

Optimal Quantum Clocks

V. Bužek,^{1,2} R. Derka,² and S. Massar³

¹*Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia*

²*Faculty of Mathematics and Physics, Comenius University, Mlynská dolina F2, Bratislava, Slovakia*

³*Service de Physique Théorique, Université Libre de Bruxelles, CP 225, Bvd du Triomphe, B1050 Brussels, Belgium*

(Received 17 August 1998)

A quantum clock must satisfy two basic constraints. The first is a bound on the time resolution of the clock given by the difference between its maximum and minimum energy eigenvalues. The second follows from Holevo's bound on how much classical information can be encoded in a quantum system. We show that asymptotically, as the dimension of the Hilbert space of the clock tends to infinity, both constraints can be satisfied simultaneously. The experimental realization of such an optimal quantum clock using trapped ions is discussed. [S0031-9007(99)08612-3]

PACS numbers: 03.67.-a, 03.65.Bz, 42.50.Ar

Recent technical advances in the laser cooling and trapping of ions suggest that coherent manipulations of trapped ions will be performed in the not too far future [1]. Apart from various important applications such as quantum information processing or improving high-precision spectroscopy these techniques also allow us to test fundamental concepts of quantum theory. In particular, much deeper insight into the problem of quantum measurement can be obtained.

In this Letter we study the problem of building an optimal quantum clock from an ensemble of N ions. To be specific let us assume an ion trap with N two-level ions all in the ground state $|\Psi\rangle = |0\rangle \otimes \dots \otimes |0\rangle$. This state is an eigenstate of the free Hamiltonian and thus cannot record time. Therefore the first step in building a clock is to bring the system to an initial state $\hat{\Omega}$ which is not an energy eigenstate. For instance one can apply a Ramsey pulse whose shape and duration is chosen such that it puts all the ions in the product state

$$\hat{\Omega}_{\text{prod}} = \hat{\rho} \otimes \dots \otimes \hat{\rho};$$

$$\hat{\rho} = \frac{1}{2} (|0\rangle + |1\rangle)(\langle 0| + \langle 1|). \quad (1)$$

We shall also consider more general states, but shall always take them to belong to the symmetric subspace of the N ions. The basis vectors of this space will be denoted $|m\rangle$, $m = 0, 1, \dots, N$. They are the completely symmetrized states of N two-level ions with m ions in the excited state and $(N - m)$ ions in the ground state. The states $|m\rangle$ have energy $E_m = m$ (this defines our unit of energy, setting $\hbar = 1$ then defines our unit of time).

The reason we can restrict ourselves to the symmetric subspace is that we can map any clock state onto the symmetric subspace without affecting its dynamics. Indeed consider an initial state $\Omega = |\psi\rangle\langle\psi|$ that does not belong to the symmetric subspace of the atoms. We can decompose $|\psi\rangle = \sum_m \sum_{\alpha} c_{m\alpha} |m, \alpha\rangle$ where $|m, \alpha\rangle$, $\alpha = 1, \dots, \binom{N}{m}$ denote a basis of the states with energy m . Consider the unitary operator \hat{U} that maps the state

$|\psi\rangle$ onto the symmetric subspace without changing its energy: $\hat{U} \sum_{\alpha} c_{m\alpha} |m, \alpha\rangle = c_m |m\rangle$ where $|m\rangle$ is as before the symmetric state with energy m . Since \hat{U} commutes with the Hamiltonian the performance of the clock based on $|\psi\rangle$ is identical to the clock whose initial state is the symmetric state $|\psi_{\text{sym}}\rangle = \hat{U}|\psi\rangle$.

After the preparation stage, the ions evolve in time according to the Hamiltonian evolution $\hat{\Omega}(t) = \hat{U}(t)\hat{\Omega}\hat{U}^{\dagger}(t)$, $\hat{U}(t) = \exp\{-it\hat{H}\}$. The task is to determine the elapsed time t by carrying out a measurement on the ions. Note that because of the indeterminism of quantum mechanics it is impossible, given a *single* set of N two-level ions, to determine the elapsed time with certainty. The best we can do is to *estimate* the elapsed time based on the result of a measurement on the system [2].

