Stabilization via covariant symmetrisation

Martin Plesch
Faculty of Mathematics and Physics, Comenius University, Mlynská dolina, Bratislava 842 48, Slovakia *
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Quantum information encoded in a pure state of a single qubit is very fragile in a sense that under the errors induced by the environment the original information is deteriorated. In this paper the protection of unknown states of qubits via symmetrisation with the help of the universal quantum entangler is discussed. It is shown, that for certain values of parameters p_x, p_y, p_z of a Pauli channel it is useful first to entangle the original state with $N-1$ ancillas so that the output state is in a completely symmetric state of N qubits and only then let the Pauli channel to act.

I. INTRODUCTION

In general, quantum information distribution is not for free. Specifically, in some situations (e.g. state swapping or quantum teleportation) a state can be transferred from one system to another system without being changed at all. But these are very special cases. Usually, any control over distribution of quantum information (encoded in unknown states of quantum systems) is accompanied by degradation of the information. This is due to the fact that the control of flow (distribution) of quantum information can only be performed by quantum devices (e.g. ancillas) which during the process of information distribution become entangled with the carriers of quantum information. This entanglement between the quantum information distributors and quantum information carriers is the reason why the information encoded in unknown states of quantum systems can be deteriorated. This source of degradation has no classical analogue and has to be considered in the quantum information processing.

To illustrate possible consequences of the fundamental problem of information degradation in specific protocols of quantum information distribution let us consider a quantum system in an unknown state. It can be a result of a quantum computation, or any other type of information. The crucial assumption is, that we have no prior knowledge about it.

Suppose, that we are asked to transmit this state to an other party, which is separated by a distance. And we keep in our disposal a quantum channel between both parties, which is however not error-free.

And just errors,caused by external influences or by the device itself, are one of the very important problems in handling quantum states and performing computation with them. In comparison with the classical systems, the error correcting schemes are more complicated. One of the reasons is, that in quantum systems there are more types of errors, which can occur.

The classical bit can only flip $0 \leftrightarrow 1$, there are no more possibilities. This error is big, or discretized, it either happens or not, there is nothing between. Quantum states can rotate by an arbitrary angle, big or small, in different directions on the Bloch sphere, there are bit flip errors and phase errors (for further information see Ref. [1]).

Another difficulty in protection of quantum states is, that there is no possibility to measure and determine an unknown state [2] without disturbing it. Therefore it is also not possible to create more copies of a state, i.e. it isn't possible to clone unknown quantum states [3] which would obviously solve the problem of protection of quantum information. On the other hand perfect cloning of unknown states would lead to violation of another fundamental properties of quantum theory. For instance, it would allow to utilize quantum entanglement for superluminal communication [4].

For small errors there are many well developed and used schemes, called Quantum Error Correction schemes. A good review of them can be found in Ref. [5]. However, these schemes work efficiently when the probability of a disturbance of a qubit is small. If the errors are large these schemes might not be efficient enough to protect quantum information processing. Therefore it is justified to study other schemes for stabilization of quantum information in a noisy environment. In particular, Barenco et al. [6] proposed a method which is based on using of the symmetric subspace of the full space of the original state and $N-1$ reference states - ancilla.

In this paper, we want to concentrate on this idea. We encode an unknown quantum state (from which we possess only one copy), to a symmetric state of N copies of the system. The new symmetric state should be more robust against stochastic influence of the environment. Besides this, in most cases an error will lead our state out from the symmetrized subspace (since the dimension of the symmetric subspace is rather small, especially for big N, in comparison to the dimension of the whole space). So a projection back to the symmetric subspace, if successful, will restore the original state.

*e-mail: pleschsavba.sk

In the case of a single qubit we can consider an arbitrary input state in the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle. \tag{1.1}$$

This state is symmetrized with $N-1$ reference qubits in a known state $|0\rangle$, which results in the state

$$|\Psi\rangle = |\psi\rangle\underbrace{|00\dots0\rangle}_{N-1} + |0\rangle|\psi\rangle|000\dots0\rangle + \dots + |000\dots0\rangle|\psi\rangle \equiv |\{\psi, 0^{\otimes(N-1)}\}\rangle \tag{1.2}$$

Unfortunately, for an unknown state $|\psi\rangle$ perfect symmetrisation $|\psi\rangle|0\rangle^{\otimes(N-1)} \to |\{\psi, 0^{\otimes(N-1)}\}\rangle$ is not possible [7]. Nevertheless, approximate symmetrisation with help of the universal quantum entanglers, see Ref. [7] is allowed. Therefore one might ask, whether this approximate covariant symmetrisation (i.e. the fidelity of the symmetrisation is constant and does not depend on the input state) would stabilize the quantum information encoded in the input qubit.

