Entangled webs: Tight bound for symmetric sharing of entanglement

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Quantum entanglement cannot be unlimitedly shared among an arbitrary number of qubits. The degree of bipartite entanglement decreases as the number of entangled pairs in an N-qubit system increases. We analyze a system of N qubits in which an arbitrary pair of particles is entangled. We show that the maximum degree of entanglement (measured in the concurrence) between any pair of qubits is 2/N. This tight bound can be achieved when the qubits are prepared in a pure symmetric (with respect to permutations) state with just one qubit in the basis state |0⟩ and the others in the basis state |1⟩.

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Schrödinger [1] has identified quantum entanglement as the key ingredient in the paradigm of quantum mechanics. Throughout the whole history of modern quantum mechanics, the mystery of quantum entanglement has puzzled generations of physicists [2]. On the other hand, in the past decade entanglement has been recognized as an important resource for quantum information processing. In particular, quantum computation [3], quantum teleportation [4], quantum dense coding [5], certain types of quantum key distributions [6], and quantum secret sharing protocols [7] are rooted in the existence of quantum entanglement. In spite of the progress in an understanding of the nature of quantum entanglement there are still open questions that have to be answered. In particular, it is not clear yet how to quantify uniquely the degree of entanglement [8–12], or how to specify the inseparability conditions for bipartite multilevel systems (qudits) [13]. A further problem that awaits a thorough illumination is that of multiparticle entanglement [14]. There are several aspects of quantum multiparticle correlations. For instance, it is the investigation of intrinsic n-party entanglement (i.e., generalizations of the Greenberger-Horne-Zeilinger state [15]). Another aspect of multiparticle entanglement is that, in contrast to classical correlation, it cannot freely be shared among many objects. In particular, Coffman et al. [16] have studied a set of three qubits A, B, and C. It has been shown that the sum of the entanglement (measured in terms of the tangle [11]) between the particles AB and the particles AC is smaller or equal to the entanglement between particle A and the subsystem BC. Wootters [17] has considered an infinite number of qubits arranged in an open line, such that each pair of nearest neighbors is entangled. In this translationally invariant entangled chain the maximal closest-neighbor (bipartite) entanglement (measured in the concurrence) is bounded by the value 1/√2 (it is not known whether this bound is achievable) [17].

In this Rapid Communication, we consider a finite system of N qubits in which each pair out of N(N−1)/2 possible pairs is entangled. We show that the maximal possible bipartite concurrence in this case is equal to 2/N. The derivation of this tight bound on the concurrence is the main result of our paper.

The problem is formally posed as follows. Suppose that N qubits, indexed by l=1,2,…,N, are in the state ρN, and we choose a basis {1},|0⟩ for each qubit. We require that the matrix form of the marginal density operator ρN,l, for a pair of qubits l and l′, represented in the chosen basis, be independent of the choice of l and l′. Note that this requirement is satisfied if ρN is invariant under any permutation of qubits. The question is to find the maximum degree of entanglement between a pair of qubits.

It is convenient to suppose that each qubit is a spin-1/2 particle with the spin operator ıS(l) (l=1, N). The Hilbert space of the subsystem composed of qubits 1 and 2 is a direct sum of the subspaces for the total spin 0 and 1, with the projectors ıP0 and ıP1 onto each subspace, respectively. Under the condition ıP1=ıP2=1/√2, we have ıP0 ıP1 = ıP1 ıP0 = 0, since these operators change their signs under the permutation of the two qubits. Let us define irreducible tensors ıT(k) ıj,q of rank k=0,1,2 and components q such that ıT(k) ıj,q = Σ(1),m,...,q=1,1,m′(Σ(k) ıj,m ıj,m′) where ıs(l)± = (ıs(l)± + is(l)±)/√2, ıs(l)± =ıs(l)±, and ⟨k,q|1,m;1,m′⟩ is the Clebsch-Gordon coefficient for forming a total spin k state from two spin-1 particles. The spin-1 part of the density operator ıρ can be expanded by ıT(k) ıj,q as ıP1(ıρ)ıP1 = ıΣ ıj,q ıα ıj,q ıT(k) ıj,q, and the coefficients ıα ıj,q are obtained by the relation ıα ıj,qTr(ıT(k) ıj,q ıP0 ıP1) =Tr(ıP1(q) ıρ ıP1) =⟨ıT(k) ıj,q⟩, where we denote Tr(⋯ ıρ ⋯) as ⟨⋯⟩. From the symmetry of ıρN, we have ⟨ıT(k) ıj,q⟩ = ⟨N−1⟩ ıj,q ıT(k) ıj,q and ⟨ıT(k) ıj,q⟩ = [N(N−1)/2] ıj,q ıT(k) ıj,q. With ıρN given, it is convenient to choose the x, y, and z axes as the principal axes for the tensor of the second-order correlation for the total spin S = Σ(k=1,2,N,−1,2,−1)ıS(l) ıl for the N qubits, namely, ı(S2) ıl,l′ + ı(S2) ıl ıl′ = ıg(ıμ,ıν) with ıg(ıμ,ıν) = ıg(ıμ) (here and henceforth, the suffixes μ and ν represent (x,y,z). Matrix elements for ıρ then take a simple form on the basis {1,1},|0⟩,|1⟩,|0⟩,|1⟩,|0⟩,|1⟩,|0⟩,|1⟩,|0⟩,|1⟩ as follows:
\[
\rho = \frac{1}{N} \begin{pmatrix}
A_x & \langle \hat{S}_y \rangle & i \langle \hat{S}_z \rangle & 0 \\
\langle \hat{S}_y \rangle & A_y & \langle \hat{S}_z \rangle & 0 \\
i \langle \hat{S}_z \rangle & \langle \hat{S}_y \rangle & A_z & 0 \\
0 & 0 & 0 & A_0
\end{pmatrix},
\] (1)

