interacting qudits can be recovered. The recovery can be done via a sequence of inverse partial-swap operations. The necessary condition for the recovery is that the *classical* information about the sequence of the interactions between the system and reservoir qudits is available.

In this paper we have also shown that in the case of continuous variables (e.g., modes of the electromagnetic field) quantum homogenization can be realized with the help of linear optical elements. Specifically, we have shown that when the input signal mode is in a Gaussian state while the reservoir modes are in the Gaussian state with the zero amplitude, then an array of quantum beam splitters with very high transmitivity realizes the quantum homogenization. It is an open question whether the quantum homogenization of electromagnetic fields can be realized with linear optical elements also for non-Gaussian states.

ACKNOWLEDGMENTS

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APPENDIX A: HOMOGENIZATION IS A CONTRACTIVE MAP

In this appendix we show that $\varrho_S^{(N)}$ monotonically converges to ξ for all parameters $\eta\neq 0$. In order to prove this convergence, we use the *Banach theorem* [8] that concerns the fixed point of a contractive transformation. Let $\mathcal{S}$ be a metric space with a distance function $D(.,.)$. The transformation $\mathcal{T}$ is called *contractive* if it fulfills the inequality $D(\mathcal{T}[\varrho],\mathcal{T}[\xi])\leqslant kD(\varrho,\xi)$ with $0\leqslant k<1$ for all $\varrho,\xi\in\mathcal{S}$. A fixed point of the transformation $\mathcal{T}$ is an element of $\mathcal{S}$ for which $\mathcal{T}[\xi]=\xi$. The Banach theorem states that a contractive map has a unique fixed point [9], and that the iteration of the map converges to it, i.e., $\mathcal{T}^N[\varrho]\rightarrow\xi$ for each $\varrho\in\mathcal{S}$. We note that contractive transformations within the context of quantum information processing have been recently discussed also in Ref. [17].

1. Definition of the distance

A natural way of how to define a distance in a Hilbert space is to use the norm induced by the scalar product of the Hilbert space. Let us consider a separable (not necessarily finite-dimensional) Hilbert space $\mathcal{H}_A$. We say that an operator $\mathbf{A}$: $\mathcal{H}_A\rightarrow\mathcal{H}_A$ is a Hilbert-Schmidt operator if the operator is bounded, and there exists orthonormal basis $\{|\varphi_i\rangle\}$ in the Hilbert space $\mathcal{H}_A$ such that

$$\sum_{i=1}^{\infty}\|\mathbf{A}|\varphi_i\rangle\|^2<\infty. \tag{A1}$$

The set of all Hilbert-Schmidt operators $\mathbf{A}$: $\mathcal{H}_A\rightarrow\mathcal{H}_A$ form a Hilbert space, denoted as $\mathcal{B}$, with scalar product defined as

$$(\mathbf{A},\mathbf{B})_{\mathcal{B}}=\mathrm{Tr}(\mathbf{A}^{\dagger}\mathbf{B}), \tag{A2}$$

where **A** and **B** are two elements of the Hilbert space $\mathcal{B}$. The norm of an element **A** of the Hilbert space $\mathcal{B}$ induced by the scalar product $(.,.)_{\mathcal{B}}$ is given by

$$\|\mathbf{A}\|_{\mathcal{B}}=\sqrt{(\mathbf{A},\mathbf{A})_{\mathcal{B}}}. \tag{A3}$$

