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Entangled webs: Tight bound for symmetric sharing of entanglement

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Quantum entanglement cannot be unlimitedly shared among an arbitrary number of qubits. The degree of bipartite entanglement decreases as the number of entangled pairs in an N-qubit system increases. We analyze a system of N qubits in which an arbitrary pair of particles is entangled. We show that the maximum degree of entanglement (measured in the concurrence) between any pair of qubits is $2/N$. This tight bound can be achieved when the qubits are prepared in a pure symmetric (with respect to permutations) state with just one qubit in the basis state $|0\rangle$ and the others in the basis state $|1\rangle$.

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Schrödinger [1] has identified quantum entanglement as the key ingredient in the paradigm of quantum mechanics. Throughout the whole history of modern quantum mechanics the mystery of quantum entanglement has puzzled generations of physicists [2]. On the other hand, in the past decade entanglement has been recognized as an important resource for quantum information processing. In particular, quantum computation [3], quantum teleportation [4], quantum dense coding [5], certain types of quantum key distributions [6], and quantum secret sharing protocols [7] are rooted in the existence of quantum entanglement.

In spite of the progress in an understanding of the nature of quantum entanglement there are still open questions that have to be answered. In particular, it is not clear yet how to quantify uniquely the degree of entanglement [8–12], or how to specify the inseparability conditions for bipartite multilevel systems (qudits) [13]. A further problem that awaits a thorough illumination is that of multiparticle entanglement [14]. There are several aspects of quantum multiparticle correlations. For instance, it is the investigation of intrinsic n-party entanglement (i.e., generalizations of the Greenberger-Horne-Zeilinger state [15]). Another aspect of multiparticle entanglement is that, in contrast to classical correlation, it cannot freely be shared among many objects. In particular, Coffman *et al.* [16] have studied a set of three qubits A,B, and C. It has been shown that the sum of the entanglement (measured in terms of the tangle [11]) between the particles AB and the particles AC is smaller or equal to the entanglement between particle A and the subsystem BC. Wootters [17] has considered an *infinite* number of qubits arranged in an open line, such that every pair of nearest neighbors is entangled. In this translationally invariant *entangled chain* the maximal closest-neighbor (bipartite) entanglement (measured in the concurrence) is bounded by the value $1/\sqrt{2}$ (it is not known whether this bound is achievable) [17].

In this Rapid Communication, we consider a *finite* system of N qubits in which each pair out of $N(N-1)/2$ possible pairs is entangled. We show that the maximal possible bipartite concurrence in this case is equal to $2/N$. The derivation of this tight bound on the concurrence is the main result of our paper.

The problem is formally posed as follows. Suppose that N qubits, indexed by $l=1,2,\ldots,N$, are in the state $\hat{\rho}_N$, and we choose a basis $\{|1\rangle,|0\rangle\}$ for each qubit. We require that the matrix form of the marginal density operator $\hat{\rho}_{ll'}$ for a pair of qubits l and l', represented in the chosen basis, be independent of the choice of l and l'. Note that this requirement is satisfied if $\hat{\rho}_N$ is invariant under any permutation of qubits. The question is to find the maximum degree of entanglement between a pair of qubits.