where we have introduced non-negative parameters

\[A_0 = \frac{N(N+2)-4\langle \hat{S}_z \rangle^2}{4(N-1)}, \quad A_\mu = \frac{N^2-4\langle \hat{S}_\mu \rangle^2}{2(N-1)} - A_0 \] (2)

that satisfy \(A_1 + A_2 + A_3 + A_0 = N\).

As a measure of entanglement between the two qubits, we use the ‘concurrence’ introduced by Hill and Wootters [11]. The concurrence of pair can be calculated as follows: Let \(\rho\) be the time reversal of \(\rho\), which is obtained by changing the sign of \(\langle \hat{S}_i \rangle\) in the expression (1). The eigenvalues of \(\rho \rho^\dagger\) are all real and non-negative, and we let the square roots of those be \(l_1, l_2, l_3,\) and \(l_4\) in decreasing order. The concurrence is then given by \(C = \max(l_1 - l_2 - l_3 - l_4, 0)\). In the present case, one of the eigenvalues of \(\rho \rho^\dagger\) is \(A_0/N)^2\). Let us denote the other three as \(A_1/N)^2, (A_2/N)^2,\) and \((A_3/N)^2\) in decreasing order, and introduce parameters \(\beta = \lambda_1 + \lambda_2 + \lambda_3\) and \(\gamma = \lambda_1 - \lambda_2 - \lambda_3\). The concurrence is then given by \(C = \max((\gamma - A_0)/N, (A_0 - \beta)/N, 0)\), where allowances are made for the possible order of \(A_1, A_2, A_3,\) and \(A_0\).

In the following, we first fix the parameters \(\hat{S}_\mu^2\) (hence \(A_\mu\) and \(A_0\)), and maximize \(\gamma\) with respect to \((X, Y, Z)\). Then we use \(\hat{S}_2^2\) to obtain the global maximum of \((\lambda_1 - A_0)/N, \) which turns out to be, as we shall see, the maximum of the concurrence. For simplicity, we assume that \(\hat{S}_2^2 > \hat{S}_3^2 > \hat{S}_1^2\) (hence \(A_2 < A_3 < A_1\)). The states with some parameters equal will be considered as the limiting cases.

There are two simple bounds for the allowed values of \((X, Y, Z)\). One is a necessary condition for \(\rho\) to be physical. The eigenvalues for \(\rho\) must be non-negative, and the boundary is given by the surface that satisfies \(\det(\rho) = 0\). Calculating from Eq. (1), this surface turns out to be a plane, and the condition for \((X, Y, Z)\) is

\[f_A = A_x A_y A_z - A_x Y - A_y Z - A_z X \geq 0.\] (3)

The other one is a requirement necessary for the spin correlations. From the inequality \(\langle [\hat{S}_i, (\hat{S}_j - \langle \hat{S}_j \rangle)/\hat{S}_i^2] \rangle \geq 0\), we obtain

\[f_S = 1 - \frac{X}{\hat{S}_x^2} - \frac{Y}{\hat{S}_y^2} - \frac{Z}{\hat{S}_z^2} \geq 0.\] (4)

Any physical state thus falls in the region \(V\), which is defined by \(f_A \geq 0, f_S \geq 0,\) and \(X, Y, Z \geq 0\).

