Entangling atoms in photonic crystals

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Abstract. We propose a method for entangling a system of two-level atoms in photonic crystals. The atoms are assumed to move in void regions of a photonic crystal. The interaction between the atoms is mediated either *via* a *defect mode* or *via* a *resonant dipole-dipole interaction*. We show that these interactions can produce pure entangled atomic states. We analyze the problem with parameters typical for currently existing photonic crystals and Rydberg atoms and we show that the atoms can emerge from photonic crystals in entangled states. Depending on the linear dimensions of the crystal we estimate that a pair of atoms entangled in a photonic crystal can be separated by tens of centimeters.

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1 Introduction

Quantum entanglement is one of the most remarkable features of quantum mechanics. Coherent control of the entanglement between quantum systems has attracted lot of attention mainly because of its potential application in quantum information processing [1]. Simultaneously, experimental investigation of the entanglement allows us to test basic postulates of quantum mechanics and to answer fundamental epistemological questions. These questions are related to the original Gedanken experiment of Einstein, Podolsky and Rosen [2] which triggered discussions about the non-locality of quantum mechanics and motivated experimental proposals to test whether quantum mechanics is a complete non-local theory. The first experimental confirmation of the violation of Bell's inequalities [3] was performed using entangled photons [4]. Recently Zeilinger *et al.* has performed several experiments testing Bell inequalities over large distances [5] as well as experiments in which three photons has been entangled [6] and entanglement swapping has been performed [7].

Another challenge is to entangle other quantum systems – such as atoms. One of the first proposals for such an experiment is described in reference [8]. Other proposals have been presented in references [9,10]. The authors of these schemes proposed techniques to create entangled atoms in microwave single-mode cavities. Recently, controlled entanglement between atoms separated approximately by 10 mm interacting with an electromagnetic field

in a high-Q cavity has been experimentally realized [11]. In addition, trapped ions have been created in entangled states [12].

In this paper we propose a simple scheme for entangling atoms in photonic crystals. Photonic crystals are artificially created three-dimensional periodic dielectric materials which exhibit a frequency gap or several gaps in spectrum of propagating electromagnetic (EM) waves [13,14]. An EM wave with its frequency from the gap can not propagate in the structure in any direction. Photonic crystals operating at microwave frequencies were successfully created in a number of laboratories [15]. They consist of a solid dielectric and empty regions. The periodicity of a photonic crystal can be destroyed by removing or adding a piece of material which creates a defect EM mode in the structure. This mode is spatially localized around the region of the defect. The frequency of the mode and the spatial modulation of its electric field amplitude depends on properties of the defect [16–19]. It means that one can adjust parameters of the defect mode by creating a suitable defect in the crystal. In particular, the spatial dependence of the mode amplitude can be adjusted to particular needs. The quality factor of a single mode in a metallic cavity can be of the order of 10^8, and similar values can be reached for a single defect mode in a photonic crystal [20]. Microwave photonic crystals could therefore be used for experiments with Rydberg atoms [20,21].

In this paper we consider two interactions *via* which one can produce entangled atoms. Firstly, we show that it is possible to generate entangled atoms without a defect mode, using the action of the resonant dipole-dipole interaction (RDDI) [22,23] mediated by off-resonant modes

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of photonic band continue. Secondly, we explore the scheme in which the atoms become mutually entangled due to the interaction with the defect-field mode.

The paper is organized as follows: basic features of the proposed setup are described in Section 2. In Section 3 we discuss how the atoms in photonic crystals can be entangled *via* the resonant dipole-dipole interaction. In Section 4 we study in detail the entanglement of atoms which interact with a single defect mode in the photonic crystal. In Section 5 we conclude the paper with some remarks.

2 Setup of the scheme

We consider two mechanisms through which a system of identical atoms can be entangled in photonic crystals. We assume that the atoms are modeled by two-level systems having their transition frequencies in a photonic bandgap (PBG).

The first mechanism is the RDDI mediated by off-resonant modes of the photonic-band continue (see the Hamiltonian (8)). This interaction has been analyzed in detail by Kurizki [24] and John and Wang [22] as well as by John and Quang [23]. These authors have considered a system of two two-level atoms. It has been been shown that if one of the atoms is excited and the other one is in its ground state, then they can exchange excitation in spite of the fact that their transitions frequencies are in a PBG and spontaneous emission is nearly totally suppressed. The RDDI can be understood as an energy exchange *via* a localized field [22]. This light tunneling (or photon-hopping conduction) can be very efficient when the distance between the atoms is much smaller than the light wavelength. The RDDI can occur either in free space or in a cavity. However, in free space the excitation is irreversibly radiated into the continuum of the field modes after a very short time (given by Fermi's Golden rule) and the entanglement between the atoms deteriorates rapidly.

The second mechanism is due to an excitation exchange *via* a defect mode which is resonant (or nearly resonant) with the atoms. This type of interaction explicitly involves a quantized defect mode and is described by the Hamiltonian (10).

These two interactions can also occur simultaneously. As we will see, the second mechanism is much more efficient and allows a coherent control over the process of entanglement. The first mechanism can be neglected in many cases, especially when the atoms have their transition frequencies near the center of a wide PBG and their distance is not much smaller than the wavelength of the resonant light.

