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# OPTIMAL ENCODING OF PHASE INFORMATION IN TRAPPED IONS *

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We propose a specific preparation of a set of $N$ two-level ions (i.e., the collective state $\hat{\Omega}$ of these ions) in which the information about the phase can be most optimally encoded. The phase $\psi$ in this context refers to a continuous parameter $\psi \in \langle 0, 2\pi \rangle$ which labels different points along the evolution path of a given state $\hat{\Omega}$ of the $N$-ion system described by a unitary one-parametric group $U(\psi) = \exp\{-i\psi\hat{H}\}$. The optimality of the preparation of the given quantum system means that using the optimal measurement scheme the phase $\psi$ encoded in the system of $N$ two-level ions can be "extracted" (decoded) with the highest mean fidelity.

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Recent technical advances in laser cooling and trapping (for a detailed review see the report by Wineland et al. [1]) improved limits with which coherent manipulations with trapped ions can be performed. Apart of various interesting physical and technical applications (such as, quantum information processing, or improving the high-precision spectroscopy) these techniques also allow us to test fundamental concepts of the quantum theory. In particular, much deeper insight into the problem of quantum statistical model of measurement can be obtained.

In this paper we study the problem of the optimal phase estimation from an ensemble of $N$ atomic ions. Specifically, we show that there exists the state of the set of $N$ ions in which the phase information can be most optimally encoded. To understand the problem more clearly, we remind us that the indeterminism of the quantum statistical model of the measurement implies that having just a *single* set of $N$ two-level ions there is no measurement by means of which the state of the system can be identified precisely (providing we have no *a priori* knowledge about the preparation of the state). The best we can do is to *estimate* the state of the quantum system based on the measured data [2, 3]. At this point we can ask a question what is the most optimal measurement which provides us with the best estimation of the state of the measured system. In our previous work [4] we have addressed this problem. In particular, we have proposed an

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algorithm for a construction of the finite positive-operator-valued-measure (POVM) measurement, which provides the optimal (best) estimation of certain parameters associated with states of quantum systems. More precisely, these parameters correspond to given symmetry groups.

As an example, one can assume the position measurement which is connected with the group of translations, or the measurement of an angle of the orientation associated with the group of rotations. In this last case one estimates the continuous parameter $\psi \in \langle 0, 2\pi\rangle$, which parameterizes the group of rotations U(1). To be specific, following Wineland et al. [1] (see also [5]) let us assume an ion trap with $N$ two-level ions initially prepared in the ground internal state

$$|\Psi\rangle = |0\rangle \otimes \ldots \otimes |0\rangle. \tag{1}$$

A Ramsey pulse of the frequency $\omega$ is applied to this collection of ions in the trap. The pulse shape and its duration is chosen so that it drives the atomic transition $|0\rangle \leftrightarrow |1\rangle$ of the natural frequency $\omega_0$ and as a result, each of the ion is prepared in the superposition state $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, that is the system of $N$ ions is in the state $\hat{\Omega}$

$$\hat{\Omega} = \hat{\rho} \otimes \ldots \otimes \hat{\rho}, \tag{2}$$

where

$$\hat{\rho} = \frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right). \tag{3}$$

Next the system evolves in time along the path generated by the unitary transformation $U(\psi) = \exp\{-i\psi\hat{H}\}$ where the Hamiltonian for each single ion is given by the relation $\hat{H}_j = E_0|0\rangle_j\langle 0| + E_1|1\rangle_j\langle 1|$ (the subscript $j$ indicates the $j$-th ion). The task is to estimate the phase $\psi$ "encoded" in the state $U(\psi)^N\hat{\Omega}U^{N\dagger}(\psi)$. In Ref. [4] we have shown how the optimal measurement for the best estimation of the phase $\psi$ can be constructed (see below). The question which we want to address in the present paper is how to choose the "reference" state $\hat{\Omega}$, so that the information about the phase is most optimally encoded into the ionic system. As we will show later the reference state (2) is not the optimal one. In what follows we will present the most optimal reference state $\hat{\Omega}$.