In what follows we will address this problem in detail. As a model of quantum errors we will consider the Pauli channel. So the scenario of stabilization via symmetrisation looks as follows: Firstly, the input qubit is entangled symmetrically (with help of universal entanglers) with $(N-1)$ ancillas prepared in a reference state $|0\rangle$. Then the set of N qubits (or, just a fraction of them) is sent through the Pauli channel. Finally, at the output the projection back in to the symmetric subspace is performed. The question is whether this process is better than just sending a single qubit through the Pauli channel.

There exists another motivation for further investigation of symmetrising. And this is the preparation and execution of general POVM's. According to the Neumarks theorem, every POVM can be carried out by a suitable transformation of the relevant state and ancilla and a subsequent projective measurement on part of whole space (of the state and ancilla together). Results of this article can be, apart from the error correction, viewed also as possible preparation of non-ideal POVM. We bind our unknown state $|\psi\rangle$ togethe with the ancilla of known qubits and investigate, how well this can bee done, if errors in the form of Pauli channel are present.

II. A MODEL

The Pauli channel is a very good approximation of most of errors, that can happen to a single qubit in an environment. Consider a quantum bit in an arbitrary state $|\psi\rangle$ which is processed by a Pauli channel. The qubit is rotated by one of the three Pauli matrices or remains unchanged: it undergoes a phase-flip (σ_z), a bit-flip (σ_x) or their combination (σ_y) with respective probabilities p_x, p_y and p_z. Thus we can write the resulting density matrix

$$|\psi\rangle \to \rho = (1-p)|\psi\rangle\langle\psi| + p_x\sigma_x|\psi\rangle\langle\psi|\sigma_x + p_y\sigma_y|\psi\rangle\langle\psi|\sigma_y + p_z\sigma_z|\psi\rangle\langle\psi|\sigma_z. \tag{2.1}$$

For the special cases like depolarizing channel ($p_i = \frac{p}{3}$) or dephasing channel ($p_i = p; p_j = p_k = 0$) is the density matrix very simple,

$$\rho_{depol} = (1 - \frac{4}{3}p)|\psi\rangle\langle\psi| + \frac{2}{3}pI \tag{2.2}$$

and

$$\rho_{dephase} = (1-p_i)|\psi\rangle\langle\psi| + p_i\sigma_i|\psi\rangle\langle\psi|\sigma_i \tag{2.3}$$

where $i = x; y; z$ respectively. All our results will be presented for this four particular cases, even though the calculations allow to handle all possibilities.

More complications arise for the 2 qubit state. Pauli channel does not cover collective effects, it influences every qubit independently of the state of other qubits in the system. The explicit form of the resulting density matrix is too complex to be presented here, but the logic of construction is exactly the same as in the 1 qubit state. With probabilities p_x^1,p_y^1,p_z^1 and p_x^2,p_y^2,p_z^2 the relevant sigma matrices act on the first or second qubit respectively, leaving the other one unchanged.

For construction of the best approximation of the symmetrized state (1.2) we use the quantum universal entangler [7]:

$$\begin{aligned}|0\rangle|N-1;0\rangle|e_0\rangle &\to [\gamma_N|N;0\rangle|e_1\rangle + \delta_N|N;1\rangle|e_2\rangle]\\ |1\rangle|N-1;0\rangle|e_0\rangle &\to [\delta_N|N;0\rangle|e_3\rangle + \gamma_N|N;1\rangle|e_1\rangle]\end{aligned} \tag{2.4}$$

where e_k are three orthogonal states of the entangler, $|N;k\rangle$ is a fully symmetric state of N qubits with k of them in state $|1\rangle$. δ_N and γ_N are given:

$$\gamma_N = \sqrt{\left(\frac{N+1}{2\left(N+1-\sqrt{N}\right)}\right)} \quad ; \quad \delta_N = \sqrt{1-\gamma_N^2}. \tag{2.5}$$

