Quantum-state synthesis of multimode bosonic fields: Preparation of arbitrary states of two-dimensional vibrational motion of trapped ions

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We present a *universal* algorithm for an efficient *deterministic* preparation of an *arbitrary* two-mode bosonic state. In particular, we discuss in detail preparation of entangled states of a two-dimensional vibrational motion of a trapped ion via a sequence of laser-stimulated Raman transitions. Our formalism can be generalized for multimode bosonic fields. We examine stability of our algorithm with respect to a technical noise. [S1050-2947(98)04209-7]

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I. INTRODUCTION

The ability to control preparation and evolution of states of quantum systems opens new horizons in experimental physics (e.g., tests of fundamental concepts of quantum mechanics) as well as in potential technical applications (such as quantum information processing). Recent advances in quantum optics (micromasers, cavity QED) [1] and atomic physics (dynamics of trapped ions) [2] have demonstrated that a macroscopic observer can effectively control dynamics as well as perform a complete measurement of states of microscopic quantum systems. In particular, preparation of an *arbitrary* quantum state of a *single*-mode electromagnetic field in micromasers and one-dimensional (1D) vibrational motion of trapped ions have been discussed in literature [3–9]. Experimental realizations of highly nonclassical states, such as Fock states, squeezed states, or Schrödinger-cat states of these *single*-mode bosonic fields have been reported [1,9]. The ability to synthesize an arbitrary motional state is a key prerequisite for a quantum measurement of an *arbitrary* motional observable of a trapped ion as proposed by Gardiner, Cirac, and Zoller [10]. These authors have generalized the 1D synthesis of motional states to 2D and higher dimensions. In spite of the conceptual elegance, the method proposed by Gardiner, Cirac, and Zoller is difficult to implement in general. The problem is that the number of laser operations required for preparation of a two-mode target state $|\Psi_{\rm target}\rangle$ of the form

$$|\Psi_{\rm target}\rangle=\sum_{m=0}^{M_{\rm max}}\sum_{n=0}^{N_{\rm max}} Q_{mn}|m,n\rangle \tag{1.1}$$

depends *exponentially* on the dimensionality of the subspace of the Fock space in which the target state is embedded. In particular, if $N_{\rm max}=M_{\rm max}$ then the number of necessary operations is proportional to $2M_{\rm max}\times 2^{M_{\rm max}}$. This exponential dependence restricts applicability of the proposed method (see below). A novel approach which overcomes this obstacle was introduced very recently by Kneer and Law [11]. The authors have considered a photon-number-dependent interaction which is induced by a detuned standing-wave laser field in y direction. It keeps on resonance only transitions between Fock states $|m,n\rangle$ with a fixed number $n=n_y$ of trap quanta in y direction while applying sequentially the 1-D preparation scheme [8] in the x direction for each particular $n=n_y$.

In this paper we propose an alternative universal algorithm for construction (synthesis) of an *arbitrary* quantum state of the two-mode bosonic field which can be straightforwardly generalized for the preparation of an arbitrary multimode system. In our algorithm the number of operations required for a preparation of the two-mode state (1.1) grows only polynomially as a function of $N_{\rm max}$ and $M_{\rm max}$. In particular, if $N_{\rm max}=M_{\rm max}$, then the number of operations is proportional to $8M_{\rm max}^2$.

To make our discussion as close as possible to experimental realization we consider the preparation of states of the two-dimensional vibrational motion of a trapped ion [12]. The choice of the system is motivated by the fact that dissipative effects can be significantly suppressed in ion traps which is important for a *deterministic* engineering of quantum states. Moreover, the quantized vibration motion of a trapped ion can be effectively controlled by a proper sequence of laser pulses tuned either to the atomic electronic transition or to resolved vibrational sidebands [2,9].

The simplest quantum-state preparation is represented by a simple *unitary* evolution of an input state of the system governed by a specific (generally, nonlinear) Hamiltonian. Obviously, the fixed Hamiltonian restricts the family of target states which can be ''generated'' from available inputs. Another way to prepare states of the system of interest (e.g., single-mode bosonic field) is to consider quantum interaction between this system and an adjoint quantum system (e.g., fermions). Dynamics of these two systems is governed by a specific interaction Hamiltonian. The desired quantum-state engineering is then achieved by *conditional* measurements performed on the adjoint quantum system [4]. This *nonunitary* selection of specific quantum trajectories allows us to synthesize essentially all quantum states of the system under consideration. But there is a price to pay—the probability of the outcome of the given conditional process can be extremely small. To overcome this problem one may consider a sequence of interactions between the original and the adjoint systems. These interactions (channels) are governed by dif-

ferent Hamiltonians with just one channel turned on at a given time. Coupling constants in these Hamiltonians and times of duration of given interactions are in this case *free* parameters which can be appropriately *tuned* so that at the output the system of interest is disentangled from the adjoint system (see below) and is prepared in the desired state. This approach has recently been utilized by Law and Eberly [8] who have shown that by a suitable switching between two channels of the atom-field interaction one can generate an arbitrary state of the single-mode cavity field. It is important to note that quantum states of 1D vibrational motion of trapped ions are experimentally created also via a sequence of laser pulses tuned either to an electronic transition or an appropriate vibrational sideband [9], that is, sequential switching of different interaction channels is used. Generalizations of the 1D vibrational quantum-state synthesis to 2D and more dimensions have been discussed recently [10,11]. In what follows we introduce a *universal* scheme which enables deterministic preparation of an *arbitrary* two-mode target state via an appropriate switching between laser-stimulated Raman transitions described by single- and two-mode interaction Hamiltonians. The method can be generalized, e.g., for a trapped ion in a 3D trap potential and other multimode bosonic systems. Even for higher dimensions the number of preparation operations scales polynomially with the dimensionality of the subspace of the Fock space in which the target state is embedded.