To proceed we firstly note that the family of operators $U(\psi)$ composes a particular unitary representation of the U(1) group whereas the initial (reference) state $\hat{\Omega}$ specifies the particular orbit in the space where this representation is acting. In our particular case the representation corresponds to a direct sum of irreducible representations of the form $(0)\oplus(1)\oplus\ldots(N)$ which act in the $N+1$ dimensional space $\mathcal{H}_+^{\otimes N}$ spanned by basis vectors $|m\rangle$, $m = 0, 1, \ldots N$. These basis vectors are completely symmetrized states of $N$ two-level ions with $m$ ions in the ground state and $(N-m)$ ions in the upper level. Obviously, the total Hamiltonian is invariant under an arbitrary permutation of the ions, therefore it always effectively acts in the $(N+1)$ dimensional space of the representation considered above.

*Optimal estimation*

As shown in the detail in Ref. [4] our task is to design the most general POVM, mathematically described as a set $\{\hat{O}_r\}_{r=1}^R$ of positive Hermitian operators such that $\sum_r \hat{O}_r = \hat{1}$ [6, 3], which when applied to the system of *all* $N$ ions provides us with the best possible estimation of the parameter $\psi$ associated with the rotation $U(\psi)$. We quantify the quality of the state (parameter

$\psi$) estimation in terms of the *mean* fidelity [2] which in the case of the reducible representation under consideration can be written as

$$\bar{f} = \sum_r \int_0^{2\pi} \mathbf{Tr}[\hat{O}_r\, U^N(\psi)\hat{\Omega}U^{N\dagger}(\psi)] f(\psi - \psi_r)\frac{d\psi}{2\pi}, \tag{4}$$

with the cost function $f(\psi - \psi_r)$ chosen as [2, 4]

$$f(\psi - \psi_r) = \mathbf{Tr}[U(\psi)\hat{\rho}U^\dagger(\psi)\; U(\psi_r)\hat{\rho}U^\dagger(\psi_r)] = \cos^2\frac{\Delta\psi}{2}, \tag{5}$$

where $\Delta\psi = \psi - \psi_r$ and the single-ion state $\hat{\rho}$ is given by Eq. (3). In Eq. (4) $U^N(\psi)$ is a new representation of the same group $U(\psi)$ equivalent to the $N$-fold symmetrized direct product [7] of the original representation $U(\psi)$. Here $U^N(\psi)$ transforms the $(N+1)$-dimensional reference state $\hat{\Omega}$ and $\psi_r$ represents our estimation of the phase when the result $O_r$ has been measured. In order to specify the most optimal measurement we have to determine the operators $\hat{O}_r$ such that the mean fidelity (4) is maximized.

As shown in Ref. [4] the optimal phase estimation scheme corresponds to the von Neumann measurement specified by the set of mutually orthogonal projectors

$$\hat{P}_j = |\Psi_j\rangle\langle\Psi_j|, \tag{6}$$

where the "phase states" $|\Psi_j\rangle$ are specified as

$$|\Psi_j\rangle = \frac{1}{\sqrt{N+1}}\sum_{m=0}^{N} \mathbf{e}^{i\theta_j m}\; |m\rangle, \tag{7}$$

and $|m\rangle$ are the completely symmetric states of ions. The parameter $\theta_j$ reads

$$\theta_j = \frac{2\pi}{N+1} j. \tag{8}$$

It is interesting to note, that when forming from the optimal POVM (6) the Hermitian operator

$$\hat{\Phi} = \sum_{j=0}^{N} \theta_j \hat{P}_j, \tag{9}$$

(the guesses prescribed to different measurement results are in the place of corresponding eigenvalues) we obtain exactly the Pegg-Barnett Hermitian phase operator [8]. From Eqs. (6) and (7) it follows that

$$\hat{\Phi}|\Psi_j\rangle = \theta_j|\Psi_j\rangle, \tag{10}$$

which is the reason why one might call $|\Psi_j\rangle$ as the phase states.