It is convenient to suppose that each qubit is a spin-1/2 particle with the spin operator $\hat{\boldsymbol{s}}^{(l)}(l=1,\ldots,N)$. The Hilbert space of the subsystem composed of qubits 1 and 2 is a direct sum of the subspaces for the total spin 0 and 1, with the projectors $\hat{P}_0$ and $\hat{P}_1$ onto each subspace, respectively. Under the condition $\hat{\rho}_{12}=\hat{\rho}_{21}(\equiv\hat{\rho})$, we have $\hat{P}_0\hat{\rho}\hat{P}_1=\hat{P}_1\hat{\rho}\hat{P}_0=0$, since these operators change their signs under the permutation of the two qubits. Let us define irreducible tensors $\hat{T}^{(k)}_{j,q}$ of rank $k=0,1,2$ and components q such that $\hat{T}^{(k)}_{j,q}\equiv\Sigma_{m,m'}\langle k,q|1,m;1,m'\rangle(\Sigma^{j}_{l=1}\hat{s}^{(l)}_{m})(\Sigma^{j}_{l'=1}\hat{s}^{(l')}_{m'})$, where $\hat{s}^{(l)}_{\pm1}\equiv\mp(\hat{s}^{(l)}_x\pm i\hat{s}^{(l)}_y)/\sqrt{2}$, $\hat{s}^{(l)}_0\equiv\hat{s}^{(l)}_z$, and $\langle k,q|1,m;1,m'\rangle$ is the Clebsh-Gordon coefficient for forming a total spin k state from two spin-1 particles. The spin-1 part of the density operator $\hat{\rho}$ can be expanded by $\hat{T}^{(k)}_{2,q}$ as $\hat{P}_1\hat{\rho}\hat{P}_1=\Sigma_{k,q}\alpha_{k,q}\hat{T}^{(k)}_{2,q}$, and the coefficients $\alpha_{k,q}$ are obtained by the relation $\alpha_{k,q}\mathrm{Tr}(\hat{T}^{(k)}_{2,-q}\hat{T}^{(k)}_{2,q})=\mathrm{Tr}(\hat{T}^{(k)}_{2,-q}\hat{\rho})=\langle\hat{T}^{(k)}_{2,-q}\rangle$, where we denote $\mathrm{Tr}(\cdots\hat{\rho}_N)$ as $\langle\cdots\rangle$. From the symmetry of $\hat{\rho}_N$, we have $\langle\hat{T}^{(1)}_{N,q}\rangle=(N/2)\langle\hat{T}^{(1)}_{2,q}\rangle$ and $\langle\hat{T}^{(2)}_{N,q}\rangle=[N(N-1)/2]\langle\hat{T}^{(2)}_{2,q}\rangle$. With $\hat{\rho}_N$ given, it is convenient to choose the x, y, and z axes as the principal axes for the tensor of the second-order correlation for the total spin $\hat{\boldsymbol{S}}\equiv\Sigma^{N}_{l=1}\hat{\boldsymbol{s}}^{(l)}$ of the N qubits, namely, $\langle\hat{S}_\mu\hat{S}_\nu+\hat{S}_\nu\hat{S}_\mu\rangle=2S^2_\mu\delta_{\mu\nu}$ with $S^2_\mu\equiv\langle\hat{S}^2_\mu\rangle$ (here and henceforth, the suffixes μ and ν represent x,y,z). Matrix elements for $\hat{\rho}$ then take a simple form on the basis $\{|\uparrow\uparrow\rangle-|\downarrow\downarrow\rangle,|\uparrow\uparrow\rangle+|\downarrow\downarrow\rangle,|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle,|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\}$ as follows:

$$\rho=\frac{1}{N}\begin{pmatrix} A_x & \langle\hat{S}_z\rangle & i\langle\hat{S}_y\rangle & 0 \\ \langle\hat{S}_z\rangle & A_y & \langle\hat{S}_x\rangle & 0 \\ -i\langle\hat{S}_y\rangle & \langle\hat{S}_x\rangle & A_z & 0 \\ 0 & 0 & 0 & A_0 \end{pmatrix}, \tag{1}$$

where we have introduced non-negative parameters

$$A_0\equiv\frac{N(N+2)-4\langle\hat{\boldsymbol{S}}^2\rangle}{4(N-1)},\quad A_\mu\equiv\frac{N^2-4S_\mu^2}{2(N-1)}-A_0 \tag{2}$$

that satisfy $A_x+A_y+A_z+A_0=N$.