In what follows we describe the basic setup of the proposed experiment in the case when the atoms interact only *via* the defect mode. We consider the photonic crystal of the geometry designed by Yablonovitch *et al.* [15,20] (see Fig. 1) although other appropriate geometries can be used as well. The periodic structure is created by drilling cylindrical holes in a bulk material. In Figure 1 we plot only cylindrical holes intersecting in the center of the (cubic)
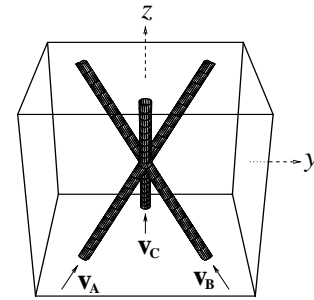

Fig. 1. A schematic description of the physical situation. We display only three of many cylindrical holes in the crystal.

crystal. We assume (except Sect. 3) that there is a defect of the crystal periodicity near the region where the displayed cylinders are crossing. This defect creates a defect mode near the center of the photonic bandgap. The defect-field mode is initially prepared in its vacuum state. This could be a problem because of the presence of microwave thermal photons. Performing the experiment at low temperatures (≤ 10 K) could, at least in principle, solve this obstacle. Let us assume that one of the three atoms (let say the atom A) is prepared initially in its excited state while the other two atoms (B and C) are initially in their ground states. After the preparation the atoms are injected into cylindrical void regions of the crystal at the same time. Firstly the atoms "fly" freely in the void cylinders outside the defect-field. Transition frequencies of the atoms lie inside the wide PBG. When the atoms enter the defect region they start to interact with the single defect-field mode. And then again, after they leave the defect region they evolve freely. States of the atoms are detected at the exit from the crystal.

We consider the following numerical values for the setup. The crystal is a cube of the side $L \approx 20$ cm. The cylindrical holes are drilled at the angle $\Theta = 35.26°$ with the vertical axis. Linear dimensions of the defect-mode region are comparable with the lattice constant of the structure. The frequencies of the defect mode and the atomic transition are $\omega_0/(2\pi) = \omega/(2\pi) = 21.50651$ GHz, *i.e.* the same as transitions used in experiments with microwave cavities [21]. This frequency lies inside the wide photonic bandgap [16] providing the crystal is made from a dielectric with a refractive index (at microwaves) 3.6, the volume filling fraction is 78% and the side of an elementary cube is $a \approx 16.3$ mm. We use this value of a to calculate the parameter $k = \pi/a$ (see Eq. (12)). These values of the parameters are feasible with current experimental techniques. In the experiments of Yablonovitch *et al.* [15,16] the side of an elementary cube was 11 mm and the diameter of the cylindrical holes was 5.16 mm. This implies that in our scheme the diameter of the cylindrical holes should be about 7.65 mm.

If the exited (ground) state of the atom j ($j = A, B, C$) is denoted as $|e_j\rangle$ ($|g_j\rangle$) and the n-photon state of the single-mode defect field is denoted as $|n\rangle$ then the initial state of the system under consideration can be written as

$$|\Psi(0)\rangle = |e_A\rangle \otimes |g_B\rangle \otimes |g_C\rangle \otimes |0\rangle \equiv |e_A, g_B, g_C, 0\rangle. \quad (1)$$

When we assume that in the defect region the atom-field interaction is governed by the Hamiltonian in the dipole and rotating wave approximations (see below in Sect. 4) then the final state of the system reads

$$|\Psi(t)\rangle = a(t)|e_A, g_B, g_C, 0\rangle + b(t)|g_A, e_B, g_C, 0\rangle$$
$$+ c(t)|g_A, g_B, e_C, 0\rangle + \gamma(t)|g_A, g_B, g_C, 1\rangle, \quad (2)$$

where t is the time at which we detect the internal states of the atoms at the exit of the crystal. The final values of the amplitudes a, b, c and γ depend on a particular setup of the experiment including the coupling parameters and velocities of the atoms. For completeness of the description we specify trajectories $\mathbf{r}_j(t)$ of the three atoms which can move along the axes of the three void regions

$$\mathbf{r}_j(t) = \mathbf{r}_j(0) + \mathbf{v}_j t; \qquad j = A, B, C \qquad (3)$$

with the vectors $\mathbf{r}_j(0)$ and $\mathbf{v}_j$ specified as

$$\mathbf{r}_A(0) = \frac{L}{4}\left\{\tan\Theta, -\sqrt{3}\tan\Theta, -2\right\},$$
$$\mathbf{v}_A = \frac{v_A}{2}\left\{-\sin\Theta, \sqrt{3}\sin\Theta, 2\cos\Theta\right\}, \qquad (4)$$

for the atom A. While for the other two atoms (B and C) we have

$$\mathbf{r}_B(0) = \frac{L}{4}\left\{\tan\Theta, \sqrt{3}\tan\Theta, -2\right\},$$
$$\mathbf{v}_B = \frac{v_B}{2}\left\{-\sin\Theta, -\sqrt{3}\sin\Theta, 2\cos\Theta\right\}, \qquad (5)$$

and

$$\mathbf{r}_C(0) = \frac{L}{2}\left\{-\tan\Theta, 0, -1\right\},$$
$$\mathbf{v}_C = v_C\left\{\sin\Theta, 0, \cos\Theta\right\}. \qquad (6)$$

Here we assume the origin of the coordinates in the center of the cube crystal with the side of the length L; Θ is the angle between the axes of the cylinders and the z direction.