Making a good quantum clock requires a double optimization. First of all one can optimize the measurement. This aspect has been studied in detail in [2] where the best measuring strategy was derived. But one can also optimize the initial state $\hat{\Omega}$ of the system. It is this second optimization that is studied in this Letter.

Before turning to the problem of optimizing the initial state $\hat{\Omega}$, it is instructive to review the fundamental limitations on the performance of quantum clocks. Let us first consider a simple classical clock that can then be generalized to the quantum case. Our classical clock consists of a set of n registers. Each register is either in the 0 or the 1 state. Thus the classical clock consists of n bits, and can be in 2^n different states. The dynamics of the clock is as follows: the first register flips from 0 to 1 or from 1 to 0 every $2\pi 2^{-n}$, the second register flips every $2\pi 2^{-n+1}$, etc. The last register flips every π . This clock thus measures time modulo 2π . Note that this clock has an inherent uncertainty since it cannot measure time with a precision better than $2\pi 2^{-n}$. Throughout this Letter the time uncertainty is defined as

$$\Delta t^2 = \overline{[(t_{\text{estimate}} - t_{\text{true}}) \pmod{2\pi}]^2}. \quad (2)$$

For the classical clock $\Delta t_{\text{class}} = \pi 2^{-n} / \sqrt{3}$.

It is straightforward to replace the classical clock by a quantum version. The quantum clock consists of n two-level systems (qubits). The first qubit has an energy splitting between the levels of 2^{n-1} so that it has the same period as the corresponding classical register. The second qubit has an energy splitting of 2^{n-2} and so on up to the last qubit that has an energy splitting of 1. Considered together these n qubits constitute a quantum system with 2^n equally spaced energy levels.

The mapping between the Hilbert space of this abstract quantum clock and the symmetric subspace of N two-level ions is straightforward when $N + 1 = 2^n$. Indeed in this case the dimension and energy spectrum of both Hilbert spaces coincide. Note, however, that this is not a mapping between the qubits of the clock and the ions individually, but between energy eigenstates. This comparison between classical and quantum clocks suggests that a quantum clock built out of N ions cannot behave better than a classical clock built out of $\ln(N + 1)$ registers. That this is indeed the case follows from two fundamental constraints:

The first constraint is a bound on the time resolution of the clock that results from its energy spectrum. Indeed the time-energy uncertainty [3,4]

$$\Delta t \Delta E \geq \frac{1}{2}, \quad (3)$$

where $\Delta E^2 = \text{Tr}(\hat{H}^2 \hat{\Omega}) - [\text{Tr}(\hat{H} \hat{\Omega})]^2$ relates the uncertainty in the estimated time [defined by Eq.(2)] to the spread in energy of the clock. In the present case there is a state with a maximum energy uncertainty, namely $|\psi_+\rangle = (1/\sqrt{2})(|N\rangle + |0\rangle)$ for which $\Delta E = N/2$. Inserting in Eq. (3) shows that for any clock built out of N ions

$$\Delta t \geq \frac{1}{N}. \quad (4)$$

Note that one cannot attain equality in Eq. (3). Indeed the state $|\psi_+\rangle$ evolves with a period $2\pi/N$, hence it necessarily has a large time uncertainty. Thus Eq. (4) is only a lower bound and maximizing the energy uncertainty as in [5] is not necessarily an optimal procedure.

The second fundamental constraint the clock must obey is a bound on the total information it can carry. Indeed Holevo [6] has shown that one cannot encode and subsequently retrieve reliably more than n bits of classical information into n qubits. Letting a clock evolve for a given time interval t can be viewed as trying to encode information about the classical variable t into the state of the clock. Hence a measurement on our model clock [consisting of $n = \ln(N + 1)$ qubits] cannot retrieve more than $\ln(N + 1)$ bits of information about t .