As a measure of entanglement between the two qubits, we use the ''concurrence'' introduced by Hill and Wootters [11]. The concurrence C can be calculated as follows. Let $\tilde{\rho}$ be the time reversal of ρ, which is obtained by changing the sign of spin $\langle\hat{\boldsymbol{S}}\rangle$ in the expression (1). The eigenvalues of $\rho\tilde{\rho}$ are all real and non-negative, and we let the square roots of those be $l_1,\ l_2,\ l_3$, and l_4 in decreasing order. The concurrence C is then given by $C=\max\{l_1-l_2-l_3-l_4,0\}$. In the present case, one of the eigenvalues of $\rho\tilde{\rho}$ is $(A_0/N)^2$. Let us denote the other three as $(\lambda_1/N)^2$, $(\lambda_2/N)^2$, and $(\lambda_3/N)^2$ in decreasing order, and introduce parameters $\beta\equiv\lambda_1+\lambda_2+\lambda_3$ and $\gamma\equiv\lambda_1-\lambda_2-\lambda_3$. The concurrence is then given by $C=\max\{(\gamma-A_0)/N,(A_0-\beta)/N,0\}$, where allowances are made for the possible order of A_0 and λ_1.

In the following, we first fix the parameters S_μ^2 (hence A_μ and A_0), and maximize γ with respect to $(X,Y,Z)\equiv(\langle\hat{S}_x\rangle^2,\langle\hat{S}_y\rangle^2,\langle\hat{S}_z\rangle^2)$. We then move S_i^2 to obtain the global maximum of $(\gamma-A_0)/N$, which turns out to be, as we shall see, the maximum of the concurrence. For simplicity, we assume that $S_z^2>S_y^2>S_x^2$ (hence $A_z<A_y<A_x$). The states with some parameters equal will be considered as the limiting cases.

There are two simple bounds for the allowed values of (X,Y,Z). One is a necessary condition for ρ to be physical. The eigenvalues for ρ must be non-negative, and the boundary is given by the surface that satisfies $\det(\rho)=0$. Calculating from Eq. (1), this surface turns out to be a plane, and the condition for (X,Y,Z) is

$$f_A\equiv A_xA_yA_z-A_xX-A_yY-A_zZ\geqslant 0. \tag{3}$$

The other one is a requirement necessary for the spin correlations. From the inequality $\langle[\Sigma_i\langle\hat{S}_i\rangle(\hat{S}_i-\langle\hat{S}_i\rangle)/S_i^2]^2\rangle\geqslant 0$, we obtain

$$f_S\equiv 1-\frac{X}{S_x^2}-\frac{Y}{S_y^2}-\frac{Z}{S_z^2}\geqslant 0. \tag{4}$$

Any physical state thus falls in the region V, which is defined by $f_A\geqslant 0,\ f_S\geqslant 0$, and $X,Y,Z\geqslant 0$.

The relations that connect (X,Y,Z) and λ_i are obtained by expanding $\det[N^2\rho\tilde{\rho}-\lambda^2 I]=\Pi_i(\lambda_i^2-\lambda^2)$ and equating the coefficients of λ^m. There are three independent equations, and it is convenient to take the following set:

$$f_0\equiv A_x^2+A_y^2+A_z^2-2X-2Y-2Z=\lambda_1^2+\lambda_2^2+\lambda_3^2,$$

$$f_A=\lambda_1\lambda_2\lambda_3,$$

$$f_B\equiv -B_xB_yB_z\left(B_x+B_y+B_z-\frac{4X}{B_x}-\frac{4Y}{B_y}-\frac{4Z}{B_z}\right)$$
$$=\lambda_1^4+\lambda_2^4+\lambda_3^4-2(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2), \tag{5}$$

where $B_\mu\equiv A_x+A_y+A_z-2A_\mu$. The relation $A_x>A_y>A_z>0$ implies that $B_z>B_y>|B_x|$. Note that the sign of f_B is the same as that of γ, since we can factorize as $f_B=\beta\gamma(\lambda_1+\lambda_3-\lambda_2)(\lambda_1+\lambda_2-\lambda_3)$.

