Optimal manipulations with qubits: Universal quantum entanglers

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(Received 21 January 2000; published 12 July 2000)

We analyze various scenarios for entangling two initially unentangled qubits. In particular, we propose an optimal universal entangler that entangles a qubit in unknown state $|\Psi\rangle$ with a qubit in a reference (known) state $|0\rangle$. That is, our entangler generates the output state that is as close as possible to the pure (symmetrized) state $(|\Psi\rangle|0\rangle+|0\rangle|\Psi\rangle)$. The most attractive feature of this entangling machine, is that the fidelity of its performance (i.e., the distance between the output and the ideally entangled—symmetrized state) does not depend on the input and takes the constant value $\mathcal{F}=(9+3\sqrt{2})/14\simeq 0.946$. We also analyze how to optimally generate from a single qubit initially prepared in an unknown state $|\Psi\rangle$ a two qubit entangled system, which is as close as possible to a Bell state $(|\Psi\rangle|\Psi^{\perp}\rangle+|\Psi^{\perp}\rangle|\Psi\rangle)$, where $\langle\Psi|\Psi^{\perp}\rangle=0$.

PACS number(s): 03.67.−a, 03.65.Bz

I. INTRODUCTION

A pure quantum state of two systems A and B is said to be entangled if it is not a product of a state for A and a state for B. Two systems in an entangled state are correlated, and these correlations are intrinsically quantum mechanical [1]. For example, one must use entangled states in order to produce violations of Bell inequalities or in the test of local realism proposed by Hardy [2,3]. Entangled states also play a key role in quantum information, in particular they are essential in quantum teleportation [4] and in superdense coding [5]. In quantum computers entanglement is one of the features of quantum mechanics that give these machines their power [6].

Here we would like to consider the problem of how to produce entanglement. In particular, if we are given particles, or systems, A and B in the pure states $|\Psi\rangle_A$ and $|\Phi\rangle_B$, respectively, we would like to produce the state $(|\Psi\rangle_A|\Phi\rangle_B+|\Phi\rangle_A|\Psi\rangle_B)$ (up to normalization). Formally, we are looking for the symmetrization map

$$\mathcal{S}:|\Psi\rangle|\Phi\rangle\rightarrow(|\Psi\rangle|\Phi\rangle+|\Phi\rangle|\Psi\rangle). \tag{1.1}$$

In what follows, where possible we omit explicit subscripts A and B. The order in which the vectors are written in the tensor products implicitly denotes to which system they belong (i.e., the left vector corresponds to the system A, while the right vector corresponds to the system B). We assume that the two quantum systems (e.g., qubits) are physically distinguishable. For instance they could be located in different regions of space. The task is to entangle their internal degrees of freedom.

That the symmetrization cannot be done perfectly via a unitary transformation can be shown by the following argument. We consider the case in which $|\Psi\rangle$ and $|\Phi\rangle$ are both qubits. A perfect transformation would have to transform the basis vectors as

$$|00\rangle|v_0\rangle\rightarrow|00\rangle|v_1\rangle,$$

$$|01\rangle|v_0\rangle\rightarrow\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)|v_2\rangle,$$

$$|10\rangle|v_0\rangle\rightarrow\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)|v_3\rangle,$$

$$|11\rangle|v_0\rangle\rightarrow|11\rangle|v_4\rangle, \tag{1.2}$$

where the $|v_j\rangle$, for $j=0,4$, are normalized ''machine'' vectors, i.e., we assume that the entangler itself has its own degrees of freedom. In addition, it is assumed that the entangler is always initially in the same state, $|v_0\rangle$. Unitarity requires that $\langle v_2|v_3\rangle=0$. Now let us consider the case where the input vectors are $|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$ and $|\Phi\rangle=|0\rangle$ (i.e., the state of the qubit A is unknown, while the qubit B is in a known state). The transformation (1.2) gives us

$$|\Psi\rangle|0\rangle\rightarrow\alpha|00\rangle|v_1\rangle+\frac{\beta}{\sqrt{2}}(|01\rangle+|10\rangle)|v_3\rangle, \tag{1.3}$$

whereas what it should produce is a vector proportional to $|\Psi\rangle|0\rangle+|0\rangle|\Psi\rangle$, which in the basis $|0\rangle$, $|1\rangle$ reads

$$|\Psi\rangle|0\rangle\rightarrow|\Psi\rangle|0\rangle+|0\rangle|\Psi\rangle=2\alpha|00\rangle+\beta(|01\rangle+|10\rangle). \tag{1.4}$$

The vectors in the right-hand sides of Eqs. (1.3) and (1.4) are clearly not the same, no matter what choice is made for $|v_1\rangle$ and $|v_3\rangle$. Therefore, we need to search for devices that will produce approximate versions of the desired state or will produce this state but with a probability that is less than one.

One way of creating a symmetrized state out of two independent systems is by means of a measurement—that is, the two systems are optimally measured and their states are estimated. Based on this estimation a two-particle entangled state is prepared. If we begin with two qubits prepared so that one of the states is known ($|0\rangle$) while the other is unknown ($|\Psi\rangle$), we need only estimate the state of one of the particles and this can be performed with a fidelity equal to

2/3 [7,8]. The information gained from the optimal measurement is then used in the preparation procedure. This is discussed in Sec. II A.

We shall present quantum-mechanical entangling transformations that generate entangled states with much higher fidelity than can be achieved by measuring the input particles. In Sec. II B we briefly discuss a *probabilistic* symmetrization (entanglement) which can be realized via a controlled-SWAP gate. The probability of success in this procedure is input-state dependent. In Sec. III we present the optimal input-state independent quantum entangler and we also study the inseparability of the outputs of this entangler. In Sec. IV we show that the universal-NOT gate [9] can also serve as a very interesting entangling device.

II. STATE-DEPENDENT SYMMETRIZATION

We shall first look at two examples of processes that produce entangled states, for which the quality of the output depends on the input state. That is, these procedures work better for some states than for others. The first is perhaps the most obvious method, we simply measure the input state. We shall consider a more limited problem in this case, entangling an unknown with a known state. The output state resulting from this procedure is only an approximation to the desired one. The second is a probabilistic method; the output when it is produced is ideal, but the probability of successfully producing it is less than one. In this case we shall consider the full problem of entangling two unknown states.

A. Entanglement via measurement

Our task is to entangle an input qubit in an unknown state with a reference qubit in a known state $|0\rangle$. That is, we want to realize the symmetrization map $|0\rangle_A|\Psi\rangle_B \to |\Psi^{(id)}\rangle_{AB}$ with the output parameterized as

$$|\Psi^{(id)}\rangle_{AB} = \frac{2\cos\frac{\vartheta}{2}|00\rangle + \sqrt{2}e^{i\varphi}\sin\frac{\vartheta}{2}|+\rangle}{\sqrt{2(1+\cos^2\vartheta/2)}}. \tag{2.1}$$

The approach we will discuss here is as follows: first, the unknown single-qubit state $|\Psi\rangle$ is measured, and then using the information gained thereby, an approximate version of the desired output is constructed. In order to specify this procedure in more detail, we must describe what measurement is to be made and how its results will be used to construct the output state. The quality of the output will be determined by calculating the fidelity between the actual output and the desired output. We shall first examine a specific strategy and then find an upper bound on the fidelity for a wide class of measurement-based procedures.