3 Entanglement via resonant dipole-dipole interaction

In this section we consider just two identical atoms (A and B) which move in the crystal as it is described above. Here we assume that there is no defect mode in the crystal. The atoms move inside the crystal with constant velocities. The recoil effect due to the interaction with the EM field is neglected because the atoms are relatively heavy particles. The interaction between the atoms and the EM field modes inside the crystal is described by the Hamiltonian in the electric-dipole approximation

$$H = \hbar\omega \sum_{j=A,B} \sigma_z^j + \hbar \sum_\lambda \omega_\lambda a_\lambda^\dagger a_\lambda$$
$$- \frac{1}{\epsilon_0}\boldsymbol{\mu}(A) \cdot \mathbf{D}(\mathbf{r}_A) - \frac{1}{\epsilon_0}\boldsymbol{\mu}(B) \cdot \mathbf{D}(\mathbf{r}_B), \qquad (7)$$

where a_λ and $a_\lambda^\dagger$ are the annihilation and creation operators of the field mode labeled by λ, $\mathbf{D}(\mathbf{r})$ is the transverse displacement-field operator, $\boldsymbol{\mu}(A)$ and $\boldsymbol{\mu}(B)$ are the atomic dipole operators. When the atomic transition frequencies are far from the regions of abrupt changes in the density of modes, and the distance R of the atoms is much less than the resonant wavelength λ, then the Hamiltonian (7) can be approximated as (for more details see Ref. [23])

$$H_{\text{eff}} = \hbar\omega \sum_{j=A,B} \sigma_z^j + \hbar\left(J_{AB}\sigma_+^A\sigma_-^B + J_{BA}\sigma_-^A\sigma_+^B\right), \quad (8)$$

where $\sigma_\pm^x$ are raising and lowering operators of the atoms ($x = A, B$) and J_{AB} is a matrix element for the effective description of the RDDI [22]. For qualitative estimations, we will use J_{AB} evaluated under the assumption that the density of the EM modes is the same as in free space. In this case we use the formula we can utilize the expression derived in reference [25]

$$\hbar J_{AB} = \mu_i^{ge}(A)\mu_j^{eg}(B)\frac{1}{4\pi\epsilon_0 R^3}$$
$$\times \left[(\delta_{ij} - 3\hat{R}_i\hat{R}_j)(\cos k_A R + k_A R \sin k_A R)\right.$$
$$\left. - (\delta_{ij} - \hat{R}_i\hat{R}_j)k_A^2 R^2 \cos(k_A R)\right], \quad (9)$$

where $k_A \equiv \omega/c$, μ^{eg} is the absolute value of the atomic dipole matrix element, $\hat{R}_i$ are the components of the unit vector starting at the position of the atom A and oriented towards the atom B. We assume the summation over the repeated indices. The above expression for J_{AB} was derived for atoms in a free space, but in the limit $R \ll \lambda$ it can be also used as a good approximation for a description of the RDDI effects in photonic crystals [22]. As we will see later the RDDI between the moving Rydberg atoms (with velocities of the order of 10^2 m s^{-1} or higher) is important only in the regime with $R \ll \lambda$. Therefore we can safely use the Hamiltonian (8) for our approximate description.

In what follows we will study the time evolution of the atoms initially prepared in the state $|\Psi(0)\rangle = |e_A, g_B\rangle$ which is governed by the effective Hamiltonian (8) with time-dependent J_{AB} (which is due to the fact that the atoms are moving through the crystal). We show that the RDDI can in principle be used for controlling the entanglement between the atoms. We have solved the corresponding Schrödinger equation numerically. We have used the physical parameters typical for Rydberg atoms and currently existing photonic crystals (see Sect. 2). In Figure 2 we plot the results for the time-dependent atomic populations. We have chosen the atomic trajectories as specified in the previous section but we added a small value (0.05–0.3 mm) to the initial $x_A(0)$ coordinate so that the trajectory of the atom A is parallel but not identical with the axis of the cylinder. This prevents the collision of the atoms. Three curves in Figure 2 correspond to three different values of $x_A(0)$ specified by the minimal distance $R_{\min}$ between the atoms when they fly through the crystal. Both atomic dipoles are oriented in the x-direction. In our calculations we have taken into account that the atoms interact
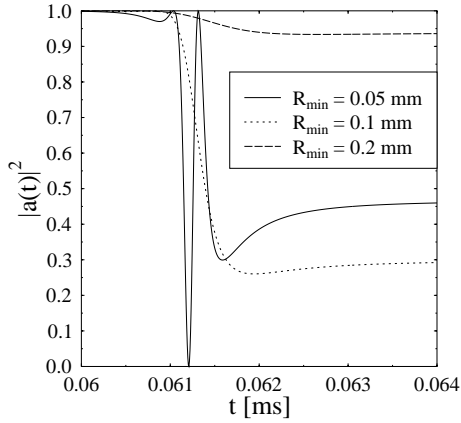
Fig. 2. The time evolution of the population of the upper level of the atom A. The atoms interact according to effective Hamiltonian (8).

only when they are near the center of the crystal. This interaction region has been taken to be the cubic region of the side 2 cm. The atomic velocities have been chosen to be $v_A = v_B = 200\,\mathrm{m\,s^{-1}}$, *i.e.* they are of the same order as in the micromaser experiments [11,21]. The atomic dipole moment has been taken to be $\mu^{eg}/e \approx 6.7 \times 10^{-7}$ m (where e is the proton charge) from the experimental parameters in reference [21].