For the input state $|\psi\rangle$, after tracing through the states of entangler, we get as a result the density matrix

$$\begin{aligned}\rho^{in} = &\left(|\alpha|^2\gamma_N^2 + |\beta|^2\delta_N^2\right)|N;0\rangle\langle N;0|\\ &+ \left(|\alpha|^2\delta_N^2 + |\beta|^2\gamma_N^2\right)|N;1\rangle\langle N;1| +\\ &+ \alpha\beta^*\gamma_N^2|N;0\rangle\langle N;1| + \alpha^*\beta\gamma_N^2|N;1\rangle\langle N;0|.\end{aligned} \tag{2.6}$$

This state is (in the sence of fidelity, defined in the next paragraph) as close as possible the to ideal symmetrised state (1.2). The fidelity is state independent and acieves the value $F = \gamma_N^2$.

To compare results of different methods for preventing errors on qubits, we need a suitable measure of success, evaluating the distance between the states. In this article, we use fidelity, defined for pure states as a squared scalar product of these states. For two states $|\psi\rangle$ and $|\phi\rangle$ we define

$$F = |\langle\psi|\phi\rangle|^2. \qquad (2.7)$$

Sometimes, it is useful to compute the average fidelity for a certain region of parameters. If we take state $|\psi\rangle$ as the input state and the superoperator P as the operator of the Pauli channel, then the mean fidelity between the original and output state $\rho = P(|\psi\rangle\langle\psi|)$ is:

$$\bar{F} = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \langle\psi|P(|\psi\rangle\langle\psi|)|\psi\rangle; \qquad (2.8)$$

where φ and θ are parameters of the input state.

Now we have defined a measure of success for the one-qubit states. This can be extended to the many-qubit states, consisting of the original qubit and ancilla, by many ways. The easiest one is to extract the information, encoded in the many-qubit state, back to one qubit. Then we can compute the fidelity of this qubit with the original one (as it is done in the section IV B). But, results using this methos depend also on the extracting procedure, what is not really what we want.

Other possibility is to compare directly many-qubit states. We define the mean fidelity

$$\bar{F} = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \langle\Phi|P(\rho)|\Phi\rangle; \qquad (2.9)$$

where $|\Phi\rangle$ is defined in (1.2) and ρ is the density matrix resulted from the universal entangler. Superoperator P now covers the action of the Pauli channel and also the projection back to the symmetric subspace. In this case, we compute the fidelity between our result and the ideal symmetrised state. We beleave, that this measure indicates better the success of the symmetrisation, since it is purged of the influence of the extraction process. This is the reason to use it throughout the most of the paper.

Part of the process of computing the fidelity is the projection to the symmetric subspace, as mentioned higher. Probability of a succesfull projection (it is a binary process, we can succeed or not) is given by the trace of the reduced resulting density matrix (containing only symmetric elements) and is rather high in all cases. In further calculations, we consider only the case of succes. As we will show, even so the resulting fidelities are to small to regard this method as useful.

III. PAULI CHANNEL ACTING ON 2 QUBITS

This scenario can represent a situation, when a qubit has to be transported from one place to another. We can either send it as it is (by sending we mean acting of the Pauli channel), or firstly project it on the symmetric subspace of a 2 qubit space, and then send this two qubits through the same channel - therefore we use a symmetric Pauli channel with equal probabilities $p_i^1 = p_i^2$. After it, the fidelities compared and the region of parameters is searched, where the 2-qubit state has better result.

The "ideal" symmetrized state of 2 qubits would be

$$|\Phi^{ideal}\rangle = \frac{(2\alpha|00\rangle + \sqrt{2}\beta|+\rangle)}{(4|\alpha|^2 + 2|\beta|^2)^{1/2}}; \qquad (3.1)$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. But since we have no prior knowledge about the input state and want to keep constant fidelity for the whole space of states, we have to use the universal entangler. As the input state we use a density matrix in the form (2.6) for $N = 2$.