Our first measurement-based scenario can then be realized in the following way. In the case of a single input qubit the *optimal* way to estimate the state, is to measure it along a randomly chosen direction in the two-dimensional Hilbert space [7,8]. Therefore, the first step in implementing the measurement-based procedure is choosing a random vector $|\eta\rangle$, where

$$|\eta\rangle = \cos\frac{\vartheta'}{2}|0\rangle + e^{i\varphi'}\sin\frac{\vartheta'}{2}|1\rangle, \tag{2.2}$$

and measuring $|\Psi\rangle$ along it. If the result is positive, then the output is taken to be $|\Phi\rangle_{AB}$, and if negative, the output is $|\tilde{\Phi}\rangle_{AB}$, where

$$|\Phi\rangle_{AB} = \frac{|0\rangle|\eta\rangle + |\eta\rangle|0\rangle}{\sqrt{2[1+\cos^2(\vartheta'/2)]}} = \frac{2\cos\frac{\vartheta'}{2}|00\rangle + \sqrt{2}e^{i\varphi'}\sin\frac{\vartheta'}{2}|+\rangle}{\sqrt{2[1+\cos^2(\vartheta'/2)]}} \tag{2.3}$$

and

$$|\tilde{\Phi}\rangle_{AB} = \frac{|0\rangle|\eta^\perp\rangle + |\eta^\perp\rangle|0\rangle}{\sqrt{2[1+\sin^2(\vartheta'/2)]}} = \frac{2e^{-i\varphi'}\sin\frac{\vartheta'}{2}|00\rangle - \sqrt{2}\cos\frac{\vartheta'}{2}|+\rangle}{\sqrt{2[1+\sin^2(\vartheta'/2)]}}, \tag{2.4}$$

where the state $|\eta^\perp\rangle$ is the state orthogonal to $|\eta\rangle$,

$$|\eta^\perp\rangle = e^{-i\varphi'}\sin\frac{\vartheta'}{2}|0\rangle - \cos\frac{\vartheta'}{2}|1\rangle. \tag{2.5}$$

For a particular orientation of the measurement apparatus, i.e., for the particular choice of the state $|\eta\rangle$, this measurement-based scenario gives the two-qubit output density matrix

$$\rho^{(out)}(\vartheta,\varphi|\vartheta',\varphi') = |\langle\Psi|\eta\rangle|^2|\Phi\rangle\langle\Phi| + |\langle\Psi|\eta^\perp\rangle|^2|\tilde{\Phi}\rangle\langle\tilde{\Phi}|. \tag{2.6}$$

To get the final output density matrix one averages this over all possible choices of the measurement (i.e., over all vectors $|\eta\rangle$)

$$\rho^{(out)}(\vartheta,\varphi) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\vartheta' \sin\vartheta' \times \rho^{(out)}(\vartheta,\varphi|\vartheta',\varphi'). \tag{2.7}$$

Finally, the fidelity can be found by computing the matrix element of this density matrix in the ideal output state $|\Psi^{(id)}\rangle_{AB}$,

$$\mathcal{F}(\vartheta,\varphi) = \langle\Psi^{(id)}|\rho^{(out)}(\vartheta,\varphi)|\Psi^{(id)}\rangle. \tag{2.8}$$

This fidelity depends on the input state, and this dependence can be eliminated if we average over all input states

$$\overline{\mathcal{F}} = \int d\Omega\, \mathcal{F}(\vartheta,\varphi). \tag{2.9}$$

This is the proper fidelity to use to judge how well our proposed strategy performs if we assume that all input states are equally probable. A more explicit expression for it is

$$\overline{\mathcal{F}}=\frac{1}{16\pi^2}\int_0^{2\pi}d\varphi\int_0^{2\pi}d\varphi'\int_0^{\pi}\sin\vartheta d\vartheta \times\int_0^{\pi}\sin\vartheta' d\vartheta'[|\langle\eta|\Psi\rangle|^2|\langle\Psi|\Phi\rangle|^2 +|\langle\eta^{\perp}|\Psi\rangle|^2|\langle\Psi|\tilde{\Phi}\rangle|^2]. \tag{2.10}$$

Explicitly evaluating this integral we find

$$\overline{\mathcal{F}}=54+112(\ln 2)^2-154.5\ln 2\simeq 0.719, \tag{2.11}$$

which is a bit larger than 2/3, the fidelity of the estimation of a state of a single qubit.

Let us now generalize this procedure. We shall again begin by choosing a random vector $|\eta\rangle$, but now according to a distribution $q(\vartheta',\varphi')$, which we shall leave unspecified for now. The output density matrix is taken to be either $\rho_1(\eta)$ if the measurement result is positive or $\rho_0(\eta)$ if it is negative, where

$$\rho_j(\eta)=\int d\Omega'' p_j(\vartheta'',\varphi''|\vartheta',\varphi')|\Gamma(\vartheta'',\varphi'')\rangle\langle\Gamma(\vartheta'',\varphi'')|, \tag{2.12}$$

with $j=0,1$, and

$$|\Gamma(\vartheta'',\varphi'')\rangle_{AB}=\cos\frac{\vartheta''}{2}|00\rangle_{AB}+e^{i\phi''}\sin\frac{\vartheta''}{2}|+\rangle_{AB}. \tag{2.13}$$

The conditional probabilities p_j will also be left unspecified; this allows us to consider a wide class of measurement-based strategies. The output density matrix, for a particular $|\eta\rangle$ is then

$$\rho(\eta)=|\langle\eta|\Psi\rangle|^2\rho_1(\eta)+|\langle\eta^{\perp}|\Psi\rangle|^2\rho_0(\eta). \tag{2.14}$$

Averaging over $|\eta\rangle$ gives us the final output density matrix

$$\rho^{(out)}(\vartheta,\varphi)=\int d\Omega'\rho(\eta)q(\vartheta',\varphi'), \tag{2.15}$$

and the fidelities for a specific input state and averaged over all input states are given by Eqs. (2.8) and (2.9), respectively, but with $\rho^{(out)}$ computed from Eq. (2.15) instead of Eq. (2.7). In particular we have that

$$\overline{\mathcal{F}}=\int\int d\Omega' d\Omega''\sum_{j=0}^{1}f_j(\vartheta'',\varphi'';\vartheta',\varphi')P_j(\vartheta'',\varphi'';\vartheta',\varphi'), \tag{2.16}$$

where

$$P_j(\vartheta'',\varphi'';\vartheta',\varphi')=p_j(\vartheta'',\varphi''|\vartheta',\varphi')q(\vartheta',\varphi') \tag{2.17}$$

for $j=0,1$, and

$$f_0=\int d\Omega\frac{1}{2[1+\cos^2(\vartheta/2)]}|2\cos\frac{\vartheta}{2}\cos\frac{\vartheta''}{2} +\sqrt{2}e^{i(\varphi''-\varphi)}\sin\frac{\vartheta}{2}\sin\frac{\vartheta''}{2}|^2|\langle\Psi|\eta^{\perp}\rangle|^2,$$

$$f_1=\int d\Omega\frac{1}{2[1+\cos^2(\vartheta/2)]}|2\cos\frac{\vartheta}{2}\cos\frac{\vartheta''}{2} +\sqrt{2}e^{i(\varphi''-\varphi)}\sin\frac{\vartheta}{2}\sin\frac{\vartheta''}{2}|^2|\langle\Psi|\eta\rangle|^2. \tag{2.18}$$

What we can now do is to find an upper bound for the fidelity $\overline{\mathcal{F}}$ for any distribution of the vector $|\eta\rangle$ and any prescription for using the result of the measurement along $|\eta\rangle$ to manufacture the entangled state. We note that for $j=0,1$:

$$1=\int d\Omega'\int d\Omega'' P_j(\vartheta'',\varphi'';\vartheta',\varphi'), \tag{2.19}$$

which implies that

$$\overline{\mathcal{F}}\leqslant\sup|f_0|+\sup|f_1|, \tag{2.20}$$

where the supremums are taken over the range $0\leqslant\vartheta',\vartheta''\leqslant\pi$ and $0\leqslant\varphi',\varphi''<2\pi$.