Taking into account that the physical conditions are chosen such that the EM field is adiabatically eliminated from the interaction (see the effective Hamiltonian (8)) the two atoms due to the unitarity of the evolution remain in a pure state $|\Psi(t)\rangle_{AB} = a(t)|e_A, g_B\rangle + b(t)|g_A, e_B\rangle$ with the amplitudes $a(t)$ and $b(t)$ which depend on the RDDI. From here it follows that due to the RDDI the two atoms become entangled. The degree of entanglement in the present case can be quantified with the help of the von Neumann entropy $S = -\mathrm{Tr}[\hat{\rho}\ln\hat{\rho}]$ of each individual atom for which we have $S = -|a(t)|^2 \ln|a(t)|^2 - |b(t)|^2 \ln|b(t)|^2$. In other words the degree of the entanglement depends on the population of internal levels of the atoms and the highest degree of entanglement is attained for $|a(t)|^2 = |b(t)|^2 = 1/2$.

As seen from Figure 2 the population of the excited state of the atom A depends on the minimal distance $R_{\min}$ between the atoms when they fly through the crystal. From our numerical investigation it follows that the atoms are most entangled for $R_{\min} \simeq 0.05$ mm. However we note that with present techniques the control over the position of the atoms in the configuration considered here is about ± 1 mm [11]. Consequently, the RDDI is not very suitable for a coherent control of entanglement between the atoms in photonic crystals. In the following section we consider entanglement through a defect mode when the currently available precision control is sufficient.

4 Entanglement via a defect mode

Let us consider the interaction of the atoms with a single defect-field mode in the dipole and the rotating-wave approximations. We assume that the distance between the atoms is always sufficiently large such that they do not interact *via* RDDI. The corresponding Hamiltonian can be written as

$$H = \hbar\omega \sum_{j=A,B,C} \sigma_z^j + \hbar\omega_0 a^\dagger a$$
$$+ \hbar \sum_{j=A,B,C} \left[G(\mathbf{r}_j)\sigma_+^j a + G^*(\mathbf{r}_j)\sigma_-^j a \right], \quad (10)$$

where ω_0 is the mode frequency (which we assume to be equal to the atomic transition frequency ω), $\sigma_\pm^j$ are the atomic raising and lowering operators and $\mathbf{r}_A$ and $\mathbf{r}_B$ are the positions of the atoms. The position dependence of the coupling parameters $G(\mathbf{r}_j)$ can be expressed as

$$G(\mathbf{r}_j) = G_0 \boldsymbol{\epsilon} \cdot \boldsymbol{\mathcal{D}}_j \, f(\mathbf{r}_j), \quad (11)$$

where $f(\mathbf{r})$ is the field-mode amplitude at the position $\mathbf{r}$, $\boldsymbol{\epsilon}$ is the electric-field polarization direction of the defect mode and $\boldsymbol{\mathcal{D}}_j$ is a unit vector in the direction of the atomic dipole matrix element of the atom j. It is known that the spatial dependence of the defect-mode amplitude is an oscillating function which decays exponentially as a function of the distance [17]. A particular profile of the spatial dependence of the defect mode can be adjusted *via* a properly generated defect of the periodicity. A rigorous calculation of the EM field in the presence of a defect in a 3D photonic crystal can be a difficult task. In this paper we use a model profile of the spatial dependence of the electric field. Similar profiles have already been created in existing photonic crystals [16–19]. We note that for the purpose of the proposed experiment a complete information about the mode shape is not needed. The results of the experiment depend only on the shape along the trajectories of the atoms. In what follows we use the profile

$$f(\mathbf{r}) = \exp\left[-\frac{|\mathbf{r} - \mathbf{R}_0|}{R_{\mathrm{def}}}\right] \sin(\mathbf{k} \cdot \mathbf{r} + \Phi), \quad (12)$$

where $\mathbf{R}_0$ is the position around which the mode is localized, R_{def} is a parameter (defect-mode radius) describing the rate of the exponential decay of the mode envelope, Φ is a phase factor and $\mathbf{k}$ is the parameter describing spatial oscillations of the field mode. We chose its magnitude to be $k = \pi/a$ where a is the value of the side of an elementary cubic cell in the photonic crystal. We consider the constant R_{def} comparable with a. We estimate the value of the coupling constant G_0 from microcavity experiments [21]

$$G_0 = \sqrt{\frac{V_{\mathrm{cav}}}{V_{\mathrm{eff}}}} \Omega, \quad (13)$$

where V_{cav} is the modal volume of the microcavity mode, V_{eff} is the effective modal volume of the defect mode and Ω is the vacuum Rabi frequency in the microwave experiment. The numerical values are [21]: $V_{\mathrm{cav}} = 11.5$ cm^3 and $\Omega = 43$ kHz. When we consider the transitions between levels 63P$_{3/2}$ and 61D$_{3/2}$ of the rubidium atoms, then the

atomic transition frequency is $\omega/(2\pi) = 21506.51$ MHz. Finally, the effective modal volume can be approximated as

$$V_{\text{eff}} = \frac{4}{3}\pi(2R_{\text{def}})^3. \qquad (14)$$

Because the atoms are moving the coupling parameters are time dependent (in what follows we will use the notation $G_j(t)$). We consider positions of the atoms given by equations (4, 5). In some cases we add a small value to $x_A(0)$ given by (4) to prevent the atoms to collide in the center of the crystal. Details of the geometry of the proposed experiment are given in Section 2 and in Figure 1.