We now "apply" the Pauli channel on it and then we make a projection back to the symmetric subspace, which actually means, that we keep only symmetric states in the density matrix and renormalize it back to unity. Resulting fidelities are presented in Fig. 1. The region, where the fidelity after symmetrisation is bigger than that one, obtained by sending the original qubit through the Pauli channel, are gray, the others are white.

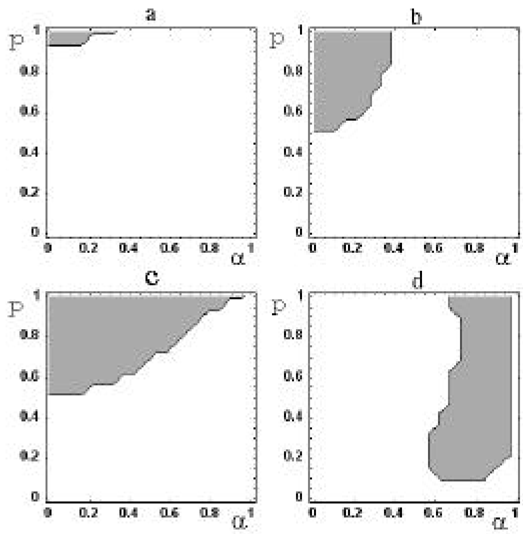

FIG. 1. Fidelity of the stabilization for depolarizing, σ_x; σ_y and σ_z channel. On the y axis is the parameter of the Pauli channel, on the x axis the parameter α from the input state.

We have to mention, that only real parameter α has been used, therefore the results for x and y channels are so different. However, still, the results are not to optimistic. They are strongly state dependent, and for the depolarizing channel only for a very small region of parameters we get better results using 2 qubits as the naive scenario of sending the one qubit information, as it is. The reason is simple: loss due to stabilization and the loss caused by errors on both qubits are bigger than the gain of projection on the symmetrized subspace.

IV. PAULI CHANNEL ACTING ON ONE QUBIT

This case represents an other situation as in the previous section. The difference is, that the Pauli channel acts only on the first qubit every time. We can imagine, that the original qubit is exposed to the influence of the environment, whereas the ancilla is in a store with no errors acting on it. The density matrix after the action of Pauli channel (for $N > 1$) is

$$\begin{aligned}
\rho_{out} = & \left[|\alpha|^2\gamma_N^2 + |\beta|^2\delta_N^2\right](1 - p + p_z)|N;0\rangle\langle N;0| + \\
& + \left[|\alpha|^2\gamma_N^2 + |\beta|^2\delta_N^2\right]\frac{p_x + p_y}{N}|N;1\rangle\langle N;1| + \\
& + \left[|\alpha|^2\delta_N^2 + |\beta|^2\gamma_N^2\right]\frac{p_x + p_y}{N}|N;0\rangle\langle N;0| + \\
& + \left[|\alpha|^2\delta_N^2 + |\beta|^2\gamma_N^2\right]\left(1 - p + p_z\left(1 - \frac{4}{N} + \frac{4}{N^2}\right)\right) \\
& \quad |N;1\rangle\langle N;1| + \\
& + \left(\alpha\beta^*\gamma_N^2\left(1 - p + p_z - \frac{2p_z}{N}\right) + \alpha^*\beta\gamma_N^2\frac{p_x - p_y}{N}\right) \\
& \quad |N;0\rangle\langle N;1| + \\
& + \left(\alpha^*\beta\gamma_N^2\left(1 - p + p_z - \frac{2p_z}{N}\right) + \alpha\beta^*\gamma_N^2\frac{p_x - p_y}{N}\right) \\
& \quad |N;1\rangle\langle N;0| + \\
& + (p_x + p_y)\left[|\alpha|^2\delta_N^2 + |\beta|^2\gamma_N^2\right]\frac{2}{N^2}|N;2\rangle\langle N;2| \\
& + \ldots
\end{aligned} \tag{4.1}$$

In the density matrix 4.1 only symmetric elements are displayed. The rest, denoted as dots, is non-symmetrical and we can project it out by a suitable measurement. In practice that means, we have to renormalize (4.1) to 1.

A. Quantum scenario

We compute now the mean fidelity of this matrix with the ideal symmetrized state (1.2). In Figs. 2,3 and 4 we can see the results.