The relations that connect \((X, Y, Z)\) and \(\lambda_i\) are obtained by expanding \(\det[N^2 \rho \rho^\dagger - \lambda^2 I] = \Pi_i (\lambda_i^2 - \lambda^2)\) and equating the coefficients of \(\lambda^m\). There are three independent equations, and it is convenient to take the following set:

\[f_0 = A_x^2 + A_y^2 + A_z^2 - 2X - 2Y - 2Z - 2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\]
\[f_A = \lambda_1 \lambda_2 \lambda_3,\]
\[f_B = -(B_x B_y B_z - \frac{4X + 4Y - 4Z}{B_x - B_y - B_z}) = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2(\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2),\] (5)

where \(B_\mu = A_x + A_y + A_z - 2A_\mu\). The relation \(\lambda_1 \lambda_2 \lambda_3 \neq 0\) implies that \(B_x > B_y > B_z\). Note that the sign of \(f_B\) is the same as that of \(\gamma\), since we can factorize as \(f_B = \beta \gamma (\lambda_1 + \lambda_2 - \lambda_3) (\lambda_1 + \lambda_2 - \lambda_3)\).

Let \(W\) be the region defined by \(f_A \geq 0\) and \(f_B \geq 0\). As a function of \((X, Y, Z)\), \(\gamma\) is continuous in the region \(W\) including the boundaries. The gradient \(\nabla \gamma = (\partial \gamma/\partial X, \partial \gamma/\partial Y, \partial \gamma/\partial Z)\) can formally be obtained by using the three relations (5). The result is

\[\frac{\partial \gamma}{\partial X} = (\gamma + B_x)(\gamma + B_y) / \kappa,\] (6)

where \(\kappa = 2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 + \lambda_3) \geq 0\), and the other two are obtained by the cyclic exchange. At the inner points of \(W\), \(\lambda_1 \lambda_2 \lambda_3 \neq 0\), since \(\gamma > 0\), and \(\lambda_1 \lambda_2 > 0\), since \(f_A > 0\). The parameter \(\kappa\) is hence positive, and the gradient \(\nabla \gamma\) exists. Since \(B_x > B_y > 0\), we have \(\partial \gamma/\partial X > 0\) for the inner points of \(W\).

The geometry of \(W\) and \(V\) is derived as follows. Let the points on which the planes \(f_B = 0, f_A = 0,\) and \(f_S = 0\) intersect the \(X, Y, Z\) axis be \(P_{BX}, P_{AX},\) and \(P_{SX},\) respectively, and denote the other six points on the \(Y\) and \(Z\) axis similarly. The relation \(A_1 A_2 - (B_x + B_y + B_z)B_z/4 = B_x B_y /4,\) and the ones obtained by the cyclic exchange of \(\{x, y, z\}\), tells us the following. When \(B_z > 0\), the triangle \(\pi_{Bz} = P_{Bx} P_{By} P_{Bz}\) does not intersect with \(\pi_A = P_{AX} P_{AY} P_{AZ}\), and lies closer to the origin. \(W\) is the sandwiched region of the two triangles. The larger \(\pi_z = P_{SX} P_{SY} P_{SZ}\) may intersect with \(\pi_A\) or \(\pi_z\) or not. When \(B_z < 0,\) \(f_B < 0\) (hence \(\gamma > 0\)) is satisfied by all the points that satisfy \(f_A \geq 0\). When \(B_z = 0\), \(\gamma = 0\) for the points on the \(Y-Z\) plane, and \(\gamma > 0\) for \(X = 0\). Combining these observations with \(\partial \gamma/\partial X > 0\), we conclude that, in \(V\), \(\gamma\) takes its maximum on points on the boundaries \(\pi_A\) or \(\pi_z\), and never on the inner points.

To determine the behavior of \(\gamma\) on the boundaries, we first derive the value of \(\gamma\) on the axes explicitly. For the points on the \(Z\) axis and satisfying the relation (3), the roots \(\{\lambda_i\}\) are found to be \(\{\lambda_i, (\gamma/4(\lambda_1 + \lambda_2)^2 + 4(\lambda_1^2 + (A_z - A_x)/2).\) The expression for \(\gamma\) depends on which is the largest root. The roots for the points on the other two axes are similarly obtained. Applying these to the vertices of \(\pi_A\), we have \(\gamma(P_{AX}) = B_y,\) and \(\gamma(P_{AY}) = \gamma(P_{AZ}) = |B_z|\). On the vertices of \(\pi_z,\) \(\gamma\) is well defined only when they are in the region \(W\). When \(P_{SZ}\) is in \(W,\)