Once we have specified all model parameters we can solve the Schrödinger equation for the system which is supposed to be initially prepared in the state $|\Psi(0)\rangle = |e_A, g_B, g_C, 0\rangle$. Due to the fact that the number of excitations is an integral of motion in the present case the state vector at time $t > 0$ has the form (2) and the corresponding Schrödinger equation can be rewritten into a set of a system of linear differential equations. These equations can be solved analytically for time-independent coupling constants $G_j(t)$ which is not our case. Therefore we have to integrate the equations numerically.

4.1 One atom

We start our discussion with a problem when just a single atom (let say the atom A) flying through the crystal is considered. We assume that the atom is on resonance with the defect mode (i.e., $\omega = \omega_0$). This corresponds to the Jaynes-Cummings model [26] with a time-dependent coupling constant. The general solution of this model for real coupling parameter was found by Sherman et al. [27]. With the initial condition $|\Psi(0)\rangle = |e_A, 0\rangle$ the solution can be expressed as

$$|\Psi(t)\rangle = \cos\left[\int_0^t G_A(t')\mathrm{d}t'\right]|e_A, 0\rangle$$
$$- \mathrm{i}\sin\left[\int_0^t G_A(t')\mathrm{d}t'\right]|g_A, 1\rangle. \qquad (15)$$

This implies for the atomic excitation

$$P_e^{(A)}(t) = \cos^2\left[\int_0^t G_A(t')\mathrm{d}t'\right]. \qquad (16)$$

In the case of the defect mode with linear dimensions much smaller than the side of the crystal we can use the approximation

$$\int_0^t G_A(t')\mathrm{d}t' \approx \int_{-\infty}^{\infty} G_A(t')\mathrm{d}t'. \qquad (17)$$

We note that this integral for a given choice of the profile function (see Eq. (12)) with the phase of the field mode $\Phi = 0$ equals to zero. This means that the atom exits the crystal in the same state as it entered it. Obviously the
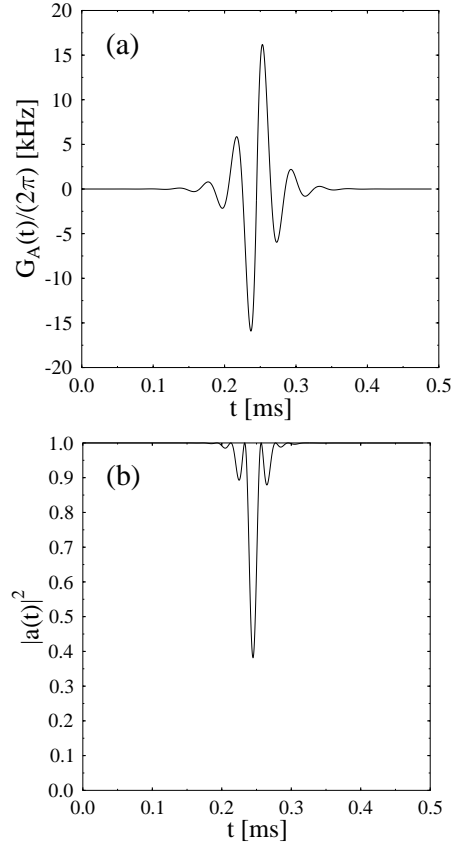

Fig. 3. (a) The time dependence of the coupling $G_A(t)$ between the defect mode and the single atom A when it moves along the axis of the cylinder inside the crystal with the velocity $v_A = 500$ m s^{-1}. (b) The time evolution of the population of the exited level of the single atom moving inside the crystal.