Our first task is to find explicit expressions for the functions f_0 and f_1. We have that

$$f_0=d_1\cos^2\frac{\vartheta''}{2}\sin^2\frac{\vartheta'}{2}+d_2\cos^2\frac{\vartheta''}{2}\cos^2\frac{\vartheta'}{2} +\frac{1}{2}d_2\sin^2\frac{\vartheta''}{2}\sin^2\frac{\vartheta'}{2}+\frac{1}{2}d_3\sin^2\frac{\vartheta''}{2}\cos^2\frac{\vartheta'}{2} -\sqrt{2}d_2\cos(\varphi''-\varphi')\cos\frac{\vartheta''}{2}\cos\frac{\vartheta'}{2}\sin\frac{\vartheta''}{2}\sin\frac{\vartheta''}{2},$$

$$f_1=d_1\cos^2\frac{\vartheta''}{2}\cos^2\frac{\vartheta'}{2}+d_2\cos^2\frac{\vartheta''}{2}\sin^2\frac{\vartheta'}{2} +\frac{1}{2}d_2\sin^2\frac{\vartheta''}{2}\cos^2\frac{\vartheta'}{2}+\frac{1}{2}d_3\sin^2\frac{\vartheta''}{2}\sin^2\frac{\vartheta'}{2} +\sqrt{2}d_2\cos(\varphi''-\varphi')\cos\frac{\vartheta''}{2}\cos\frac{\vartheta'}{2}\sin\frac{\vartheta''}{2}\sin\frac{\vartheta''}{2}, \tag{2.21}$$

where

$$d_1=2\ln 2-1,\quad d_2=3-4\ln 2,\quad d_3=8\ln 2-5. \tag{2.22}$$

From the above equations it is clear that in order to maximize f_0 we need to choose $\varphi''-\varphi'=\pi$ and to maximize f_1 we need to choose $\varphi''-\varphi'=0$. Making these choices and simplifying the resulting expressions we find that

$$f_0(\vartheta'',\pi;\vartheta',0)=\tfrac{1}{4}[1+c_1\cos\vartheta''-c_2\cos\vartheta''\cos\vartheta' +c_3\sin\vartheta''\sin\vartheta'],$$

$$f_1(\vartheta'',0;\vartheta',0)=\tfrac{1}{4}[1+c_1\cos\vartheta''+c_2\cos\vartheta''\cos\vartheta' +c_3\sin\vartheta''\sin\vartheta'], \tag{2.23}$$

where

$$c_1=3-4\ln 2,$$
$$c_2=12\ln 2-8, \tag{2.24}$$
$$c_3=\sqrt{2}(3-4\ln 2).$$

These functions can now be maximized. The maximum of f_0 occurs at $\vartheta'=\pi$ and $\vartheta''=0$, and the maximum of f_1 occurs when $\vartheta'=0$ and $\vartheta''=0$. The maximum values of both functions are the same and are approximately equal to 0.386. This implies that the fidelity for this kind of a measurement-based strategy must satisfy

$$\overline{\mathcal{F}}\leqslant 4\ln 2-2\cong 0.773. \tag{2.25}$$

As we shall see, a method that maintains quantum coherences at all stages of the process can do better than this.

B. Controlled-SWAP gate

We now begin with systems A and B of the same physical origin. Their pure states are described by vectors in the D-dimensional Hilbert space $\mathcal{H}$, so that both together are described by $\mathcal{H}\otimes\mathcal{H}$. Let $\{|u_j\rangle|j=1,\ldots,D\}$ be an orthonormal basis for $\mathcal{H}$. System A is in the state

$$|\Psi\rangle_A=\sum_{j=1}^{D} c_j|u_j\rangle_A, \tag{2.26}$$

and system B is in the state

$$|\Phi\rangle_B=\sum_{j=1}^{D} d_j|u_j\rangle_B. \tag{2.27}$$

Our objective is to produce the (entangled) symmetrized state [see Eq. (1.1)]

$$|\Psi\rangle|\Phi\rangle+|\Phi\rangle|\Psi\rangle=\sum_{j=1}^{D}\sum_{k=1}^{D}(c_jd_k+c_kd_j)|u_j\rangle|u_k\rangle \tag{2.28}$$

(here we omit the normalization factor).

Recently Barenco *et al.* [10] have shown that the entanglement (symmetrization) of the form (1.1) can be performed when the two input qubits interact via a controlled-SWAP (Fredkin) gate with an ancilla initially prepared in a specific state. The entanglement is achieved when a conditional measurement is performed on the ancilla. Exactly the same scenario can be used not only for qubits but for arbitrary quantum systems. To show this we briefly review the operation of the controlled-SWAP gate.

This gate has three inputs. The first, the control bit, is a qubit. The second and third are for D-dimensional systems. The control bit is unaffected by the action of the gate. If the control bit is $|0\rangle$, then the gate does nothing, i.e., the output state is the same as the input state. If the control bit is $|1\rangle$, then the two D-dimensional states are swapped. This can be accomplished by the following explicit unitary transformation:

$$|0\rangle|u_j\rangle|u_k\rangle\rightarrow|0\rangle|u_j\rangle|u_k\rangle,$$
$$|1\rangle|u_j\rangle|u_k\rangle\rightarrow|1\rangle|u_k\rangle|u_j\rangle. \tag{2.29}$$

Summarizing, the action of our controlled-SWAP gate is,

$$|0\rangle|\Psi\rangle|\Phi\rangle\rightarrow|0\rangle|\Psi\rangle|\Phi\rangle,$$
$$|1\rangle|\Psi\rangle|\Phi\rangle\rightarrow|1\rangle|\Phi\rangle|\Psi\rangle. \tag{2.30}$$

We now define the qubit states

$$|v_+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\quad |v_-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle), \tag{2.31}$$

and take the input state of the controlled-SWAP gate to be $|v_+\rangle|\Psi\rangle_A|\Phi\rangle_B$. Using the SWAP transformation (2.30) we find that the output state is

$$|\Psi^{(out)}\rangle=\frac{1}{\sqrt{2}}(|0\rangle|\Psi\rangle|\Phi\rangle+|1\rangle|\Phi\rangle|\Psi\rangle)$$
$$=\frac{1}{2}|v_+\rangle(|\Psi\rangle|\Phi\rangle+|\Phi\rangle|\Psi\rangle)$$
$$+\frac{1}{2}|v_-\rangle(|\Psi\rangle|\Phi\rangle-|\Phi\rangle|\Psi\rangle). \tag{2.32}$$

If we now measure the qubit in the $|v_\pm\rangle$ basis we obtain the states $(|\Psi\rangle|\Phi\rangle\pm|\Phi\rangle|\Psi\rangle)$ with probabilities $(1\pm|\langle\Psi|\Phi\rangle|^2)/2$, respectively. As we see the probability of generation of a particular entangled state explicitly depends on the (unknown) states of the two systems. In particular, let us assume we begin with two orthogonal qubits, $|\Psi\rangle$ and $|\Psi^\perp\rangle$. Then either of the maximally entangled states $(|\Psi\rangle|\Psi^\perp\rangle\pm|\Psi^\perp\rangle|\Psi\rangle)/\sqrt{2}$ can be prepared with probability 1/2.

We stress that the probability of the success in this entanglement (symmetrization) procedure is input-state dependent. In what follows our task will be to find a ''machine'' that entangles the input with a *constant* (i.e., input-state independent) fidelity. This covariance property of the entangler with respect to unitary transformations performed on the input qubits makes the entangler universal.