\[0 \leq 4(A_x A_y - S_z^2) = (A_x + A_0)(A_z + A_0 - 2) - (A_z - A_0)^2,\] (7)

and hence \(A_z > 2 - A_0\). Then \(\gamma\) is written as

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similarly, define directions on $\pi_{SV}$ as $p^{x\gamma} = (-S_{z}^{2}, 0, 0)$, $p^{x^{\gamma}} = (0, 0, S_{z}^{2})$, and $p^{x^{\gamma}} = (-S_{z}^{2}, 0, 0)$. Let $\pi_{SV}$ be the intersection of $\pi_{S}$ and $W$. On $\pi_{SV}$, we have $p^{x\gamma} \cdot \gamma = (y + B_{y})(\gamma - y_{m})(S_{z}^{2} - S_{z}^{2})/\kappa$, $p^{2\gamma} \cdot \gamma = (y + B_{y})(\gamma - y_{m})(S_{z}^{2} - S_{z}^{2})/\kappa$, and $p^{2\gamma} \cdot \gamma = (y + B_{y})(\gamma - y_{m})(S_{z}^{2} - S_{z}^{2})/\kappa$. When $B_{y} < 0$, $\gamma(\pi_{SV}) = \gamma(P_{AZ}) = |B_{y}|$, and Eq. (6) implies that $\gamma = |B_{y}|$ on the $Y-Z$ plane, and $\gamma + B_{y} > 0$ for $X > 0$. Since we have seen that $\gamma > y$ on the vertices on $\pi_{SV}$, $p^{x\gamma} \cdot \gamma > 0$ and $p^{x\gamma} \cdot \gamma > 0$ everywhere on $\pi_{SV}$, and $p^{2\gamma} \cdot \gamma > 0$ on $\pi_{SV}$ except for the segment $P_{SV}P_{SZ}$.

Now we are in a position to find the maximum of $\gamma$. We must consider the following four cases separately (see Fig. 1). (i) $A_{1}A_{2} \approx S_{z}^{2}$ and $A_{3}A_{2} \approx S_{z}^{2}$. In this case, $A_{2} > 2 - A_{0}$ is necessary. $\gamma$ takes its maximum on $P_{SZ}$ and the value is given by Eq. (8) and $\gamma < 2 - A_{0}$. $\gamma$ approaches 2 only in the limit of $A_{0} \rightarrow 0$, $A_{2} \rightarrow 2$, $B_{y} \rightarrow 2$, and $B_{z} \rightarrow 2$. This limit can be taken only if $N \gg 6$, since $A_{1} \gg A_{2}$, must hold in the limit. When $N > 6$, $P_{SZ}$ is the only point that attains $\gamma = 2$ since $p^{x\gamma} \cdot \gamma > 0$ and $p^{2\gamma} \cdot \gamma > 0$ still hold in the limit. When $N = 6$, $\gamma = 2$ everywhere on $\pi_{SV}$, but these states are equivalent in the sense that they are related by the rotation of the whole system. (ii) $A_{1}A_{3} < S_{z}^{2}$ and $A_{2}A_{3} < S_{z}^{2}$. In this case, the relation $0 \leq 4(S_{z}^{2} - A_{2}A_{3}) = - (B_{y} + A_{2} - 2)(B_{y} - A_{2} - 2)(B_{y} + A_{2}) = (B_{y} + A_{2}) (B_{y} - A_{2})$. This limit can be taken only if $N < 6$, since $A_{2} \gg A_{1}$ must hold in the limit. When $N < 6$, $P_{0}$ coincides with $P_{AX}$ in the limit and $P_{AX}$ is thus the only point that attains $\gamma = 2$. When $N = 6$, the limit is the same as in case (i). (iii) $A_{1}A_{3} \approx S_{z}^{2}$ and $A_{2}A_{3} \approx S_{z}^{2}$. The maximum is the larger of $\gamma(P_{AX})$ and $\gamma(P_{SZ})$. Depending on $N$, one of them or both can approach 2. The limit is the same as described in cases (i) and (ii). (iv) $A_{1}A_{3} < S_{z}^{2}$ and $A_{2}A_{3} > S_{z}^{2}$. In this case, from the relation $C(P_{AX}) = C(P_{SX})$, we have $B_{y} > y_{m}$. We also have $A_{0} > 2$ since $(2 - A_{0})(B_{y} + A_{0}) = 4(S_{z}^{2} - A_{2}A_{3}) + 2A_{2} + 2B_{y} = 4S_{z}^{2} + 4y_{m}(N - 1) > 0$. $\gamma$ takes its maximum at $P_{1}$, that is, the intersection of $P_{AX}$ and $P_{SZ}P_{AX}$. Since $\lambda_{3} = 0$ at $P_{1}$, $\gamma = \lambda_{1} - \lambda_{2}$ and $\gamma = \lambda_{1} + \lambda_{2}$. Then we have $\beta^{2} = \beta^{2} = 2f_{0}$ and $\beta^{2} \gamma = f_{0}$. This implies that $\gamma^{2}$ is the smaller of the two roots $t = t_{a}, t_{b}$ of the equation $t^{2} - 2f_{0} + f_{0} = 0$. The coordinates of $P_{1}$ can explicitly be obtained by solving $f_{a} = 0$ and $f_{b} = 0$ with $Y = 0$. Substituting the result into $f_{0}$ and $f_{b}$ in the equation of $t$, we finally obtain $t_{b} = B_{y}$ and