defect mode also remains in its initial (vacuum) state. In Figure 3a we plot the time dependence of the coupling constant between the atom A and the defect mode. While in Figure 3b we present the time evolution of the population of the excited level of the atom. It is assumed that the defect is located at the center $\mathbf{R}_0 = \mathbf{0}$ of the crystal. The atom moves along the axis of the cylindrical cavity with the velocity $v_A = 500$ m s^{-1}. The other parameters are chosen such that (see Eqs. (11, 12)) $\Phi = 0$ rad, $\mathbf{k} = (0,0,k)$, $\mathcal{D}_A = \boldsymbol{\epsilon} = (1,0,0)$, $R_{\text{def}} = 10$ mm. The parameter $\mathbf{k} = (0,0,\pi/a)$ with $a \approx 16.3$ mm. The integral (17) in this case is equal to zero. Consequently, the atom at the exit of the crystal is again in its initial state. From Figure 3a we clearly see that the atom on its way through the crystal interacts with the defect mode just around the center of the crystal. The other important feature is seen from Figure 3b, i.e. the atom is transiently entangled with the defect mode in the center of the crystal. Nevertheless it leaves the crystal in a pure (unentangled) state. This effect of "spontaneous" disentanglement of the atom from the defect mode is very important when we consider creation of pure entangled state of two atoms.
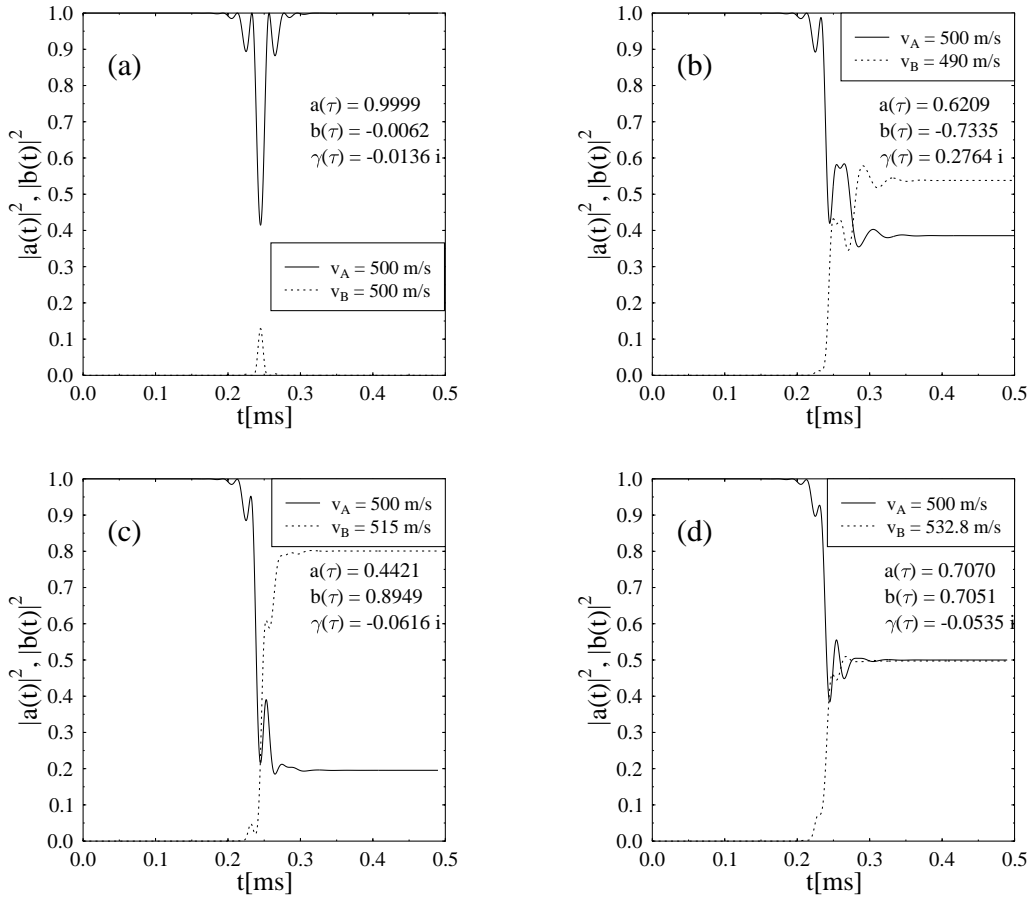
Fig. 4. The time evolution of the populations of excited levels of the atoms A (solid line) and B (dashed line). The parameters of the defect mode and the internal atomic parameters are chosen same as in Figure 3. The final values of the probability amplitudes are presented directly in the figures.

4.2 Two atoms

Let us consider a situation when two atoms interact with the same defect mode as in the previous case. The atoms have their dipoles oriented along the direction ϵ of the electric-field polarization. The velocity of the atom A is 500 m s^{-1}. The time evolution of the corresponding atomic populations for various velocities of the atom B are plotted in Figure 4. The atom A enters the crystal in the excite state, while the atom B is initially in the ground state. The four plots corresponds to four different velocities of the atom B (their numerical values are shown in the figures).

Firstly, we consider both atoms to have the same velocity (see Fig. 4a). In this case we assume that the atom A is displaced from the axis of the cylindrical hole through which it flies (*i.e.* we add 0.3 mm to $x_A(0)$ given by (4)) to avoid the influence of the RDDI between the atoms and their collision. We see that the atoms strongly interact with the field in the region of the defect. However, after this interaction the initial state of the system is approximately restored (see the "stationary" values of the probability amplitudes $a(\tau)$, $b(\tau)$ and $\gamma(\tau)$ which are displayed in the figures). It is interesting to compare Figure 3b with Figure 4a to see how the time evolution of the population of the atom A is modified due to the presence of the

atom B. We see that for the given set of parameters the presence of the atom B does not influence dynamics of the atom A significantly.

Now we will study how the level population depends on the velocity of the atom B. From Figure 4 we see that for properly chosen velocity the interaction between the atoms mediated by the defect field can be pronounced. For instance, from Figure 4b (here $v_B = 490 \text{ m s}^{-1}$) we see that not only the excitation of the atom B can be higher than the population of the atom A, but also the defect mode becomes partially excited and entangled with the atomic system.