III. UNIVERSAL ENTANGLERS

Suppose we again consider the problem of constructing a device that will entangle a qubit in an arbitrary unknown state $|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$ with a qubit in a known, reference state, which we shall take to be the basis state $|0\rangle$. Before we proceed further we have to specify properties of the entangling map. In fact, we can consider two maps. The symmetrization map

$$\mathcal{S}:|0\rangle_A|\Psi\rangle_B\rightarrow|\Psi^{(id)}\rangle_{AB}=N_s(|\Psi\rangle|0\rangle+|0\rangle|\Psi\rangle), \quad (3.1)$$

and the antisymmetrization map

$$\mathcal{A}:|0\rangle_A|\Psi\rangle_B\rightarrow|\overline{\Psi}^{(id)}\rangle_{AB}=N_a(|\Psi\rangle|0\rangle-|0\rangle|\Psi\rangle), \quad (3.2)$$

where $N_{a,s}$ are corresponding normalization factors. As we have shown in the introduction, perfect entanglers for arbitrary unknown states cannot be constructed. So the task of the physically realizable symmetric (antisymmetric) entangler is to produce outputs as close as possible to the ideally entangled states $|\Psi^{(id)}\rangle_{AB}$ ($|\overline{\Psi}^{(id)}\rangle_{AB}$). In what follows we will quantify the quality of the performance of the universal entangler with the help of the fidelity

$$\mathcal{F}:=\langle\Psi^{(id)}|\rho^{(out)}|\Psi^{(id)}\rangle. \quad (3.3)$$

We shall impose the condition that the value of this fidelity does not depend on the input. The fidelity (3.3) is a good measure of the accuracy with which the entangler produces the desired output state, but we would also like to evaluate the degree of entanglement of the actual output state. Here, however, we have a problem that is due to the fact that it is still not clear how to quantify the entanglement of a quantum system which is in a mixed state. When a bipartite system is in a pure state, then the von Neumann entropy of subsystems can serve as a measure of entanglement. In the case of impure states more sophisticated measures are required (see, for instance, Refs. [11–13]).

In terms of the basis vectors, the input state is $\alpha|00\rangle+\beta|01\rangle$, and the ideal output state in the case of symmetrization is

$$|\Psi^{(id)}\rangle=\frac{(2\alpha|00\rangle+\sqrt{2}\beta|+\rangle)}{(4|\alpha|^2+2|\beta|^2)^{1/2}}, \quad (3.4)$$

while in the case of the antisymmetrization we have

$$|\overline{\Psi}^{(id)}\rangle=|-\rangle, \quad (3.5)$$

where $|\pm\rangle$ are symmetric and antisymmetric Bell states in the given basis

$$|\pm\rangle=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle). \quad (3.6)$$

In what follows we will briefly discuss the antisymmetric entangler and then we will concentrate on the symmetric entangler.

A. Entanglement via antisymmetrization

Recently Alber [14] studied a quantum entangler that takes as an input a quantum-mechanical system prepared in an *unknown* pure state $|\Psi\rangle_A$ and a reference (known) state (let us say $|0\rangle_A$) and at the output generates a two-particle entangled state $\rho_{AB}^{(out)}$ that is optimally entangled. Alber imposed two constraints on the output of the universal quantum entangler

$$\mathrm{Tr}_A[\rho_{AB}^{(out)}]=\mathrm{Tr}_B[\rho_{AB}^{(out)}]=\frac{\mathbb{1}}{D} \quad (3.7)$$

and

$$S[\rho_{AB}^{(out)}]\rightarrow\text{minimum}. \quad (3.8)$$

Where D is the dimensionality of the Hilbert space of the system A (B) and S is the von Neumann entropy $S=-\mathrm{Tr}\,\rho\ln\rho$ associated with a given density operator ρ. The first condition corresponds to the requirement that the subsystems at the output are in the maximally mixed state while the second condition guarantees that the whole system is as close as possible to a pure two-particle state. Alber has found the solution for this problem. It turns out that the two-particle state that is produced by the optimal (with respect to the above conditions), universal entangler is *independent* of the input state $|\Psi\rangle$ and is equal to a maximally disordered mixture of all possible antisymmetric Bell states. In the case of qubits ($D=2$) there is only one possible antisymmetric Bell state $|-\rangle$. That is, Alber's machine realizes the antisymmetric entangler. We see that the universality of Alber's entangler means that all inputs are mapped to a single output (the antisymmetric Bell state $|-\rangle$), so the ideal output state is *a priori* known, and one could instead build a device that just prepares the known output state. In the antisymmetric entangler the information initially encoded in the qubit A is completely lost. But our task is different, we want to redistribute the initial *unknown* information encoded in the state of the qubit A, into the entangled state of two qubits. Therefore we will analyze universal entanglement via symmetrization, because the ideal state (3.4) directly contains information about the initial state of the qubit A. In other words, we consider the entangling procedure not only as the way to generate the state with highest possible entanglement but also we require that this state contains as much information about the input(s) as possible.

B. Entanglement via symmetrization

Let us now construct a machine that entangles an unknown state with the known state $|0\rangle$. Taking into account the basic features of the symmetrization transformation (3.1) we can assume that the basis vectors transform as

$$|00\rangle|v_0\rangle\rightarrow|00\rangle|w_0\rangle+|+\rangle|x_0\rangle,$$

$$|01\rangle|v_0\rangle\rightarrow|00\rangle|w_1\rangle+|+\rangle|x_1\rangle, \quad (3.9)$$

where $|w_0\rangle$, $|w_1\rangle$, $|x_0\rangle$, and $|x_1\rangle$ are states of the entangler itself. The entangler is initially always prepared in the state $|v_0\rangle$.

We want to impose the condition that the fidelity between the actual output state and the ideal output state be *independent* of the state $|\Psi\rangle$, but before doing so let us state the restrictions that unitarity places on the machine vectors. These are

$$\|w_0\|^2+\|x_0\|^2=1,$$
$$\|w_1\|^2+\|x_1\|^2=1, \tag{3.10}$$
$$\langle w_0|w_1\rangle+\langle x_0|x_1\rangle=0,$$

where $\|x\|^2\equiv\langle x|x\rangle$. We now calculate the output two-qubit density matrix $\rho^{(out)}$ by using the transformation in Eq. (3.9) to find the full output density matrix and then tracing out the machine degrees of freedom. We then find the fidelity (3.3) by taking the matrix element of this density matrix in the ideal output state. Our task is to find the machine vectors $|x_j\rangle$ and $|w_j\rangle$ $(j=0,1)$ such that the fidelity $\mathcal{F}$ does not depend on the input state $|\Psi\rangle$ and simultaneously is as close as possible to unity.

We find that if we choose $|x_0\rangle$ to be orthogonal to each of the other machine vectors and $|w_1\rangle$ to be orthogonal to $|x_0\rangle$ and $|w_0\rangle$, then the output fidelity will be independent of the phases of α and β. Making these choices we find that

$$\mathcal{F}=N^{-1}\{2|\alpha|^4\|w_0\|^2+|\beta|^4\|x_1\|^2+|\alpha|^2|\beta|^2[\sqrt{2}(\langle w_0|x_1\rangle+\langle x_1|w_0\rangle)+2\|w_1\|^2+\|x_0\|^2]\}, \tag{3.11}$$

where $N=2|\alpha|^2+|\beta|^2$.