Let W be the region defined by $f_A\geqslant 0$ and $f_B\geqslant 0$. As a function of (X,Y,Z), γ is continuous in the region W including the boundaries. The gradient $\boldsymbol{\nabla}\gamma\equiv(\partial\gamma/\partial X,\partial\gamma/\partial Y,\partial\gamma/\partial Z)$ can formally be obtained by using the three relations (5). The result is

$$\frac{\partial\gamma}{\partial X}=\frac{(\gamma+B_y)(\gamma+B_z)}{\kappa}, \tag{6}$$

where $\kappa\equiv 2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2+\lambda_3)\geqslant 0$, and the other two are obtained by the cyclic exchange. At the inner points of W, $\lambda_1>\lambda_2$, since $\gamma>0$, and $\lambda_3>0$, since $f_A>0$. The parameter κ is hence positive, and the gradient $\boldsymbol{\nabla}\gamma$ exists. Since $B_z>B_y>0$, we have $\partial\gamma/\partial X>0$ for the inner points of W.

The geometry of W and V is derived as follows. Let the points on which the planes $f_B=0,\ f_A=0$, and $f_S=0$ intersect the X axis be $P_{BX},\ P_{AX}$, and P_{SX}, respectively, and denote the other six points on the Y and Z axis similarly. The relation $A_xA_y-(B_x+B_y+B_z)B_z/4=B_xB_y/4$, and the ones obtained by the cyclic exchange of $\{x,y,z\}$, tells us the following. When $B_x>0$, the triangle $\pi_B\equiv P_{BX}P_{BY}P_{BZ}$ does not intersect with $\pi_A\equiv P_{AX}P_{AY}P_{AZ}$, and lies closer to the origin. W is the sandwiched region of the two triangles. The triangle $\pi_S\equiv P_{SX}P_{SY}P_{SZ}$ may intersect with π_A and π_B or not. When $B_x<0,\ \ f_B>0$ (hence $\gamma>0$) is satisfied by all the points that satisfy $f_A\geqslant 0$. When $B_x=0,\ \ \gamma=0$ for the points on the Y-Z plane, and $\gamma>0$ for $X>0$. Combining these observations with $\partial\gamma/\partial X>0$, we conclude that, in V, γ takes its maximum on points on the boundaries π_A or π_S, and never on the inner points.

To determine the behavior of γ on the boundaries, we first derive the value of γ on the axes explicitly. For the points on the Z axis and satisfying the relation (3), the roots $\{\lambda_i\}$ are found to be $\{A_z,[\sqrt{(A_x+A_y)^2-4\langle\hat{S}_z\rangle^2}\pm(A_x-A_y)]/2\}$. The expression for γ depends on which is the largest root. The roots for the points on the other two axes are similarly obtained. Applying these to the vertices of π_A, we have $\gamma(P_{AX})=B_y$ and $\gamma(P_{AY})=\gamma(P_{AZ})=|B_x|$. On the vertices of π_S, γ is well defined only when they are in the region W. When P_{SZ} is in W,

$$0\leqslant 4(A_xA_y-S_z^2)=(A_z+A_0)(A_z+A_0-2)-(A_x-A_y)^2, \tag{7}$$

and hence $A_z>2-A_0$. Then γ is written as

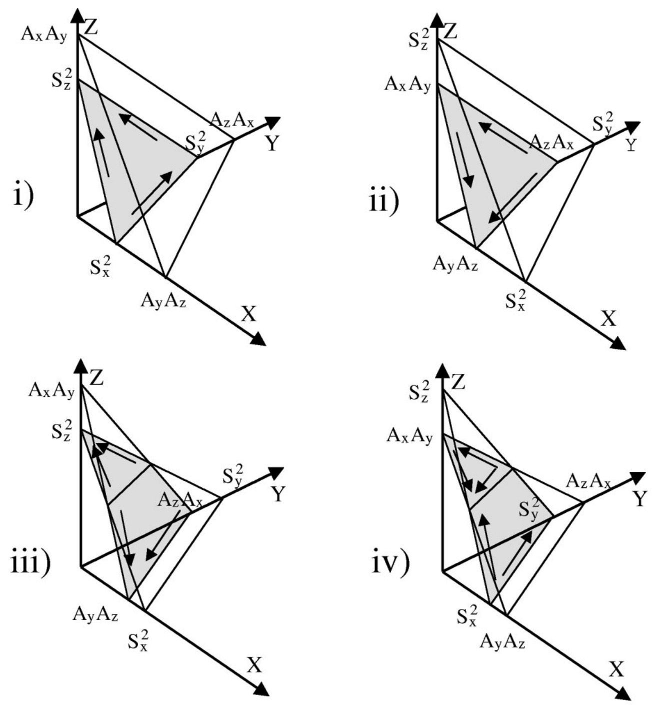

FIG. 1. Allowed regions for the parameters $\{X,Y,Z\}$. The shaded surfaces are boundaries for the physically allowed states. The arrows show the directions in which the concurrence increases.