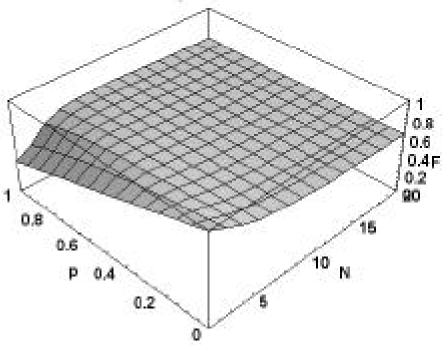

FIG. 2. Fidelity as the function of the probability of depolarasing Pauli channel and the number of qubits used. For N=1 it is a simple acting of the Pauli channel on one qubit.

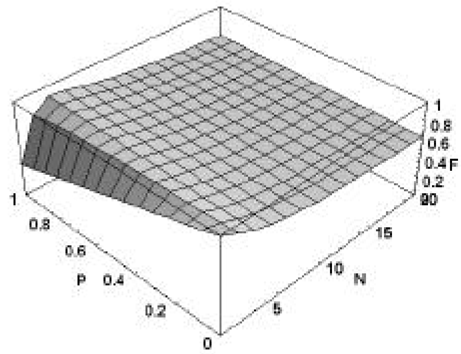

FIG. 3. The same as figure 2 but for σ_z channel.

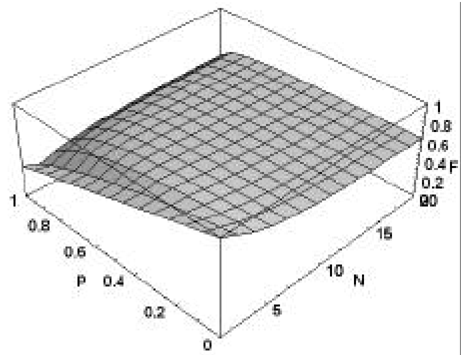

FIG. 4. The same as figure 2 but for σ_x. We do not introduce σ_y, since the results are completely the same as those for σ_x.

N	p	p_z	p_x
2	0:22	0:10	–
3	0:29	0:44	0:35
4	0:48	0:86	0:41
5	0:63	–	0:52
6	0:76	–	0:60
7	0:86	–	0:66
8	0:95	–	0:71

TABLE I. In the Quantum scenario, the fidelity for fixed p (fixed channel) reaches its maximum for a certain N. As N is a natural number, in most cases the actual maximum is not reachable. It is only possible to find the number, which is closest to it. In the table we state the marginal parameters p for three types of Pauli channel, where the maximum of fidelity is reached for two values of N. That means, e.g. for $p = 0{:}22$ in depolarizing channel the maximum is reached for one and two qubits, for $0{:}22 < p < 0{:}29$ we shall use two qubits etc. In depolarizing channel we use maximally eight qubits (the maximum is never reached for more qubits), in σ_z channel four qubits. In σ_x channel we never use two qubits (for $p = 0{:}35$ the maximum is reached for one and three qubits), but the maximal number of qubits was not reached within the searched region.

As we see, the fidelity, especially for bigger probabilities, is growing with the number of qubits N. For bigger N, as expected, there is only a small dependence on the probability, because the one qubit, on that the channel acts, is more and more negligible in comparison to other qubits. But, the error produced by entangling is also growing with N. So, for defined probabilities it is possible to find the optimal number of qubits in ancilla (see Table I), which has to be used to obtain the optimal fidelity.

B. Measurement scenario

The mean fidelity helps us to see, how well we can conserve the information encoded in many-qubit symmetrized state, but the extraction of the information is problematic. We can, at least for the σ_z channel, use the measurement scenario, as described in Ref. [8]. We introduce two states

$$|\Psi_0\rangle = \cos\frac{\theta'}{2}|N;0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|N;1\rangle \qquad (4.2a)$$

$$|\Psi_1\rangle = e^{i\phi'}\sin\frac{\theta'}{2}|N;0\rangle - \cos\frac{\theta'}{2}|N;1\rangle \qquad (4.2b)$$

where θ' and ϕ' are randomly chosen orientations. Now we can compute a density matrix of a one-qubit state

$$\varrho^M = |\langle\Psi_0|\rho^{out}|\Psi_0\rangle|.|n_0\rangle\langle n_0| + |\langle\Psi_1|\rho^{out}|\Psi_1\rangle|.|n_1\rangle\langle n_1| \qquad (4.3)$$

where $|n_j\rangle$ are the same states as (4.2) with $N = 1$.