In order for this expression to be independent of $|\alpha|$ and $|\beta|$ it is necessary that the expression in the curly brackets be proportional to

$$(2|\alpha|^2+|\beta|^2)(|\alpha|^2+|\beta|^2)=2|\alpha|^4+3|\alpha|^2|\beta|^2+|\beta|^4. \tag{3.12}$$

Comparing this expression to Eq. (3.11) we see that

$$\|w_0\|=\|x_1\|, \tag{3.13}$$
$$3\|w_0\|^2=\sqrt{2}(\langle x_1|w_0\rangle+\langle w_0|x_1\rangle)+2\|w_1\|^2+\|x_0\|^2.$$

If these conditions are satisfied, then the fidelity is simply equal to $\|w_0\|^2$, so that we want to make this quantity as large as possible. If we now make use of the unitarity conditions and the two equations above, we find that

$$1-\frac{2}{3}\sqrt{2}\cos\mu=\frac{1-\|w_0\|^2}{\|w_0\|^2}, \tag{3.14}$$

where

$$\cos\mu=\frac{\langle x_1|w_0\rangle+\langle w_0|x_1\rangle}{2\|w_0\|^2}. \tag{3.15}$$

From Eq. (3.14) we see that $\|w_0\|^2$ will be a maximum when $\cos\mu=1$, which implies that $|w_0\rangle$ and $|x_1\rangle$ are parallel. When this condition is satisfied, we find that

$$\mathcal{F}=\|w_0\|^2=\frac{9+3\sqrt{2}}{14}, \tag{3.16}$$

which gives 0.946 as the approximate value of the fidelity. This means that the output state $\rho^{(out)}$ is indeed very close to the ideal state, and it should be remembered that this fidelity is the same for all input states.

We can summarize our results for the machine vectors as follows. From the above analysis we see that we can take the machine state space to be *three* dimensional. Define

$$\cos\theta=\left[\frac{9+3\sqrt{2}}{14}\right]^{1/2},\quad \sin\theta=\left[\frac{5-3\sqrt{2}}{14}\right]^{1/2}, \tag{3.17}$$

and let $\{|v_j\rangle|j=1,\ldots,3\}$ be an orthonormal basis for the machine vector space. We then have

$$|w_0\rangle=\cos\theta|v_1\rangle,$$
$$|w_1\rangle=\sin\theta|v_2\rangle,$$
$$|x_0\rangle=\sin\theta|v_3\rangle, \tag{3.18}$$
$$|x_1\rangle=\cos\theta|v_1\rangle,$$

and our transformation in terms of basis vectors becomes

$$|00\rangle|v_0\rangle\to\cos\theta|00\rangle|v_1\rangle+\sin\theta|+\rangle|v_3\rangle,$$
$$|01\rangle|v_0\rangle\to\sin\theta|00\rangle|v_2\rangle+\cos\theta|+\rangle|v_1\rangle. \tag{3.19}$$

By construction this is the *optimal* entangling transformation that entangles an unknown pure state with a known reference state.

Alternatively, for $|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$ we can rewrite this transformation in the form

$$|0\rangle|\Psi\rangle|v_0\rangle\to\cos\theta(\alpha|00\rangle+\beta|+\rangle)|v_1\rangle+\sin\theta(\alpha|+\rangle|v_3\rangle+\beta|00\rangle|v_2\rangle). \tag{3.20}$$

When the trace over the entangler is performed we obtain the density operator $\rho_{AB}^{(out)}$ describing the two qubits A and B at the output of the quantum entangler

$$\rho_{AB}^{(out)}=(|\alpha|^2\cos^2\theta+|\beta|^2\sin^2\theta)|00\rangle\langle 00|+(|\alpha|^2\sin^2\theta+|\beta|^2\cos^2\theta)|+\rangle\langle +|+\cos^2\theta(\alpha\beta^*|00\rangle\langle +|+\alpha^*\beta|+\rangle\langle 00|). \tag{3.21}$$

It is important to stress that the fidelity (3.3) associated with the output state (3.20) is input-state independent.

C. Remarks

Throughout this paper we have utilized the fidelity (3.3) as the measure of the performance of the quantum entangler. The universality (covariance) of the entangler is expressed in the fact that the value of the fidelity $\mathcal{F}$ is equal for all input states. We note that this covariance constraint is equivalent to the requirement that the Bures distance [15] defined as

$$d_B(\rho_1,\rho_2)=\sqrt{2}(1-\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}})^{1/2}, \qquad (3.22)$$

between the ideal state $|\Psi^{(id)}\rangle$ and the output of the entangler $\rho_{AB}^{(out)}$ is constant. In our particular case we find the Bures distance to be

$$d_B=2\sin(\theta/2)\simeq 0.0541 \qquad (3.23)$$

for all inputs. This distance is very small indeed. It is important to note that the Hilbert-Schmidt norm

$$d_{HS}(\rho_1,\rho_2)=[\mathrm{Tr}(\rho_1-\rho_2)^2]^{1/2}, \qquad (3.24)$$

which in our case can be expressed as

$$d_{HS}=[1-2\mathcal{F}+\mathrm{Tr}(\rho_{AB}^{(out)})^2]^{1/2}, \qquad (3.25)$$

is *not* input-state independent because $\mathrm{Tr}(\rho_{AB}^{(out)})^2$ depends on the initial state. This is closely related to the fact that the von Neumann entropy of the state $\rho_{AB}^{(out)}$ is state dependent (see below).

D. Inseparability of the output qubits

We note that the entanglement between the two qubits prepared in the state $|\Psi^{(id)}\rangle$ depends on the particular form of the state $|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$. Because $|\Psi^{(id)}\rangle$ is a pure state we can quantify the degree of entanglement via the von Neumann entropy S of one of the two qubits under consideration, i.e., $S_A=-\mathrm{Tr}[\rho_A\ln\rho_A]$ (obviously $S_A=S_B$). For $\alpha=1$ the entropy is equal to zero, which corresponds to a completely disentangled state (we note that in this case $|\Psi^{(id)}\rangle=|0\rangle|0\rangle$). The entropy takes the maximal value $S=\ln 2$ for $\alpha=0$ when $|\Psi^{(id)}\rangle=(|0\rangle|1\rangle+|1\rangle|0\rangle)/\sqrt{2}$. We plot this entropy in Fig. 1 (see line 1). The entropy of the individual particle (qubit) at the output of the entangler, i.e., $\rho_A^{(out)}=\mathrm{Tr}\rho_{AB}^{(out)}$ is always larger than in the ideal case (see line 2 in Fig. 1). Nevertheless, for the case $\alpha=0$, we have in this case $S(\alpha=0)=0.998\ln 2$, i.e., this entropy is very close to the entropy of a qubit in the ideal case. Unfortunately, this entropy in the case of an impure two-particle state cannot be used as a measure of entanglement.

It is interesting to find the entropy of the two-particle state $\rho_{AB}^{(out)}$ at the output of the entangler as a function of the initial state (in the ideal case the two-particle system is always considered to be in a pure state with $S=0$). We plot this entropy in Fig. 2. We see that the total entropy of the output is state-dependent and it takes the minimal value for $\alpha^2=1/2$. Therefore the entropy of the subsystems does not indicate whether they are entangled.

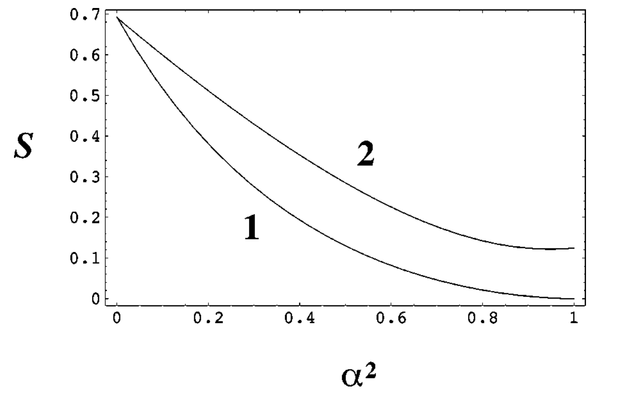

FIG. 1. The von Neumann entropy of the single-qubit state ρ_A when the two-qubit system is in an ideally entangled state $|\Psi^{(id)}\rangle$ (line 1) and when the output state $\rho_{AB}^{(out)}$ is given by Eq.(3.21) (line 2). In both cases we assume α and β to be real.