$$\gamma(P_{SZ})=A_z-\sqrt{(A_x+A_y)^2-4S_z^2}$$
$$=A_z-\sqrt{(A_z+A_0)(A_z+A_0-2)}. \qquad (8)$$

This is a decreasing function of A_z, and noting $S_z^2>0$, we have $2-A_0>\gamma(P_{SZ})>N/(N-1)-A_0\equiv\gamma_m$. Similarly, $\gamma(P_{SX})>\gamma_m$ and $\gamma(P_{SY})>\gamma_m$ if those points are in W.

Next, we will evaluate the gradient on the boundaries. On the boundary π_A, κ is not always positive, but $\kappa=0$ is possible only for the following two cases. (a) $\lambda_1=\lambda_2$. This occurs only when $f_A=f_B=0$, or equivalently, on the segment $P_{AY}P_{AZ}$ when $B_x=0$. (b) $\lambda_2=\lambda_3=0$. This occurs if and only if $f_A=0$ and $(f_0)^2-f_B=0$. The intersection of $f_A=0$ and $(f_0)^2-f_B=0$ is a parabola. The point $P_0(A_z^2(A_x-A_y)/(A_x-A_z),0,A_x^2(A_y-A_z)/(A_x-A_z))$ is on this parabola. At P_0, $\boldsymbol{u}=(A_x,A_y,A_z)$ is normal to $f_A=0$, and $\boldsymbol{v}=(-A_xB_y,A_yB_y-2A_xA_z,-A_zB_y)$ is normal to $(f_0)^2-f_B=0$. Since $\boldsymbol{u}\times\boldsymbol{v}$ has a vanishing Y component, the parabola is tangent to the X-Z plane at P_0. Another point $(A_y^2(A_x-A_z)/(A_x-A_y),-A_x^2(A_y-A_z)/(A_x-A_y),0)$, on which $Y<0$, is also on the parabola. We thus conclude that the gradient exists on π_A except for the segment $P_{AY}P_{AZ}$ and P_0.

Let us define three particular directions on π_A as $\boldsymbol{q}^{yx}\equiv(A_yA_z,-A_zA_x,0)$, $\boldsymbol{q}^{zx}\equiv(A_yA_z,0,-A_xA_y)$, and $\boldsymbol{q}^{yz}\equiv(0,-A_zA_x,A_xA_y)$. The differential coefficients of γ for these directions are calculated to be $\boldsymbol{q}^{yx}\cdot\nabla\gamma=A_z(B_z^2-\gamma^2)(A_x-A_y)/\kappa$, $\boldsymbol{q}^{zx}\cdot\nabla\gamma=A_y(B_y^2-\gamma^2)(A_x-A_z)/\kappa$, and $\boldsymbol{q}^{yz}\cdot\nabla\gamma=A_x(\gamma^2-B_x^2)(A_y-A_z)/\kappa$. Noting that $\gamma(P_{AX})=B_y$ and $\gamma(P_{AY})=\gamma(P_{AZ})=|B_x|$, we conclude that $\boldsymbol{q}^{yx}\cdot\nabla\gamma>0$, $\boldsymbol{q}^{zx}\cdot\nabla\gamma>0$ and $\boldsymbol{q}^{yz}\cdot\nabla\gamma>0$ on π_A except for on the segment $P_{AY}P_{AZ}$ and P_0P_{AX}. On P_0P_{AX}, $\boldsymbol{q}^{zx}\cdot\nabla\gamma=0$, so that γ is constant.