We remind us that the optimal POVM (6) has been established in two steps: First the solution of the following system of equations was found

$$\left[\hat{F} - \hat{L}\right]|\Psi\rangle = 0; \qquad |\langle m|\Psi\rangle|^2 = 1; \quad \forall m, \tag{11}$$

where the operator $\hat{F}$ is calculated as

$$\begin{aligned}\hat{F}_{m,n} &= \int_0^{2\pi} \frac{d\psi}{2\pi} \frac{[\hat{\Omega}]_{m,n}}{2} \,\mathrm{e}^{i\psi(n-m)} \,(1+\cos\psi) \\ &= \frac{[\hat{\Omega}]_{m,n}}{4} \left(2\delta_{m,n} + \delta_{m,n+1} + \delta_{m+1,n}\right), \end{aligned} \tag{12}$$

and reflects the structure of the reference state $\hat{\Omega}$ into which during the evolution the phase $\psi$ is encoded. Second, the optimal POVM (6) was uniquely constructed from the solution for $|\Psi\rangle$ determined by the system of equations (11). The mean fidelity $\bar{f}$ quantifying the quality of the estimation was given as the trace $\mathrm{Tr}[\hat{L}]$ of the diagonal operator $\hat{L}$.

In the particular case when the initial (reference) state $\hat{\Omega}$ has the factorized form (2), i.e. its matrix elements read

$$[\hat{\Omega}]_{mn} = \frac{\sqrt{\binom{N}{N-m}\binom{N}{N-n}}}{2^N}, \tag{13}$$

the optimal mean fidelity $\bar{f}$ is given as the sum

$$\bar{f} = 1/2 + 1/2^{N+1} \sum_{i=0}^{N-1} \sqrt{\binom{N}{i}\binom{N}{i+1}} \tag{14}$$

(see the dot-dashed line in Fig. 1). We note that the optimality of the measurement scheme (6) is universal and independent of the choice of the reference state $\hat{\Omega} \in \mathcal{H}_+^{\otimes N}$. Obviously, the particular value of the mean fidelity depends on $\hat{\Omega}$.

*Optimal reference states*

Now it is legitimate to ask another question: Which state one has to select as the reference state $\hat{\Omega}$, if he wants to use it as an ideal carrier of the information about $\psi$? That is, given the number $N$ of trapped ions which fixes the form $(0) \oplus (1) \oplus \ldots (N)$ of the representation of the group U(1), which state $\hat{\Omega}$ provides us with the highest mean fidelity $\bar{f}$ of the subsequent optimal estimation.

To answer this question we have to assume the most general pure state $|\alpha\rangle = \sum_{m=0}^N \alpha_m |m\rangle$ in the $(N+1)$-dimensional Hilbert space $\mathcal{H}_+^{\otimes N}$, with the corresponding density matrix $\hat{\Omega}$ specified by its matrix elements $[\hat{\Omega}]_{mn} = \alpha_n \alpha_m^*$. The goal is to determine the numbers $\alpha_n$ so that the solution of the system of Eqs. (11) and (12) provides the maximal mean fidelity $\bar{f} = \mathrm{Tr}[\hat{L}]$. First we note, that there is no need to assume that $\alpha_n$ are complex. One can always make them real by redefining the basis vectors $|m\rangle$. This redefinition would have no influence on any of the previous conclusions. It is straightforward to formulate mathematically the problem as the maximization of the mean fidelity $\bar{f}(\vec{\alpha}) = \frac{1}{2}(1 + \sum_{i=0}^{N-1} \alpha_i \alpha_{i+1})$ with the normalization condition $\sum_{i=0}^N \alpha_i^2 = c(\vec{\alpha}) = 1$. This leads to a system of Lagrange equations $\frac{\partial}{\partial \alpha_i}(\bar{f} - \lambda c) = 0$. They can be explicitly written as