We need to check whether the two qubits A and B at the output are indeed quantum mechanically entangled. Quantum-mechanical entanglement of two qubits formally means that the density operator of these two qubits is represented by an inseparable matrix (see Ref. [1]). It follows from the Peres-Horodecki theorem that [16,17] the necessary and sufficient condition of inseparability of the two-qubit density matrix ρ_{AB}, is that the corresponding partially transposed matrix $\rho_{AB}^{T_2}$ has at least one negative eigenvalue.

For instance, let us consider the state $|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$ with real amplitudes α and β. The partially transposed matrix corresponding to the state $|\Psi^{(id)}\rangle$ given by Eq. (3.4) has one negative eigenvalue

$$E(\alpha)=\frac{\alpha^2-1}{2(\alpha^2+1)}. \qquad (3.26)$$

We plot this eigenvalue in Fig. 3 (see line 1). We see that the eigenvalue is negative for all values of α except $\alpha=1$ when $|\Psi^{(id)}\rangle=|0\rangle|0\rangle$. The minimal value of the eigenvalue is achieved for $\alpha=0$ when the two qubits are in the maximally entangled state $(|01\rangle+|10\rangle)/\sqrt{2}$.

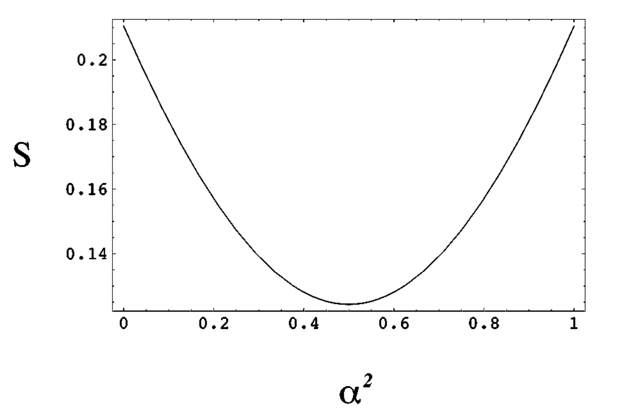

FIG. 2. The von Neumann entropy of the two-qubit state $\rho_{AB}^{(out)}$ at the output of the entangler [see Eq. (3.21)] as a function of α^2. We assume α and β to be real.

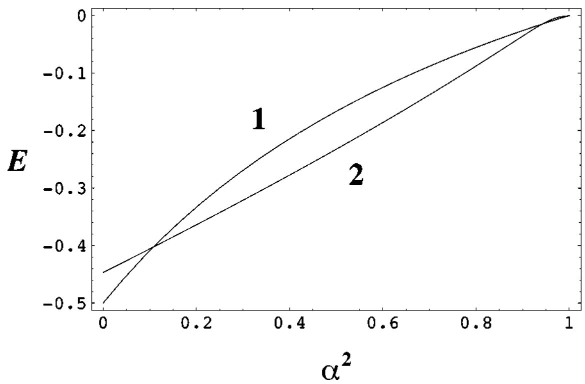

FIG. 3. Here we plot the negative eigenvalue [Eq. (3.26)] of the partially transposed matrix of the density operator $\rho_{AB}^{(ideal)}$ when the state $|\Psi\rangle$ has real amplitudes α and β (see line 1). The negative eigenvalue of the partially transposed matrix associated with the density operator $\rho_{AB}^{(out)}$ given by Eq. (3.21) as functions of α is presented by line 2. (We assume α and β to be real.)

Now we utilize the Peres-Horodecki theorem to check whether the state $\rho_{AB}^{(out)}$ given by Eq. (3.21) describes an entangled state of two qubits. First, we find that the partially transposed matrix corresponding to the density operator (3.21) has one eigenvalue that is negative for all values of α (here we assume α and β to be real). In particular, this eigenvalue for $\alpha=0$ is

$$E(\alpha=0)=\tfrac{1}{2}[\cos^2\theta-(\cos^4\theta+\sin^4\theta)^{1/2}], \quad (3.27)$$

which is the minimal value ($\simeq -0.447$) of the negative eigenvalue. On the other hand the maximal value ($\simeq -0.001$) is attained for $\alpha=1$,

$$E(\alpha=1)=\tfrac{1}{2}[\sin^2\theta-(\cos^4\theta+\sin^4\theta)^{1/2}]. \quad (3.28)$$

The complete dependence of $E(\alpha)$ is shown in Fig. 3. From this figure we clearly see that the output density operator is inseparable for an arbitrary input considered in this section. We note, that if the entanglement is measured in terms of the tangle as introduced by Hill and Wootters [13] then the negative eigenvalues E of the partially transposed density operators perfectly reflect the degree of entanglement between the two qubits in our cases.

By construction the fidelity of the entangler in this case is constant but the actual degree of entanglement is state dependent. This suggests that it would be interesting to find an entangler whose output states have the same degree of entanglement irrespective of the input, yet still carry information about the input.

IV. ENTANGLEMENT VIA UNIVERSAL NOT GATE

Even though the negative eigenvalue of the partially transposed density matrix cannot be directly used as the measure of entanglement, we see that the degree of entanglement between two qubits generated in the entangler (3.19) depends on the input state. In what follows we describe a different type of the entangler, which out of a single qubit $|\Psi\rangle$ generates a two-qubit state as close as possible to the state

$$|\Psi\rangle\rightarrow|\{\Psi,\Psi^{\perp}\}\rangle\equiv(|\Psi\rangle|\Psi^{\perp}\rangle+|\Psi^{\perp}\rangle|\Psi\rangle)/\sqrt{2}. \quad (4.1)$$

We will present an entangler that not only produces the state that is as close as possible to the ideal state $|\{\Psi,\Psi^{\perp}\}\rangle$ but also has the property that the fidelity does not depend on the input state. In addition, the degree of entanglement also does not depend on the input. This type of the entangler implicitly assumes creation of the state $|\Psi^{\perp}\rangle$ from the input $|\Psi\rangle$. That is, we face the problem of creating an orthogonal state from unknown input.

It is not a problem to complement a classical bit, i.e., to change the value of a bit, a 0 to a 1 and vice versa. This is accomplished by a NOT gate. Complementing a qubit, however, is another matter. The complement of a qubit $|\Psi\rangle$ is the qubit $|\Psi^{\perp}\rangle$ that is orthogonal to it. But it is not possible to build a device that will take an *arbitrary* (unknown) qubit and transform it into the qubit orthogonal to it. As shown in Ref. [9] the ideal universal-NOT (U-NOT) operation corresponds to the *inversion of the Bloch (Poincaré) sphere*. This inversion preserves angles (related in a simple way to the scalar product $|\langle\Phi|\Psi\rangle|$ of rays), so by Wigner's theorem the ideal U-NOT must be implemented either by a unitary or by an antiunitary operation. Unitary operations correspond to proper rotations of the Poincaré sphere, whereas antiunitary operations correspond to orthogonal transformations with determinant -1. Clearly, the U-NOT operation is of the latter kind, and an antiunitary operator Θ (unique up to a phase) implementing it is

$$\Theta(\alpha|0\rangle+\beta|1\rangle)=\beta^*|0\rangle-\alpha^*|1\rangle. \quad (4.2)$$

The difficulty with antiunitarily implemented symmetries is that they are not completely positive, i.e., they cannot be applied to a small system, leaving the rest of the world alone.