Similarly, define directions on π_S as $\boldsymbol{p}^{xy}\equiv(-S_x^2,S_y^2,0)$ $\boldsymbol{p}^{yz}\equiv(0,-S_y^2,S_z^2)$ and $\boldsymbol{p}^{xz}\equiv(-S_x^2,0,S_z^2)$. Let π_{SV} be the intersection of π_S and W. On π_{SV}, we have $\boldsymbol{p}^{xy}\cdot\nabla\gamma=(\gamma+B_z)(\gamma-\gamma_m)(S_y^2-S_x^2)/\kappa$, $\boldsymbol{p}^{xz}\cdot\nabla\gamma=(\gamma+B_y)(\gamma-\gamma_m)(S_z^2-S_x^2)/\kappa$, and $\boldsymbol{p}^{yz}\cdot\nabla\gamma=(\gamma+B_x)(\gamma-\gamma_m)\times(S_z^2-S_y^2)/\kappa$. When $B_x<0$, $\gamma(P_{AY})=\gamma(P_{AZ})=|B_x|$ and Eq. (6) implies that $\gamma=|B_x|$ on the Y-Z plane, and $\gamma+B_x>0$ for $X>0$. Since we have seen that $\gamma>\gamma_m$ on the vertices on π_{SV}, $\boldsymbol{p}^{xz}\cdot\nabla\gamma>0$ and $\boldsymbol{p}^{xy}\cdot\nabla\gamma>0$ everywhere on π_{SV}, and $\boldsymbol{p}^{yz}\cdot\nabla\gamma>0$ on π_{SV} except for the segment $P_{SY}P_{SZ}$.

Now we are in a position to find the maximum of γ. We must consider the following four cases separately (see Fig. 1).

(i) $A_xA_y\geqslant S_z^2$ and $A_yA_z>S_x^2$. In this case, $A_z>2-A_0$ is necessary. γ takes its maximum on P_{SZ} and the value is given by Eq. (8) and $\gamma<2-A_0$. γ approaches 2 only in the limit of $A_0\to0$, $A_z\to2$, $B_x\to2$, and $B_y\to2$. This limit can be taken only if $N\geqslant6$, since $A_y\geqslant A_z$ must hold in the limit. When $N>6$, P_{SZ} is the only point that attains $\gamma=2$ since $\boldsymbol{p}^{yz}\cdot\nabla\gamma>0$ and $\boldsymbol{p}^{xz}\cdot\nabla\gamma>0$ still hold in the limit. When $N=6$, $\gamma=2$ everywhere on π_A, but these states are equivalent in the sense that they are related by the rotation of the whole system.

(ii) $A_xA_y<S_z^2$ and $A_yA_z\leqslant S_x^2$. In this case, from the relation $0\leqslant4(S_x^2-A_yA_z)=-(B_y+A_0-2)(B_y+A_0)-(B_z-B_y)(B_y+A_0-1)$, $B_y<2-A_0$ is necessary. γ takes its maximum on P_0P_{AX} and the value is $\gamma=B_y<2-A_0$. γ approaches 2 only in the limit of $A_0\to0$, $A_x\to2$, $B_y\to2$, and $B_z\to2$. This limit can be taken only if $N\leqslant6$, since $A_x\geqslant A_y$ must hold in the limit. When $N<6$, P_0 coincides with P_{AX} in the limit and P_{AX} is thus the only point that attains $\gamma=2$. When $N=6$, the limit is the same as in case (i).

(iii) $A_xA_y\geqslant S_z^2$ and $A_yA_z\leqslant S_x^2$. The maximum is the larger of $\gamma(P_{AX})$ and $\gamma(P_{SZ})$. Depending on N, one of them or both can approach 2. The limit is the same as described in cases (i) and (ii).