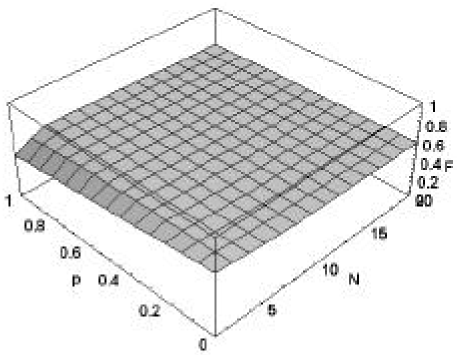

FIG. 5. Fidelity of the measurement density matrix, averaged over all possible orientations of the input state and the estimated state.

Now we are able to compute the real one-qubit state mean fidelity (integrating through angles of the input state AND of the estimated state) between this density matrix and the input state $|\psi\rangle$. The result are shown in the figure 5.

As we see, the shape of the plot is nearly the same as in the quantum scenario. The differences between the fidelities for different parameters are much smaller and we lose fidelity even when no ancilla and no errors are present. This is due to the extraction procedure and shows, that for detection of the effort of the stabilisation scheme is this fidelity not very suitable.

V. CONCLUSIONS

The scenario was as follows: we took an unknown qubit and entangled it with the ancilla of $N - 1$ qubits. symmetrisation has been performed with the help of the covariant (input-state independent) device. In the first case, for $N = 2$, the obtained density matrix was sent through a symmetric Pauli channel acting on both qubits. This corresponds to a scenario of sending an information from one place to another, where someone can, with local measurements, gain the information back.

In the second case, we sent only one of the qubits through the Pauli channel, the rest was considered as to be in a perfect store without influence of the environment. This scenario corresponds to an authentication protocol, where we send only one qubit there and back. Again, we can with local measurements gain the information.

In both cases, after the action of the Pauli channel and before we have made any measurement, we projected the density matrix on the symmetric subspace (since we know, that everything non-symmetric is an error). The results show us, in which region of parameters it is useful to make this procedure of symmetrisation (in the first case) or, tell us, how many qubits are optimal for the ancilla (in the second case). As we see, especially for small values of the probability of Pauli channel, the optimal number of qubits in both cases is 1, that means the utilization of ancillas would not help us to protect quantum information. The loss of the fidelity due to the covariant symmetrisation is bigger than the gain due to the better stability of the symmetrized system.

We investigated a method for stabilization of quantum information via symmetrisation. Although we found some regions of parameters, where this method is giving interesting results (see TABLE I), in general are the outcomes rather negative. The reason is, that it is not possible, for an unknown input state, to produce a perfect symmetrized state, as it was expected in the method. Furthermore, in the case where we consider errors acting on the original qubit AND ancilla, the probability of errors is summing up for all the qubits.

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[1] J. Preskill, Lecture Notes on Quantum Computation, http://www.theory.caltech.edu/people/preskill/ph229/#lecture

[2] We assume that no prior knowledge about the state of quantum system is available. In this case that prior distribution in the state space of quantum system, e.g. a Bloch sphere in the case of a qubit, is constant. For more details see and references therein V.Buzek, G. Drobný, R.Derka, G.Adam, and H.Wiedeman, Chaos, Solitons & Fractals 10, 981{1074 (1999).

[3] W. K. Wootters and W. H. Zurek, Nature 299, 802 (1982).

[4] N. Herbert, Found. Phys. 12, 1171 (1982); N. Gisin, Phys. Lett. A 242, 1 (1998).

[5] D. Gottesman, Los Alamos e-print arXiv quant-ph/0004072 (2000).

[6] A. Barenco, A. Berthiaume, D. Deutsch, A. Ekert, R. Jozsa, Ch. Macchiavello, Los Alamos e-print arXiv quant-ph/9604028 (1996).

[7] V. Buzek and M. Hillery, Phys. Rev. A 62, 022303 (2000)

[8] V. Buzek and M. Hillery, Phys. Rev. A 62, 052303 (2000)