Let us note that the norm in Eq. (A3) may be infinite even though the operator **A** is a bounded operator **A**: $\mathcal{H}_A\rightarrow\mathcal{H}_A$. For example, if the element **A** is a unitary operator acting in an infinite-dimensional Hilbert space $\mathcal{H}_A$ the norm of **A** in Eq. (A1) is infinite (and consequently no unitary operators belong to the Hilbert space $\mathcal{B}$). One should not forget that the norm of **A** considered as an element of the Hilbert space $\mathcal{B}$ may not equal to the norm of the same element **A** considered as a bounded operator **A**: $\mathcal{H}_A\rightarrow\mathcal{H}_A$. It can be shown (see Ref. [18]) that the convex set of all density operators $\mathcal{S}$ is a subset of $\mathcal{B}$. Now the distance between two elements ρ,σ of a given Hilbert space $\mathcal{B}$ can be defined with the help of the norm presented in Eq. (A3) as follows:

$$D(\rho,\sigma)\equiv\|\rho-\sigma\|_{\mathcal{B}}. \tag{A4}$$

Using expression (A2) for the scalar product one may derive a more convenient form for the distance now expressed via the trace operation,

$$D^2(\rho,\sigma)=(\rho-\sigma,\rho-\sigma)_{\mathcal{B}}=\mathrm{Tr}(\rho^{\dagger}\rho-\rho^{\dagger}\sigma-\sigma^{\dagger}\rho+\sigma^{\dagger}\sigma).$$

2. Stability of the reservoir

The first condition of homogenization (1.3) requires that all reservoir qudits after the interaction remain in the δ neighborhood of their initial state ξ. That is, states of the individual qudits of the reservoir are "stable" during the system-reservoir interactions. As we apply sequentially the same unitary transformation U for all reservoir qudits it holds that if

$$D(\xi_1',\xi)\leqslant\delta, \tag{A5}$$

for all initial states ρ of the system then

$$D(\xi_i',\xi)\leqslant\delta;\quad \forall \;\; i=1\cdots N. \tag{A6}$$

Apparently more natural reasoning would be to exploit the convergence $\rho^{(n)}\rightarrow\xi$ which follows from the contractivity of the map $\mathcal{T}$ demonstrated below. If the state $\rho^{(n)}$ converges monotonously to the state ξ then

$$D(\xi_1',\xi)\leqslant D(\xi_2',\xi)\leqslant D(\xi_3',\xi)\cdots \tag{A7}$$

However, one does not need the convergence to prove it. It simply follows from the fact that relation (A5) must hold for all initial states of the system ρ and that we sequentially apply the same unitary transformation U [see Eqs. (2.6) and (2.7)]. The important point is that it is sufficient to estimate only the distance $D(\xi_1',\xi)$. Using expression (2.5) for the state ξ_1', one finds

$$D(\xi_1',\xi)=\|s^2\rho-s^2\sigma+ics[\rho,\xi]\|_{\mathcal{B}}. \tag{A8}$$

The last result can be further simplified. Recall that for any two elements **A** and **B** from $\mathcal{B}$ the equation

$$\begin{aligned}\|\mathbf{A}+i\mathbf{B}\|_{\mathcal{B}}^2&=(\mathbf{A}+i\mathbf{B},\mathbf{A}+i\mathbf{B})_{\mathcal{B}}\\&=(\mathbf{A},\mathbf{A})_{\mathcal{B}}+(\mathbf{B},\mathbf{B})_{\mathcal{B}}+i[(\mathbf{A},\mathbf{B})_{\mathcal{B}}-(\mathbf{B},\mathbf{A})_{\mathcal{B}}]\\&=\|\mathbf{A}\|_{\mathcal{B}}^2+\|\mathbf{B}\|_{\mathcal{B}}^2+i[(\mathbf{A},\mathbf{B})_{\mathcal{B}}-(\mathbf{B},\mathbf{A})_{\mathcal{B}}]\end{aligned} \tag{A9}$$

holds. If the scalar product $(\mathbf{A},\mathbf{B})_{\mathcal{B}}$ is real, then the expression in the brackets equals to zero and one obtains

$$\|\mathbf{A}+i\mathbf{B}\|_{\mathcal{B}}^2=\|\mathbf{A}\|_{\mathcal{B}}^2+\|\mathbf{B}\|_{\mathcal{B}}^2. \tag{A10}$$