When the atom B has the velocity $v_B = 515 \text{ m s}^{-1}$ (see Fig. 4c) then the defect mode in the stationary limit is in the vacuum state $[\gamma(\tau) \simeq -0.0616\text{i}]$ and is (with the high precision) disentangled from the atomic system. It is interesting to note that in this particular situation the defect mode mediates transfer of most of the excitation from the atom A to the atom B.

Let us assume now the velocity of the atom B to be $v_B = 532.8 \text{ m s}^{-1}$ (see Fig. 4d). In this case the defect mode in the stationary limit is again in the vacuum state and is completely disentangled from the atomic system. The amplitudes $a(\tau)$ and $b(\tau)$ are in this case almost equal,
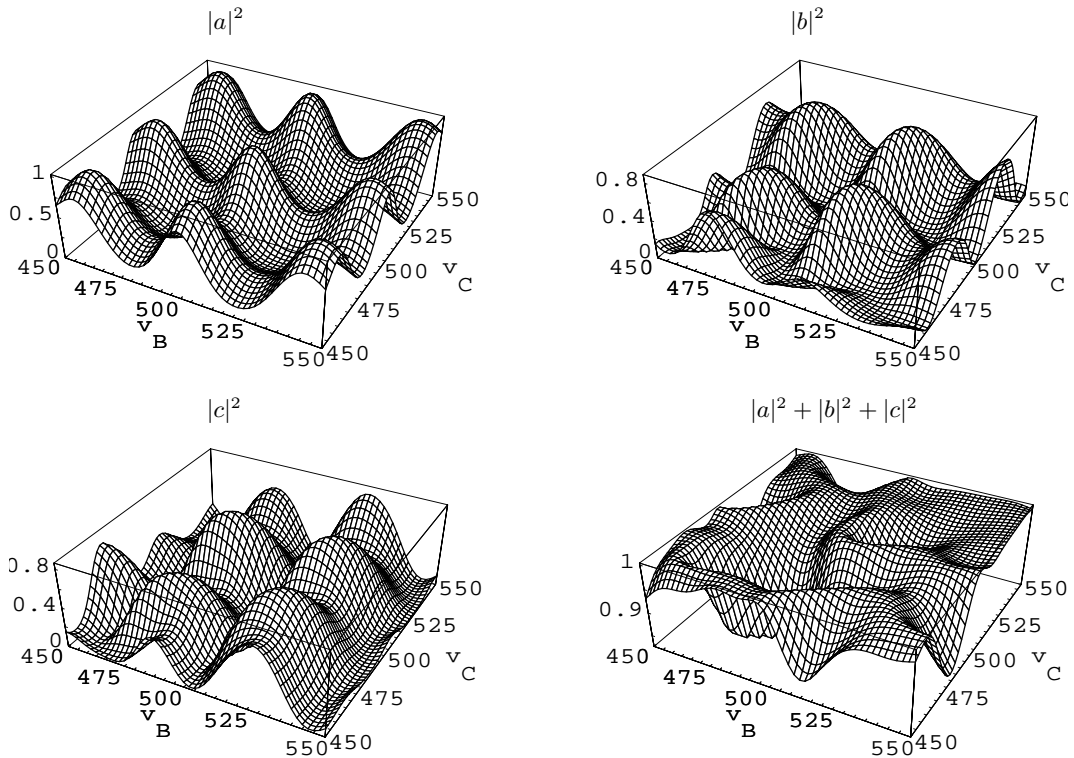
Fig. 5. The final atomic populations of the three atoms which interact with the defect mode in the crystal *versus* velocities v_B, v_C while $v_A = 500$ m s^{-1}. The defect region is displaced from the center with $\mathbf{R}_0 = (1, -3, 2)$ mm. We chose the phase $\Phi = 0$ and the other parameters (a and $R_{\rm def}$) are the same as in Figure 3.

which means that the atoms at the exit from the crystal are in the state $|\Psi\rangle = (|e_A, g_B\rangle + |g_A, e_B\rangle)/\sqrt{2}$, *i.e.* they are prepared in a pure maximally entangled state.

In the cases presented in Figure 4 the phase factor Φ of the defect mode is set to zero so that the integrals of the coupling constants $G_A(t)$ and $G_B(t)$ over the trajectories of the atoms are equal to zero. The defect-mode radius is taken to be $R_{\rm def} = 10$ mm. We have also studied dynamics for other values of Φ, when the integrals of the coupling constants differ from zeros. In this case the disentanglement of the defect mode and the atoms is not so well pronounced, *i.e.* the defect mode becomes excited. We have also found a general feature: if the integrals of the coupling constants are zeros and the coupling constants are small enough then the defect mode after the interaction is left in the vacuum state. However, if we increase the couplings (by decreasing the mode volume $V_{\rm eff}$) the defect mode can be left in an excited state (*i.e.* $\gamma(\tau) \neq 0$; see the expression for the state vector (2)). Consequently, the atoms are left in a mixed state.

We have also analyzed the situation when the defect mode is not located directly in the center of the crystal. In addition we have assumed that $\Phi \neq 0$. We have found that even in this case it is possible to find a value v_B at which the atoms exit the crystal in a nearly pure maximally entangled state.