$$4\lambda \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}. \tag{15}$$

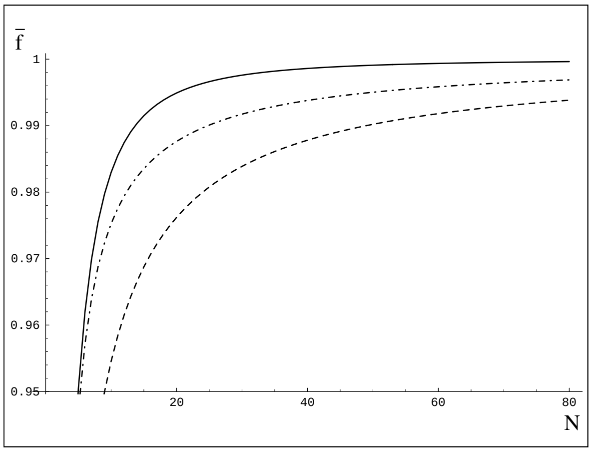


Fig. 1. We plot the mean fidelity $\bar{f}$ as a function of the number $N$ of the trapped ions. The dot-dashed line corresponds to estimation based on the non-entangled reference state given by Eq. (2), while the dashed line shows the mean fidelity for the phase state (7). In this particular case one can find a simple analytical expression, i.e. $\bar{f} = 1 - 1/[2(N+1)]$. The solid line corresponds to the optimal choice of the reference state obtained as the solution of Eq. (15).

Among the $N+1$ different solutions we select that eigenvector $\vec{\alpha}$ which corresponds to highest eigenvalue $\lambda_{\max}$. This is because we are looking for the global maximum of the mean fidelity which is given as

$$\bar{f} = \frac{1}{2} + \lambda_{\max}. \tag{16}$$

In Fig. 1 for comparison we plot the mean fidelity $\bar{f}$ of the estimation of the phase for three different reference states $\hat{\Omega}$: the product state (2), the phase state (7) and the optimal state given as the solution of Eq. (15). We see that even for a relatively small number of trapped ions the phase $\psi$ can be estimated with a very high fidelity. It is also interesting to note, that to reach a specific value of the fidelity close to unity one needs significantly more atoms prepared in the reference state (2) or the phase state (7) than when the optimal state (15) is considered.

In conclusion we would like to make three comments.

*Comment 1.*

The Lagrange equation (15) can be expressed as the eigenvalue equation of the form

$$4\lambda|\alpha\rangle = \left(\hat{A} + \hat{A}^{\dagger}\right)|\alpha\rangle, \tag{17}$$

where the operator $\hat{A}$ reads

$$\hat{A} = \sum_{m=1}^{N} |m-1\rangle\langle m|. \tag{18}$$

We note that the exponent of the phase operator $\hat{\Phi}$ (9) can be expressed as [9]

$$\exp(i\hat{\Phi}) = \sum_{m=1}^{N} |m-1\rangle\langle m| + |N\rangle\langle 0|. \tag{19}$$

We see that it differs from the operator $\hat{A}$ by the term $|N\rangle\langle 0|$. This difference results in the fact that the most optimal phase-information carrier in the case of $N$ trapped ions is not one of the phase states $|\Psi_j\rangle$ given by Eq. (7) [i.e. eigenstates of the phase operator $\hat{\Phi}$ (9)] but the state obtained as the solution of Eq. (15) with the maximum value of $\lambda$. Obviously, the reason is that we consider the phase $\psi$ to be a continuous parameter rather then a discrete parameter $\theta_j$ given by Eq. (8).