The scalar product $(s^2\rho-s^2\sigma,cs[\rho,\xi])_{\mathcal{B}}$ is apparently real and from Eqs. (A9) and (A10), it follows that

$$D^2(\xi_1',\xi)=\|s^2\rho-s^2\sigma\|_{\mathcal{B}}^2+\|ics[\rho,\xi]\|_{\mathcal{B}}^2. \tag{A11}$$

Using the "law of parallelogram" (which holds for the norm induced by a scalar product)

$$\|\mathbf{A}+\mathbf{B}\|^2+\|\mathbf{A}-\mathbf{B}\|^2=2\|\mathbf{A}\|^2+2\|\mathbf{B}\|^2, \tag{A12}$$

the first term in Eq. (A11) can be rewritten as

$$\|s^2\rho-s^2\sigma\|_{\mathcal{B}}^2=s^4(2\|\rho\|_{\mathcal{B}}^2+2\|\sigma\|_{\mathcal{B}}^2-\|\rho+\sigma\|_{\mathcal{B}}^2). \tag{A13}$$

The elements ρ and σ are density matrices, i.e., positive operators with unit trace that gives certain bounds on the terms on the left side of Eq. (A13),

$$\begin{aligned}&\|\rho\|_{\mathcal{B}}^2=\mathrm{Tr}(\rho\rho)\leqslant 1,\\&\|\sigma\|_{\mathcal{B}}^2\leqslant 1,\\&\|\rho+\sigma\|_{\mathcal{B}}^2\geqslant\|\rho\|_{\mathcal{B}}^2+\|\sigma\|_{\mathcal{B}}^2.\end{aligned} \tag{A14}$$

Inserting these results into Eq. (A13) we can estimate the first term on the right side of Eq. (A11) as $\|s^2\rho-s^2\sigma\|_{\mathcal{B}}^2\leqslant 2s^4$. In the same way it can be shown that $\|ics[\rho,\xi]\|_{\mathcal{B}}^2\leqslant 2c^2s^2$, and we finally obtain

$$D^2(\xi_1',\xi)\leqslant 2s^4+2c^2s^2=2s^2. \tag{A15}$$

3. Contractivity

Consider two elements $\rho,\sigma\in\mathcal{S}$, i.e., two density matrices and denote their difference as $\rho-\sigma=\mathbf{A}$. The element **A** is Hermitian (this follows from the fact that ρ and σ are density operators, that are bounded and self-adjoint), and the distance between the two elements ρ and σ reads

$$D^2(\rho,\sigma)=\|\mathbf{A}\|_{\mathcal{B}}^2=\sum_{i,j}|\langle i|\mathbf{A}|j\rangle|^2. \tag{A16}$$

Let us consider now two elements $\rho^{(1)}$ and $\sigma^{(1)}$, i.e., $\rho^{(1)}=\mathcal{T}[\rho]$ and $\sigma^{(1)}=\mathcal{T}[\sigma]$. Using expression (A4) for the distance $D(.,.)$ and expression (2.4) for $\rho^{(1)}$ and $\sigma^{(1)}$, the distance between the two elements is given by

$$D(\rho^{(1)},\sigma^{(1)})^2=\|c^2\rho^{(0)}-c^2\sigma^{(0)}+ics[\xi,\rho^{(0)}-\sigma^{(0)}]\|_{\mathcal{B}}^2.$$

The last result can be again simplified by using Eqs. (A9) and (A10). It is easy to check that

$$(c^2\rho^{(0)}-c^2\sigma^{(0)},ics[\xi,\rho^{(0)}-\sigma^{(0)}])_{\mathcal{B}}=ic^3s\times\text{const}$$

so that the scalar product $(c^2\rho^{(0)}-c^2\sigma^{(0)},cs[\xi,\rho^{(0)}-\sigma^{(0)}])_{\mathcal{B}}$ is real and consequently

$$D^2(\rho^{(1)},\sigma^{(1)})=c^4\|\rho^{(0)}-\sigma^{(0)}\|_{\mathcal{B}}^2+c^2s^2\|[\xi,\rho^{(0)}-\sigma^{(0)}]\|_{\mathcal{B}}^2. \tag{A17}$$