(iv) $A_xA_y<S_z^2$ and $A_yA_z>S_x^2$. In this case, from the relation $C(P_{AX})>C(P_{SX})$, we have $B_y>\gamma_m$. We also have $A_0<2$ since $(2-A_0)(B_y+A_0+2A_z)=4(S_z^2-A_xA_y)+2A_z+B_y(N-2+4S_x^2)/(N-1)>0$. γ takes its maximum at P_1, that is, the intersection of $P_{AZ}P_{AX}$ and $P_{SZ}P_{SX}$. Since $\lambda_3=0$ at P_1, $\gamma=\lambda_1-\lambda_2$ and $\beta=\lambda_1+\lambda_2$. Then we have $\beta^2+\gamma^2=2f_0$ and $\beta^2\gamma^2=f_B$. This implies that γ^2 is the smaller of the two roots $t=t_\alpha,t_\beta$ of the equation $t^2-2f_0t+f_B=0$. The coordinates of P_1 can explicitly be obtained by solving $f_A=0$ and $f_S=0$ with $Y=0$. Substituting the result into f_0 and f_B in the equation of t, we finally obtain $t_\alpha=B_y^2$ and

$$t_\beta=\frac{4A_xA_z}{(B_y-\gamma_m)(N-1)}+A_0(A_0-2)$$
$$=(2-A_0)^2+2A_0\frac{B_y+A_0-2}{B_y-\gamma_m}$$
$$-\frac{2(B_y+A_0-2)(B_y+N)+(B_z-B_y)(B_y-B_x)}{(N-1)(B_y-\gamma_m)}. \qquad (9)$$

When $B_y \geq 2 - A_0$, we have $t_\beta < 4 - A_0(2 - A_0) < 4$. We thus conclude that $\gamma < 2$ for all values of B_y. When $N < 6$, γ approaches 2 only in the limit of $A_0 \to 0$, $B_y \to 2$, $B_z \to 2$, and $A_x \to 2$. P_1 approaches P_{AX} in this limit, so the limit is the same as in case (ii). When $N > 6$, $\gamma \to 2$ only in the limit of $A_0 \to 0$, $B_y \to 2$, $B_x \to 2$, $A_z \to 2$, and $P_1 \to P_{SZ}$. This is the same limit as in case (i). When $N = 6$, $\gamma \to 2$ in the limit of $A_0 \to 0$ and all A_μ approaching 2.

Finally, we show that $A_0 - \beta$ is smaller than 2. In W, f_0 takes its minimum on P_{AZ}, and its value is $(A_x - A_y)^2 + A_z^2$. We thus have $A_0 - \beta \leq A_0 - \sqrt{f_0} \leq A_0 - A_z = (N - 2S_x^2 - 2S_y^2)/(N-1) < N/(N-1) < 2$ for $N \geq 3$.

Combining all cases, we reach a conclusion that the maximum value of the concurrence is $2/N$, and this value is reached only by the state satisfying the constraints $\langle \hat{S}_z^2 \rangle = (N/2 - 1)^2$, $\langle \hat{S}_z \rangle = N/2 - 1$, $\langle \hat{S}_x^2 \rangle = \langle \hat{S}_y^2 \rangle = (3N-2)/4$, and $\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = 0$, if the z axis is suitably chosen. Such a state exists — it is the eigenstate of $\hat{S}_z$ with eigenvalue $N/2 - 1$ with total spin $N/2$. This state is the equally weighted in-phase superposition of any one qubit that is in the state $|0\rangle$ and the other $N-1$ qubits in the state $|1\rangle$. This is a permutationally invariant pure state that highly (in the order of N) breaks the symmetry between the basis states $|0\rangle$ and $|1\rangle$.

It is worth noting that the optimal state for our problem was found to be a pure state. This is not trivial because the symmetry is required for density operators. Obviously, possible mixed states span a larger Hilbert space than that spanned by possible pure states. For infinite entangled chains, the best state so far is given as a *mixed state* [17]. It will be interesting to investigate entangled chains (loops) composed of a *finite* number of qubits and to see whether the optimum value is attained by mixed states or pure states (or by both). O'Connor and Wootters have found [18] a series of *pure* states in N-qubit finite entangled loops and showed that the limiting value of the nearest-neighbor concurrence in $N \to \infty$ is the same as the one obtained for the infinite chain in [17]. It is still an open question whether these states are optimum or not.

Note added. Recently we became aware that Dür [19] has independently conjectured the upper bound on the concurrence in entangled webs to be $2/N$.

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