*Comment 2.*

For large enough number $N$ the mean fidelity $\bar{f}$ is close to unity and using the expression for the cost function (5) we can simply relate $\bar{f}$ to the mean variance $\overline{\Delta\psi}$ in the phase distribution, i.e. $\bar{f} = \cos^2(\overline{\Delta\psi}/2)$. The mean variance $\overline{\Delta\psi}$ itself can be associated with the precision with which the time estimation can be performed via the phase measurement described above. In this case the mean variance of the time distribution $\overline{\Delta t}$ is

$$\overline{\Delta t}^2 = \frac{(\overline{\Delta\psi})^2}{\omega^2} \simeq \frac{4(1-\bar{f})}{\omega^2}. \tag{20}$$

At this point we can derive a qualitative time-energy "uncertainty" relation which relates the maximal internal energy difference in the system of $N$ trapped ions, i.e. $\Delta E = N\hbar\omega$, and the mean variance in the time (phase) distribution:

$$U = (\Delta E)^2 \overline{\Delta t}^2 = 4(1-\bar{f})\,\hbar^2 N^2. \tag{21}$$

We plot this parameter as a function of $N$ in Fig. 2. From the figure it is seen that for the product state (2) and the phase state (7) the precision in the time estimation is scaled as $\overline{\Delta t} \simeq 1/\sqrt{N}$. On the other hand for the optimal reference state (15) we have $\overline{\Delta t} \simeq 1/N$ which is reflected by the fact that the corresponding uncertainty parameter $U$ (solid line in Fig. 2) is constant. This once again illustrates the efficiency of the phase encoding into the optimal reference state (15) and the optimality of the proposed phase measurement.

*Comment 3.*

For a practical realization of the presented optimal phase estimation with trapped ions one has to keep in mind two important points. Firstly, it is a problem of the preparation of the reference state $\hat{\Omega}$. The coherent manipulation (preparations) with trapped ions is one of the main tasks in trapped-ions experiments. The present state of the art is very promising [1, 10] and it can be expected that for systems with the moderate number of trapped ions this will be feasible and an arbitrary state $|\alpha\rangle$ in the Hilbert space $\mathcal{H}_+^{\otimes N}$ would be possible to prepare. Once this is possible then it is enough to leave the system to evolve freely so the phase can be accumulated (encoded). The other problem then is how to construct the projective measurement (6). We note that to perform this projective measurement it is necessary to know how to perform a projective measurement associated with the projectors $|m\rangle\langle m|$. That is, it is necessary that we are able to perform a measurement of the number of excited ions in the given state. Exactly this type of the measurement has recently been considered by Turchette et al. [11]. Obviously, there is still

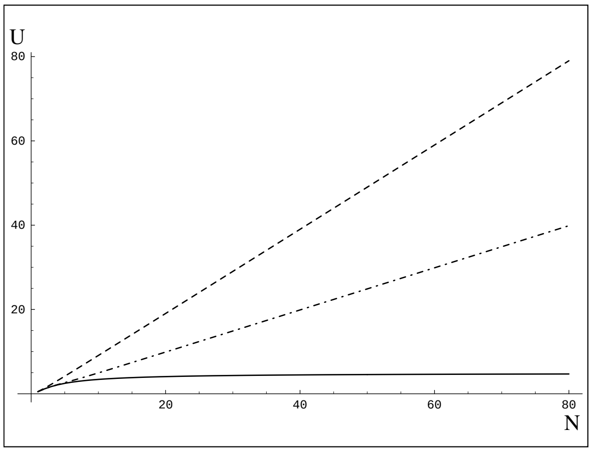


Fig. 2. We plot the uncertainty measure $U$ as a function of the number $N$ of the trapped ions. The dot-dashed line corresponds to estimation based on the non-entangled reference state given by Eq. (2), while the dashed line shows the mean fidelity for the phase state (7). The solid line corresponds to the optimal choice of the reference state obtained as the solution of Eq. (15). We see that in this case $U$ is approximately constant and does not depend on $N$ (we assume $\hbar = 1$).

a long way before the optimal estimation of phase "encoded" in a finite ensemble of trapped ions would be possible to perform. In this paper we have considered the optimal scenario for this particular task.

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