Together these two bounds imply that the quantum clock cannot perform better than the corresponding classical clock: it cannot carry more information and it cannot

have better resolution. But is it possible, by making an optimal measurement and choosing in an astute manner the initial state of the clock, to make a quantum clock with similar performances to the classical one? Our main result is to show that this is indeed the case for clocks built out of symmetric states of N ions and to provide an algorithm for constructing such an optimal clock.

The problem of constructing quantum clocks has been considered previously in [7,8]. However, the best clocks considered in these papers are based on the phase state $|\Psi_0\rangle$ described below. As we shall see for these clocks the time uncertainty is very large $\Delta t \simeq (1/\sqrt{n})$ and is very far from reaching equality in Eq. (4). Recently, Vaidman and Belkind [9] considered the problem of a clock for which equality holds in Eq. (3). They showed that in the limit of large N the product states satisfy this condition. However, for the product state the energy uncertainty is very small: $\Delta E = \sqrt{N}/2$ hence they also do not saturate Eq. (4). Furthermore, clocks based on product state also do not attain Holevo's bound.

A similar approach to the one used here, namely optimizing both the initial state and the measurement on a system of N ions was considered in [10] with the aim of using the ions as an improved frequency standard. This problem can be rephrased in the following way: one disposes of a classical but noisy clock which provides some *a priori* knowledge about the time t and one wants to improve the knowledge of t by using the N ions. On the other hand, in the present Letter we suppose that there is no prior knowledge about t . The other difference with the present work is that our aim is to study the fundamental structure of quantum mechanics. We therefore neglect the effect of noise during the preparation and measurement stages and decoherence during the evolution. On the other hand, taking these effects into account was central to [10].

In order to find how to build an optimal clock we must delve in detail into the functioning of this device. We first recall Holevo's results concerning the optimal measurement strategy [2]. The measurement is described by a positive operator measurement (POVM), that is a set $\{\hat{O}_r\}_{r=1}^R$ of positive Hermitian operators such that $\sum_r \hat{O}_r = 1$. To each outcome r of the measurement we associate an estimate t_r of the time elapsed. The difference between the estimated time t_r and the true time t is measured by a cost function $f(t_r - t)$. Here we note that because of the periodicity of the clock, f has to be periodic. We also take $f(t)$ to be an even function. The task is to minimize the mean value of the cost function

$$\bar{f} = \sum_r \int_0^{2\pi} \text{Tr}[\hat{O}_r \hat{\Omega}(t)] f(t_r - t) \frac{dt}{2\pi}. \quad (5)$$

To proceed we expand the cost function in Fourier series:

$$f(t) = w_0 - \sum_{k=1}^{\infty} w_k \cos kt. \quad (6)$$

The essential hypothesis made by Holevo is positivity of the Fourier coefficients: $w_k \geq 0$, ($k = 1, 2, \dots$). He also supposes that the initial state $\hat{\Omega} = |\omega\rangle\langle\omega|$, $|\omega\rangle = \sum_m a_m |m\rangle$ is a pure state (and makes a phase convention such that a_m is real and positive). He then shows that

$$\bar{f} \geq w_0 - \frac{1}{2} \sum_{k=1}^{\infty} w_k \sum_{\substack{m, m' \\ |m-m'|=k}} a_m a_{m'}, \quad (7)$$

and equality is attained only if the measurement is of the form

$$\hat{O}_r = p_r |\Psi_r\rangle\langle\Psi_r|; \quad p_r \geq 0; \quad |\Psi_r\rangle = e^{it_r \hat{H}} |\Psi_0\rangle, \\ |\Psi_0\rangle = \frac{1}{\sqrt{N+1}} \sum_{m=0}^N |m\rangle, \quad (8)$$

with the completeness relation $\sum_r \hat{O}_r = \hat{1}$.