4.3 Three atoms

Let us consider the same setup as in our previous discussion except we assume three atoms flying through the crystal (see Fig. 1). These three two-level Rydberg atoms (A, B and C) are injected into the holes at the bottom side of the crystal simultaneously. The atom A is initially in its upper level $|e_A\rangle$ while atoms B and C are initially in their lower states $|g_B\rangle$ and $|g_C\rangle$. The single defect mode is initially prepared in its vacuum state $|0\rangle$. The atoms move along the axes of the holes and interact with the defect mode in the central region of the crystal. The electric-field amplitude of the mode is given by equation (12). We consider slightly asymmetric position of the defect mode in the crystal (the reason is explained below). In Figure 5 we present plots of the final atomic populations versus velocities v_B and v_C while v_A is fixed at the value 500 m s^{-1}. These plots show that by adjusting the atomic velocities we can obtain required probabilities such that in the final state (2) the probability amplitude $\gamma(\tau)$ is equal to zero, which means that the defect mode is decoupled from the atomic system. The atoms are then in a pure superposition state. In particular, if we select the velocities as $v_B = 536.4$ m s^{-1} and $v_C = 527.4$ m s^{-1}, we obtain a final state with equal probabilities $|a(\tau)|^2 = |b(\tau)|^2 = |c(\tau)|^2 \approx 0.33$ (see Fig. 6), while $|\gamma(\tau)| \simeq 0.02$. It means that the atomic subsystem is in a good approximation decoupled from the field subsystem.
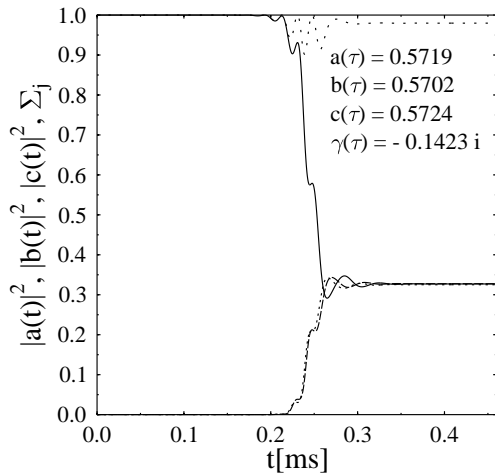
Fig. 6. The time evolution of the atomic populations $|a(t)|^2$ (solid line), $|b(t)|^2$ (long dashed line), and $|c(t)|^2$ (short dashed line) for specially chosen velocities (see Sect. 4). All other parameters are the same as in Figure 5. We plot also the sum $|a(t)|^2 + |b(t)|^2 + |c(t)|^2$ (dotted line) which is close to unity.

We have chosen an asymmetric position of the defect mode with respect to the center of the crystal because for the symmetric position we were able to obtain the "symmetric" result $|a(\tau)|^2 = |b(\tau)|^2 = |c(\tau)|^2 \approx 0.33$ only when two of the velocities are equal. In this case we face the problem that the atoms can collide. We expect that a better choice of the defect geometry might produce a final state more disentangled from the field as is the case presented in Figure 6.

We see from Figures 5 that the final atomic populations are rather robust with respect to changes in the velocities, *i.e.* velocity fluctuations (which in experiments can be reduced up to 0.4 m s^{-1} [11]) do not deteriorate the predicted entanglement.

5 Conclusions

In this paper we have shown that atoms can be entangled in the photonic crystals *via* the dipole interaction mediated through the off-resonant modes or *via* the interaction with a single defect mode. In the first mechanism (RDDI) the atoms can coherently exchange excitation while only a very small part of this energy is radiated into the field. However, this interaction might not be easy to control in an experiment because it requires a precise control of the position of the atoms. The second mechanism (*via* a single resonant defect mode) is experimentally more promising because it can be realized with currently available microwave photonic crystals and with highly excited Rydberg atoms.

We have shown that atoms can be prepared in pure entangled states and that the probability amplitudes of the generated superposition states of the atoms can be coherently controlled by varying the velocities of the atoms or by varying the orientations of the atomic dipoles. In our

scheme of entanglement the distance between the entangled atoms is given by linear dimensions of the crystal providing the lifetime of the Rydberg states is large enough to fly through the crystal. For the parameters used in this paper the distance between the entangled atoms is of the order of tens of centimeters.

Our scheme for entangling atoms can also be realized in metallic cavities. In fact, Cirac and Zoller proposed a similar scheme [9] which has been experimentally realized [11]. In this paper, however, we consider that two (or three) atoms interact with the field simultaneously while in the other schemes (see [9]) the atoms fly through a cavity sequentially. There are other differences between conditions of similar experiments in cavities and in microwave photonic crystals. In particular, the shape of the modal function of a defect mode is different from a cavity-mode profile. The defect-mode shape can be relatively easily designed and tuned in a photonic crystal by adding or removing a piece of material of the crystal. While inside a cavity there are several modes which can interact with the microwave atomic transitions, a photonic crystal can be designed with only a single defect mode inside the gap. Given the size of the microwave photonic crystals, the distance between entangled atoms can be larger than in the case of microcavities. Nevertheless, there is a serious problem with thermal effects when studying atomic transitions at microwave frequencies. To avoid the influence of thermal photons one has to consider the photonic crystal to be cooled down to very low temperatures (< 10 K). In spite of this problem we believe that the investigation of dynamics of Rydberg atoms in photonic crystals might provide us with new possibilities how to manipulate with atomic states and eventually how to create multiparticle entangled states of atoms.

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