Because we cannot design a perfect universal-NOT gate, we have introduced in Ref. [9] an approximate *optimal* U-NOT gate (an analogous spin-flip operation has recently been introduced by Gisin and Popescu [18]). This device takes as an input the qubit A in the state $|\Psi\rangle$ and generates at the output a qubit in a mixed state as close as possible to the orthogonal state $|\Psi^{\perp}\rangle$. The role of the U-NOT gate is played by two additional (ancilla) qubits B and C. So, all together the transformation involves three qubits and it can be explicitly written as

$$|\Psi\rangle_A|X\rangle_{BC}\rightarrow\gamma_0|\Psi,\Psi\rangle_{AB}|\Psi^{\perp}\rangle_C+\gamma_1|\{\Psi,\Psi^{\perp}\}\rangle_{AB}|\Psi\rangle_C, \quad (4.3)$$

where $|X\rangle_{BC}$ is the initial state of the U-NOT gate; $\gamma_0=\sqrt{2/3}$ and $\gamma_1=-\sqrt{1/3}$. In this particular transformation the qubit C at the output is in the state that is as orthogonal as possible to the input state. The fidelity of this transformation is input-state independent and is equal to $\mathcal{F}=2/3$.

A. U-NOT gate as the entangler

It is interesting to note that the two-qubit state $\rho_{AB}^{(out)}$ at the output of the U-NOT gate (4.3) has the form

$$\rho_{AB}^{(out)} = \gamma_1^2 |\{\Psi,\Psi^\perp\}\rangle\langle\{\Psi,\Psi^\perp\}| + \gamma_0^2 |\Psi\Psi\rangle\langle\Psi\Psi|. \tag{4.4}$$

The mean fidelity between the state $\rho_{AB}^{(out)}$ and the ideal output (4.1) is input-state independent and takes the value $\mathcal{F} = 1/3$. This again corresponds to the fact that the Bures distance between the actual output of the entangler and the ideal output is input-state independent and equal to $d_B = (2 - 2/\sqrt{3})^{1/2}$. We can easily check that the partially transposed matrix corresponding to the density operator (4.4) has one negative eigenvalue $E = (2 - \sqrt{5})/6$ that is constant and does not depend on the initial input state $|\Psi\rangle$.

We note that the universal NOT gate (4.3) acts also a quantum cloner, i.e., the two qubits A and B are the optimal clones of the input (for details see Refs. [19] and [20]). It is the optimality of the transformation (4.3) with respect to cloning and the generation of the optimally orthogonal state (i.e., the universal NOT gate) which indicates that the transformation (4.3) also serves as the optimal universal entangler.

B. Proof of optimality

Our proof of the optimality of the entangler (4.1) via the U-NOT gate is based on the recent idea of Gisin [21,22] that the impossibility of instantaneous signaling generates upper bounds on the fidelity of particular quantum-mechanical processes. To be more specific, we have shown earlier that the impossibility of the ideal (perfect) entangler is due to the linearity of quantum mechanics. On the other hand, another consequence of the linearity of quantum mechanics is the fact that the entangled quantum-mechanical states cannot be used for super-luminal communication. Gisin [21] has shown that this no-signaling constraint implies bounds on the fidelity of universal cloning and the universal U-NOT gate. In the case of cloning the bound on fidelity is $\mathcal{F} = 5/6$, while in the case of the U-NOT gate the bound is $\mathcal{F} = 2/3$. We note that the transformation (4.3) achieves both these bounds when used as the cloner or the U-NOT gate, respectively. Recently Alber [14] used this idea of Gisin to prove that the upper bound in the fidelity of the antisymmetric entangling is equal to unity. The no-signaling constraint can also be used to derive an upper bound on the fidelity of the entangling operation given in Eq. (4.1) [22]. We will present a proof, which is based on the methods developed in Ref. [21], that this upper bound is $\mathcal{F} = 1/3$, which means that the U-NOT gate (4.3) serves as the *optimal* universal entangler in the sense of Eq. (4.1).

We consider a process in which a single-particle input state is mapped into a two-particle output state. The input state can be represented as

$$\rho^{(in)}(\vec{m}) = \tfrac{1}{2}(\mathbb{1} + \vec{m}\cdot\vec{\sigma}), \tag{4.5}$$

where $\vec{m}$ is a real vector whose length is less than or equal to unity. The most general two-particle output state, which is Hermitian and has a trace equal to one, can be expressed as

$$\rho^{(out)}(\vec{m}) = \frac{1}{4}\left[\mathbb{1} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{b}\cdot\vec{\sigma} + \sum_{j,k=x,y,z} t_{jk}\sigma_j\otimes\sigma_k\right], \tag{4.6}$$

where $\vec{a}$, $\vec{b}$, and t_{jk} are functions of $\vec{m}$. The requirement that the reduced density matrixes of the two output particles be the same, that we shall impose, implies that $\vec{a} = \vec{b}$.

We now want to impose the requirement of covariance. This means that if $\rho^{(in)}(\vec{m})$ is mapped onto $\rho^{(out)}(\vec{m})$, and if u is a matrix in $SU(2)$, then the input state $u\rho^{(in)}(\vec{m})u^{-1}$ will be mapped onto the output state $u\otimes u\rho^{(out)}(\vec{m})u^{-1}\otimes u^{-1}$. Another way of stating this condition is obtained by noting that if we express u as

$$u = \exp(-i\theta\hat{e}\cdot\vec{\sigma}/2), \tag{4.7}$$

where $\hat{e}$ is a unit vector corresponding to the rotation axis and θ is the rotation angle, then

$$u(\vec{m}\cdot\vec{\sigma})u^{-1} = \vec{m}'\cdot\vec{\sigma}, \tag{4.8}$$

where $\vec{m}' = R(\hat{e},\theta)\vec{m}$. The rotation matrix, $R(\hat{e},\theta)$, is the 3×3 matrix that rotates a vector about the axis $\hat{e}$ by an angle θ, and it is given explicitly by

$$R(\hat{e},\theta) = \exp(\theta\hat{e}\cdot\vec{K}), \tag{4.9}$$

where

$$K_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$K_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$K_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

We have that

$$u\rho^{(in)}(\vec{m})u^{-1} = \rho^{(in)}(R\vec{m}), \tag{4.11}$$

which will be mapped to $\rho^{(out)}(R\vec{m})$, so that the covariance condition can now be expressed as

$$\rho^{(out)}(R\vec{m}) = u\otimes u\rho^{(out)}(\vec{m})u^{-1}\otimes u^{-1}. \tag{4.12}$$

Now let us examine the consequences of this relation. We shall first consider the terms linear in $\vec{\sigma}$ and let R be a rotation about $\vec{m}$ by a very small angle θ. We have that

$$\vec{a}(R\vec{m}) = R\vec{a}(\vec{m}), \tag{4.13}$$

which for our choice of rotation becomes

$$\vec{a}(\vec{m}) = (\mathbb{1} + \theta \hat{m}\cdot\vec{K})\vec{a}(\vec{m}), \tag{4.14}$$

or

$$\hat{m}\cdot\vec{K}\vec{a}(\vec{m}) = \vec{0}, \tag{4.15}$$

where $\hat{m}$ is a unit vector in the direction of $\vec{m}$. This implies that $\hat{m}\times\vec{a}(\vec{m}) = \vec{0}$, so that $\vec{a}(\vec{m})$ is parallel to $\vec{m}$, and we can write $\vec{a}(\vec{m}) = a(\vec{m})\vec{m}$. If we now substitute this result back into Eq. (4.13) and consider a general rotation R, we have that

$$a(R\vec{m}) = a(\vec{m}). \tag{4.16}$$

This implies that $a(\vec{m})$ is a constant, which, following, Ref. [21], we shall denote by η.