Several remarks about this result are called for:

(1) Holevo supposed that the initial state is a pure state. If the initial state is mixed, $\hat{\Omega} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, then one finds that the corresponding cost is bounded by the average of the bounds Eq. (7). This shows that in building a good quantum clock one should always take the initial state to be pure.

(2) Holevo considered covariant measurements in which the times t_r takes the continuum of values between 0 and 2π . But as shown in [11] the completeness relation can also be satisfied by taking a discreet set of times $t_j = \frac{2\pi j}{N+1}$, $j = 0, \dots, N$. These ‘‘phase’’ states $|\Psi_j\rangle$ [12] form an orthonormal basis of the Hilbert space, and this measurement is therefore a von Neumann measurement. Thus it is not necessary to use an ancilla to make the optimal measurement in this case.

(3) In Eq. (7) only the values $k = 0, \dots, N$ intervene because of the condition $|m - m'| = k$. That is, only the first $N + 1$ Fourier coefficients of the cost function are meaningful.

(4) Because of the condition of positivity of w_k , not all cost functions are covered by this result, but several important examples are $4 \sin^2 \frac{t}{2} = 2(1 - \cos t)$, $|t \pmod{2\pi}|$, $|\sin \frac{t}{2}|$, $-\delta[t \pmod{2\pi}]$. The most notable absence from this list is the quadratic deviation t^2 [as defined in Eq. (2)] but it can be well approximated by the first cost function since $4 \sin^2 \frac{t}{2} \approx t^2$ for $|t| \ll \pi$.

We now turn back to the central problem of this Letter, namely how to optimize the initial state of the system so that the time estimate is as good as possible. This corresponds to minimizing the right-hand side of Eq. (7) with respect to a_m . To this end let us define the matrix

$$F_{mm'} = w_0 \delta_{m,m'} - \frac{1}{2} \sum_{k=1}^N w_k (\delta_{m,m'+k} + \delta_{m+k,m'}). \quad (9)$$

We must then minimize $\bar{f} = a^T \hat{F} a$ under the condition $a^T a = 1$. Using a Lagrange multiplier we find the eigenvalue equation $(\hat{F} - \lambda \hat{1})a = 0$, and the task is therefore to find the smallest eigenvalue and eigenvector of \hat{F} .

Let us first consider the cost function $f = 4 \sin^2 t/2$. The advantage of this cost function is that the matrix F is particularly simple in this case. Furthermore, for errors much smaller than 2π , f and Δt^2 as defined in Eq. (2) coincide. Hence the first constraint on clock resolution Eq. (4) can be approximately replaced by $\bar{f} \geq 1/N^2$.

If the initial state is the product state Eq. (1) then the mean cost is given by the expression $\bar{f} = 2[1 - 2^{-N} \sum_{i=0}^{N-1} \sqrt{\binom{N}{i} \binom{N}{i+1}}]$ which for large N decreases as $\bar{f} \approx 1/N$. If the initial state is the phase state $|\Psi_0\rangle$, then direct calculation shows that $\bar{f} = \frac{2}{N+1}$. Thus for both states $\Delta t \approx N^{-1/2}$ and one is very far from attaining equality in Eq. (4).

However, neither of these two states is optimal. To find the optimum we note that the matrix $\hat{F} = 2\delta_{mm'} - \delta_{mm'+1} - \delta_{m+1m'}$ can be viewed as the discretized second derivative operator $\hat{F} \approx -d^2/dx^2$ with von Neumann boundary conditions. The lowest eigenvalue of \hat{F} is therefore approximately $\lambda_{\min} \approx \pi^2/(N+1)^2$ and the corresponding eigenvector is

$$|\Psi_{\text{opt}}\rangle \approx \frac{\sqrt{2}}{\sqrt{N+1}} \sum_m \sin \frac{\pi(m+1/2)}{N+1} |m\rangle. \quad (10)$$

Thus in this case the cost decreases for large N as $\bar{f}_{\text{opt}} \approx \frac{\pi^2}{(N+1)^2}$ corresponding to $\Delta t_{\text{opt}} \approx \frac{\pi}{(N+1)}$. Therefore, up to a factor of π the optimal clock attains the bound Eq. (4).