Recall that $\rho^{(0)}$ is in fact ρ (in same way $\sigma=\sigma^{(0)}$) so that $\rho^{(0)}-\sigma^{(0)}=\mathbf{A}$ and

$$D^2(\rho^{(1)},\sigma^{(1)})=c^4\|\mathbf{A}\|_{\mathcal{B}}^2+c^2s^2\|[\xi,\mathbf{A}]\|_{\mathcal{B}}^2. \tag{A18}$$

The second term in the last expression can be rewritten with the help of the scalar product [see Eq. (A3)] or directly using a more convenient form in Eq. (A5),

$$\|[\xi,\mathbf{A}]\|_{\mathcal{B}}^2=2\ \mathrm{Tr}(\xi^2\mathbf{A}^2)-2\ \mathrm{Tr}(\xi\mathbf{A}\xi\mathbf{A}). \tag{A19}$$

The operator ξ is a density matrix, which implies that it is a compact operator. Every nonzero element of the spectrum of a compact operator is an eigenvalue. It means that every density operator ξ can be written as $\xi=\Sigma_i\lambda_i|i\rangle\langle i|$, where λ_i are nonzero eigenvalues of the operator ξ and $|i\rangle$ are the corresponding eigenvectors. Let us perform the trace using a basis consisting of the eigenvectors $|i\rangle$ of the density matrix ξ [19],

$$\|[\xi,\mathbf{A}]\|_{\mathcal{B}}^2=2\sum_i \langle i|\xi^2\mathbf{A}^2|i\rangle-2\sum_i \langle i|\xi\mathbf{A}\xi\mathbf{A}|i\rangle$$

$$=2\sum_{i,j} \lambda_i^2|\langle i|\mathbf{A}|j\rangle|^2-2\sum_{i,j} \lambda_i\lambda_j|\langle i|\mathbf{A}|j\rangle|^2.$$

Since the operator $\mathbf{A}$ is Hermitian it follows that $|\langle i|\mathbf{A}|j\rangle|^2=|\langle j|\mathbf{A}|i\rangle|^2$ and

$$\|[\xi,\mathbf{A}]\|_{\mathcal{B}}^2=\sum_{i,j} \lambda_i^2|\langle i|\mathbf{A}|j\rangle|^2+\sum_{i,j} \lambda_j^2|\langle i|\mathbf{A}|j\rangle|^2$$

$$-2\sum_{i,j} \lambda_i\lambda_j|\langle i|\mathbf{A}|j\rangle|^2$$

$$=\sum_{i,j} (\lambda_i-\lambda_j)^2|\langle i|\mathbf{A}|j\rangle|^2. \tag{A20}$$

Recall that λ_i are nonzero eigenvalues of the density matrix ξ, i.e., they are positive and $\lambda_i\leqslant 1$ for all i. It follows that:

$$|\lambda_i-\lambda_j|\leqslant 1;\quad \forall\quad i,j, \tag{A21}$$

and

$$\|[\xi,\mathbf{A}]\|_{\mathcal{B}}^2\leqslant\|\mathbf{A}\|_{\mathcal{B}}^2. \tag{A22}$$

Inserting the last result into Eq. (A18) together with the expression $\mathbf{A}=\rho-\sigma$ for the element $\mathbf{A}$, we obtain the following relation:

$$D^2(\rho^{(1)},\sigma^{(1)})\leqslant c^4\|\rho^{(0)}-\sigma^{(0)}\|_{\mathcal{B}}^2+c^2s^2\|\rho^{(0)}-\sigma^{(0)}\|_{\mathcal{B}}^2$$

$$\leqslant c^2D^2(\rho^{(0)},\sigma^{(0)}), \tag{A23}$$

which implies that the map $\mathcal{T}$: $\rho^{(0)}\rightarrow\rho^{(1)}$ is contractive iff $|c|<1$.

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