Now let us see what covariance implies about the terms quadratic in $\vec{\sigma}$. Application of the covariance condition, Eq. (4.12), to these terms gives

$$t_{jk}(R\vec{m}) = \sum_{j',k'} R_{jj'}R_{kk'}t_{j'k'}(\vec{m}). \tag{4.17}$$

If we again choose R to be a rotation about $\vec{m}$ by a small angle θ, we find the condition

$$0 = \sum_{j'} (\hat{m}\cdot\vec{K})_{jj'}t_{j'k}(\vec{m}) + \sum_{k'} (\hat{m}\cdot\vec{K})_{kk'}t_{jk'}(\vec{m}). \tag{4.18}$$

If we choose $\vec{m}$ to be in the z direction, in particular $\vec{m} = \hat{z}$, we find, as did Gisin, that $t_{xx} = t_{yy}$, $t_{xy} = -t_{yx}$, and $t_{xz} = t_{zx} = t_{yz} = t_{zy} = 0$, where all of these are evaluated at $\vec{m} = \hat{z}$. We now want to impose the no-signaling condition

$$\rho^{(out)}(\hat{z}) + \rho^{(out)}(-\hat{z}) = \rho^{(out)}(\hat{x}) + \rho^{(out)}(-\hat{x}), \tag{4.19}$$

and to do so we need to find all of the density matrices in the above equation in terms of $t_{jk}(\hat{z})$. This can be done by applying the covariance condition, Eq. (4.12), to $\rho^{(out)}(\hat{z})$ and making the proper choice of R. When these results are substituted into Eq. (4.19) we find that $t_{xx}(\hat{z}) = t_{yy}(\hat{z}) = t_{zz}(\hat{z})$, and we shall designate this common value by $t(\hat{z})$. We then have that

$$\rho^{(out)}(\hat{z}) = \frac{1}{4}\begin{pmatrix} 1+2\eta+t & 0 & 0 & 0 \\ 0 & 1-t & 2(t+it_{xy}) & 0 \\ 0 & 2(t-it_{xy}) & 1-t & 0 \\ 0 & 0 & 0 & 1-2\eta+t \end{pmatrix}. \tag{4.20}$$

The basis in which the matrix is expressed is $\{|+\hat{z},+\hat{z}\rangle, |+\hat{z},-\hat{z}\rangle, |-\hat{z},+\hat{z}\rangle, |-\hat{z},-\hat{z}\rangle\}$, where $\sigma_z|\pm\hat{z}\rangle = \pm|\hat{z}\rangle$. This matrix must be positive, which implies that the eigenvalues

$$\tfrac{1}{4}(1\pm 2\eta + t) \quad \text{and} \quad \tfrac{1}{4}(1 - t \pm 2\sqrt{t^2 + t_{xy}^2}) \tag{4.21}$$

must be non-negative.

For an input state $\rho^{(in)}(\hat{z})$ our desired output state is $(|+\hat{z},-\hat{z}\rangle + |-\hat{z},+\hat{z}\rangle)/\sqrt{2}$, and this implies that the fidelity of $\rho^{(out)}$ is

$$F = \frac{1+t}{4}. \tag{4.22}$$

This is clearly maximized when t is as large as possible, and examining the eigenvalues of $\rho^{(out)}$, this happens when $t_{xy} = 0$ and $t = 1/3$. Substituting this into the expression for the fidelity, we see that the maximum fidelity is 1/3. This means that the no-signaling constraint specifies the upper bound on the fidelity of the symmetric entangling that is exactly the same one as achieved by the U-NOT gate. This proves that the entangling via the NOT gate is optimal.

C. Remark

We note that using the universal NOT gate one can also produce an entangled state of the form (3.1). Specifically, the U-NOT gate allows Charlie to produce an entangled state, consisting of $|\Psi\rangle$ and one of two known states, which is shared by Alice and Bob. In order to see how this can be accomplished it is useful to express the state on the right-hand side of Eq. (4.3) as

$$\sqrt{\tfrac{1}{3}}(|\Psi\rangle_A|\Phi_-\rangle_{BC} + |\Psi\rangle_B|\Phi_-\rangle_{AC}), \tag{4.23}$$

where

$$|\Phi_-\rangle = \frac{(|\Psi\rangle|\Psi^\perp\rangle - |\Psi^\perp\rangle|\Psi\rangle)}{\sqrt{2}} = \frac{(|0\rangle|1\rangle - |1\rangle|0\rangle)}{\sqrt{2}} \tag{4.24}$$

is the singlet state. Charlie now measures his particle along the axis corresponding to the states $|0\rangle$ and $|1\rangle$. Whatever result he obtains for his particle, the other two particles will be in an entangled state shared by Alice and Bob. For example, if Charlie finds his particle in the state $|1\rangle$, Alice and Bob share the state in Eq. (3.1). Note that Charlie can choose the states with which the state $|\Psi\rangle$ will be entangled by choosing the axis along which to measure his particle.

This implies that if we want to produce either the entangled state of $|\Psi\rangle$ with $|0\rangle$ or the entangled state of $|\Psi\rangle$ with $|1\rangle$, and we don't care which one we get, this can be done with perfect fidelity. Perhaps a better way of stating this is that if we want to entangle $|\Psi\rangle$ with one of two orthogonal states, this can be done perfectly, and we will know with which state it is entangled.

V. CONCLUSIONS

In this paper we have studied various possibilities for entangling two qubits so the initial information about their preparation is preserved. We have studied a specific situation when the state of one of the qubits is known while the second state is arbitrary. We have shown that entanglement via symmetrization in this case can be performed with a very high fidelity (much higher than the fidelity of estimation). This type of entanglement can be very useful for stabilization of the storage of an (unknown) quantum state of one qubit against environmental interaction and a random imprecision [10]. We have shown that the U-NOT gate *optimally* implements the entanglement transformation $|\Psi\rangle \rightarrow |\Psi\rangle|\Psi^{\perp}\rangle + |\Psi^{\perp}\rangle|\Psi\rangle$. This means that the transformation (4.3) is very special indeed—it describes the optimal cloning, the optimal U-NOT transformation, as well as the optimal entangler.

ACKNOWLEDGMENTS

We thank Nicolas Gisin and Christoph Simon for helpful discussions. This work was supported by the National Science Foundation under Grant No. PHY-9970507 and by the IST project EQUIP under Contract No. IST-1999-11053.

[1] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).
[2] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992).
[3] L. Hardy, Phys. Lett. A **167**, 17 (1992).
[4] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[5] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[6] A.M. Steane, Rep. Prog. Phys. **61**, 117 (1998).
[7] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
[8] R. Derka, V. Bužek, and A.K. Ekert, Phys. Rev. Lett. **80**, 1571 (1998).
[9] V. Bužek, M. Hillery, and R.F. Werner, Phys. Rev. A **60**, R2626 (1999).
[10] A. Barenco, A. Berthiaume, D. Deutsch, A. Ekert, R. Jozsa, and C. Macchiavello, SIAM J. Comput. **26**, 1541 (1997).
[11] V. Vedral, M.M. Plenio, M.A. Ripin, and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[12] N. Linden and S. Popescu, Phys. Rev. A **56**, R3319 (1997).
[13] S. Hill and W.K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997); W.K. Wootters, *ibid.* **80**, 2245 (1998).
[14] G. Alber, e-print quant-ph/9907104.
[15] D. Bures, Trans. Am. Math. Soc. **135**, 199 (1969); A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976); **24**, 229 (1986); see also W.K. Wootters, Phys. Rev. D **23**, 357 (1981).
[16] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[17] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[18] N. Gisin and S. Popescu, Phys. Rev. Lett. **83**, 432 (1999).
[19] V. Bužek and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998).
[20] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[21] N. Gisin, Phys. Lett. A **242**, 1 (1998).
[22] N. Gisin (private communication).