It is also interesting to consider the situation where the cost function is the delta function $f = -\delta[t \pmod{2\pi}]$. In this case $F_{mm'} = -\frac{1}{2\pi}$ for all m, m' . One checks that the phase state $|\Psi_0\rangle$ is the eigenvector of \hat{F} with minimal eigenvalue $\lambda = -\frac{N}{2\pi}$. Note that one could also have taken a smeared delta function since only the first $N + 1$ terms intervene in Eq. (7). The smeared delta function is approximately zero everywhere except in an interval of about $1/N$ around zero where it is equal to N . Thus for this cost function one wants to maximize the frequency with which the estimated value of t is within about $\frac{1}{N}$ of the true value. But there is no extra cost if the estimated value is very far from the true one. It is for this reason that taking $|\Psi_0\rangle$ as the initial state when the cost function is $4 \sin^2 t/2$ is bad since making estimates that are wildly off is strongly penalized in that case.

The mean value of a cost function gives only partial information about the sensitivity of a clock. The full information is encoded in the probability $P(t|t_r) = P(\hat{O}_r | t) P(t) / P(\hat{O}_r) = P(\hat{O}_r | t) \frac{N+1}{2\pi}$ that the true time is t given that the readout of the measurement is t_r . In the case of the optimal state $|\Psi_{\text{opt}}\rangle$, one finds that

$$P_{\text{opt}}(t|t_r) \\ \approx \mathcal{N} \frac{[1 + \cos(N+1)T](1 + \cos T)}{[1 - \cos(T + \frac{\pi}{N+1})][1 - \cos(T - \frac{\pi}{N+1})]}, \quad (11)$$

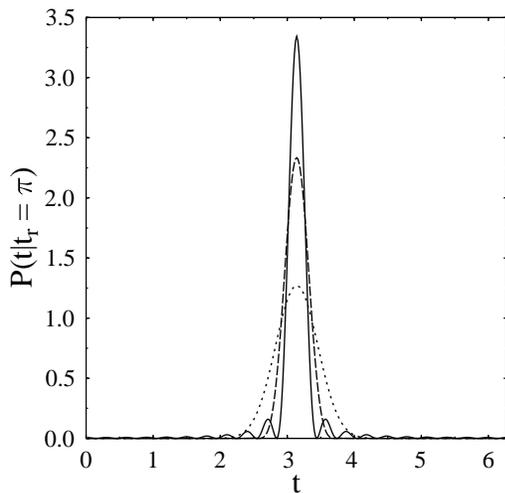


FIG. 1. A plot of the *a posteriori* probability $P(t|t_r)$ that the time was t given that the measurement yielded outcome $t_r = \pi$ for different initial states ($N = 20$). The dotted line corresponds to the product state Eq. (1), the solid line to the phase state $|\Psi_0\rangle$ (8), and the dashed line to the optimal state $|\Psi_{\text{opt}}\rangle$ (10).

where $\mathcal{N} \simeq \frac{\pi}{4(N+1)^3}$ and $T = t - t_r$. This distribution (see Fig. 1) has a central peak of width $3\pi/(N+1)$ and tails which decrease for $2\pi \gg |t - t_r| \gg N^{-1}$ as $P_{\text{opt}}(t|t_r) \simeq N^{-3}|t - t_r|^{-4}$.

For the phase state $|\Psi_0\rangle$ one finds

$$P_{|\Psi_0\rangle}(t|t_r) = \frac{1}{2\pi(N+1)} \frac{[1 - \cos(N+1)T]}{(1 - \cos T)}. \quad (12)$$

This distribution has a slightly tighter central peak of width $2\pi/(N+1)$ but the tails of the distribution decrease as $N^{-1}|t - t_r|^{-2}$. It is these slowly decreasing tails that give the main contribution to $\Delta t \simeq N^{-1/2}$.