$$t_{a} = \frac{4A_{2}A_{3}}{B_{y} - y_{m}(N - 1)} + A_{0}(A_{0} - 2)$$

$$= \frac{2(A_{0} - 2) + 2A_{0}}{B_{y} - y_{m}}$$

$$= \frac{2(B_{y} + A_{0} - 2)(B_{y} - N + 2) + 2(B_{y} - B_{y})}{(N - 1)(B_{y} - y_{m})}.$$

$\pi_{AX}$ and $P_{0}P_{AX}$. On $P_{0}P_{AX}$, $q^{x\gamma} \cdot \gamma = 0$, so that $\gamma$ is constant.
When \( B_1 \geq 2 - A_0 \), we have \( t_\beta < 4 - A_0 (2 - A_0) < 4 \). We thus conclude that \( \gamma < 2 \) for all values of \( B_1 \). When \( N < 6 \), \( \gamma \) approaches 2 only in the limit of \( A_0 \rightarrow 0 \), \( B_1 \rightarrow 2 \), \( B_2 \rightarrow 2 \), and \( A_1 \rightarrow 2 \). \( P_1 \) approaches \( P_{AX} \) in this limit, so the limit is the same as in case (ii). When \( N \rightarrow 6 \), \( \gamma \rightarrow 2 \) only in the limit of \( A_0 \rightarrow 0 \), \( B_1 \rightarrow 2 \), \( B_2 \rightarrow 2 \), \( A_1 \rightarrow 2 \), and \( P_1 \rightarrow P_{SZ} \). This is the same limit as in case (i). When \( N = 6 \), \( \gamma \rightarrow 2 \) in the limit of \( A_0 \rightarrow 0 \) and all \( A_\mu \) approaching 2.

Finally, we show that \( P_{AX} \) is optically invariant. Pure states are reached only by the state satisfying the constraints

\[
\begin{align*}
\langle \hat{S}_z \rangle &= (N/2 - 1)^2, \\
\langle \hat{S}_x \rangle &= N/2 - 1, \\
\langle \hat{S}_y \rangle &= (3N/2 - 2)/4, \\
\langle \hat{S}_z \rangle &= 0, \\
\end{align*}
\]

if the \( z \) axis is suitably chosen. Such a state exists — it is the eigenstate of \( \hat{S}_z \) with eigenvalue \( N/2 - 1 \) with total spin \( N/2 \). This state is the equally weighted in-phase superposition of any one qubit that is in the state \( |0 \rangle \) and the other \( N - 1 \) qubits in the state \( |1 \rangle \). This is a permutation-invariant pure state that highly (in the order of \( N \)) breaks the symmetry between the basis states \( |0 \rangle \) and \( |1 \rangle \).

It is worth noting that the optimal state for our problem was found to be a pure state. This is not trivial because the symmetry is required for density operators. Obviously, possible mixed states span a larger Hilbert space than that spanned by possible pure states. For infinite entangled chains, the best state so far is given as a mixed state \([17]\). It will be interesting to investigate entangled chains (loops) composed of a finite number of qubits and to see whether the optimum value is attained by mixed states or pure states (or by both). O’Conner and Wootters have found \([18]\) a series of pure states in \( N \)-qubit finite entangled loops and showed that the limiting value of the nearest-neighbor concurrence in \( N \rightarrow \infty \) is the same as the one obtained for the infinite chain in \([17]\). It is still an open question whether these states are optimum or not.

Note added. Recently we became aware that Dürr \([19]\) has independently conjectured the upper bound on the concurrence in entangled webs to be \( 2/N \).

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\[\text{References} \]