For the product state Eq. (1) the distribution $P_{\text{prod}}(t|t_r)$ has a very wide central peak of width $\simeq \frac{1}{\sqrt{N}}$. In this case it is the wide central peak that gives rise to the large time uncertainty.

Using these distributions it is possible to calculate the number of bits of information about time that is encoded in the outcomes of the measurement. This is given by the mutual information

$$I = - \int dt P(t) \ln P(t) + \sum_r P(t_r) \int dt P(t|t_r) \ln P(t|t_r). \quad (13)$$

In all cases the integral in the second term is dominated by the central peak. Thus one finds that for the product state only $\frac{1}{2} \ln N$ bits of information about time are obtained, whereas for both the states $|\Psi_0\rangle$ and $|\Psi_{\text{opt}}\rangle$ one obtains approximately $\ln N$ bits, thereby saturating Holevo's bound.

In summary, we have seen that for different cost functions there are different optimal clocks. There is, of course, no cost function that is in an absolute sense better than another, and the choice of a particular cost function depends on the physical context. Nevertheless, the experimental realization of a quantum clock based on the state $|\Psi_{\text{opt}}\rangle$ seems particularly desirable because it combines the attractive features that the *a posteriori* probability $P(t|t_r)$ has a tight central peak and rapidly decreasing tails.

Carrying out such an experiment with trapped ions presents two main difficulties. The first is the preparation of the initial state $\hat{\Omega}$. Such coherent manipulation of trapped ions is one of the current experimental challenges. A possibly important simplifying feature of the quantum clock is that since it is symmetric in the N ions one does not need to address each ion individually. The second problem is to realize the optimal phase measurement as discussed in the present Letter. Recent experiments [13] suggest that this type of coherent preparation and specific projective measurements are possible for systems with a moderate number of trapped ions. Therefore, one may hope that it will be feasible to make optimal quantum clocks in the not too distant future.

Our special thanks to Lev Vaidman for his criticism and careful rereading of the manuscript. We also thank Win van Dam, Susana Huelga, Peter Knight, and Christopher Monroe for helpful discussions. This work was in part supported by the Royal Society and by the Slovak Academy of Sciences. V.B. and S.M. thank the Benasque Center for Physics where part of this work was carried out. S.M. would like to thank Utrecht University where most of this work was carried out; he is a chercheur qualifié du FNRS.

-
- [1] D. J. Wineland *et al.*, quant-ph/9710025.
 - [2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
 - [3] L. Mandelstam and I. E. Tamm, *Izv. Akad. Nauk. SSSR Fiz.* **9**, 122 (1945); *J. Phys. (Moscow)* **9**, 249 (1945).
 - [4] S. L. Braunstein and C. M. Caves, *Phys. Rev. Lett.* **72**, 3439 (1994); S. L. Braunstein, C. M. Caves, and G. J. Milburn, *Ann. Phys.* **247**, 135 (1996).
 - [5] J. J. Bollinger *et al.*, *Phys. Rev. A* **54**, R4649 (1996).
 - [6] A. S. Holevo, *Probl. Pereda. Inf.* **9**, 3 (1973) [*Probl. Inf. Transm.* **9**, 177 (1973)].
 - [7] H. Salecker and E. P. Wigner, *Phys. Rev.* **109**, 571 (1958).
 - [8] A. Peres, *Am. J. Phys.* **48**, 552 (1980).
 - [9] L. Vaidman (private communication); O. Belkind, M.Sc. thesis, Tel Aviv University, 1998.
 - [10] S. F. Huelga *et al.*, *Phys. Rev. Lett.* **79**, 3865 (1997).
 - [11] R. Derka, V. Bužek, and A. K. Ekert, *Phys. Rev. Lett.* **80**, 1571 (1998).
 - [12] D. T. Pegg and S. M. Barnett, *Europhys. Lett.* **6**, 483 (1988).
 - [13] Q. A. Turchette *et al.*, *Phys. Rev. Lett.* **81**, 3631 (1998).