The paper is organized as follows. In Sec. II we introduce our algorithm for 2D quantum-state synthesis. In Sec. III we discuss a physical realization of the preparation scheme. In Sec. IV we examine the stability of our algorithm with respect to a technical noise. Fidelity between outputs and specific target states is evaluated. We finish our paper with conclusions.

II. QUANTUM-STATE SYNTHESIS

Without any loss of generality let us assume that we want to generate an arbitrary two-mode state given as a finite superposition of number (Fock) states (1.1). To describe the preparation algorithm we first split the whole Hilbert space into an appropriate set of finite-dimensional subspaces labeled by a specific quantum number. Then we introduce two sorts of interaction channels. The first set of Hamiltonians will be responsible for dynamics within a given subspace, while the second one will realize a ''transfer'' of probability amplitudes between different subspaces. In particular, let us consider vibrational states of a trapped ion, confined in 2D harmonic potential. Excitations of two vibrational modes are described by creation and annihilation operators $\hat{a}_\mu$ and $\hat{a}_\mu^\dagger$ $(\mu=x,y)$. The Hilbert space $\mathcal{H}_{\rm vib}$ of all two-mode vibrational states can be divided to subspaces with constant total number of vibrational quanta, i.e., $\mathcal{H}_{\rm vib}=\oplus\mathcal{H}_J$, $J=0,1,\ldots,\infty$, where $\mathcal{H}_J$ is spanned by two-mode Fock states $|J,0\rangle,|J-1,1\rangle,\ldots,|0,J\rangle$. The subspaces $\mathcal{H}_J$ are dynamically independent (invariant) for those Hamiltonians which do not change the total number of vibrational quanta $\hat{J}=\hat{a}_x^\dagger\hat{a}_x+\hat{a}_y^\dagger\hat{a}_y$. However, to realize controlled manipulations with states within a given subspace $\mathcal{H}_J$ we need an interaction of the 2D bosonic field with another quantum system. In the case of trapped ions it is natural to assume the coupling to internal energy levels of ions. In particular, we use three internal electronic states $|i\rangle$ (in Λ or Ξ configurations) with energies $\hbar\omega_i$ $(i=a,b,c)$ which form the basis of the Hilbert subspace $\mathcal{H}_{\rm in}$. Due to quantum interaction between the vibrational and internal degrees of freedom the total state vector of the composed system with the given maximum number of vibrational quanta $J_{\max}=M_{\max}+N_{\max}$ at time t reads

$$|\Psi(t)\rangle=\sum_{m=0}^{M_{\max}}\sum_{n=0}^{N_{\max}}\sum_{i=a,b,c}Q_{m,n;i}(t)|m,n\rangle\otimes|i\rangle$$
$$=\sum_{J=0}^{J_{\max}}\sum_{k=0}^{J}\sum_{i=a,b,c}Q_{k,J-k;i}(t)|k,J-k\rangle\otimes|i\rangle, \tag{2.1}$$

which reflects quantum-mechanical entanglement between the two subsystems. We assume that initially the ion is in an internal state $|a\rangle$ and the vibrational motion is cooled to the ground state $|0,0\rangle$. That is, the initial state vector of the composed system $|\Psi(t=0)\rangle=|0,0\rangle\otimes|a\rangle$ is factorized. Our task is to find a unitary evolution $\hat{U}^\dagger$, such that at some time $t=T$ the state vector (2.1) can again be factorized, while the vibrational state of the ion is described by the target vector (1.1), i.e., $|\Psi(t=T)\rangle=\hat{U}^\dagger|0,0\rangle\otimes|a\rangle=|\Psi_{\rm target}\rangle\otimes|a\rangle$.

In what follows we prove that the unitary transformation $\hat{U}$ can be represented by a sequence of five ''elementary'' unitary transformations $\hat{U}^{(p)}$ $(p=1,\ldots,5)$ which act in the product Hilbert space $\mathcal{H}_{\rm vib}\otimes\mathcal{H}_{\rm in}$ and which correspond to five interaction channels associated with interaction Hamiltonians $\hat{H}^{(p)}$:

$$\hat{U}=\hat{A}_0\left\{\prod_{J=1}^{J_{\max}}\hat{C}_J\hat{B}_{J-1}\hat{A}_J\right\},$$

$$\hat{A}_J=\hat{U}^{(1)}_{|J,0,b\rangle}\hat{U}^{(3)}_{|J-1,1,a\rangle}\cdots\hat{U}^{(1)}_{|2,J-2,b\rangle}\hat{U}^{(3)}_{|1,J-1,a\rangle}\hat{U}^{(1)}_{|1,J-1,b\rangle}\hat{U}^{(3)}_{|0,J,a\rangle},$$

$$\hat{B}_J=\hat{U}^{(2)}_{|J,0,c\rangle}\hat{U}^{(4)}_{|J-1,1,b\rangle}\hat{U}^{(2)}_{|J-1,1,c\rangle}\cdots\hat{U}^{(4)}_{|1,J-1,b\rangle}$$
$$\times\hat{U}^{(2)}_{|1,J-1,c\rangle}\hat{U}^{(4)}_{|0,J,b\rangle}\hat{U}^{(2)}_{|0,J,c\rangle}, \tag{2.2}$$

$$\hat{C}_J=\hat{U}^{(5)}_{|J,0,a\rangle}.$$

The operators $\hat{A}_J$, $\hat{B}_J$, $\hat{C}_J$ are built up from ''elementary'' unitary transformations $\hat{U}^{(p)}_{|k,J-k,i\rangle}$ where superscript p determines the interaction channel and subscript indicates that parameters of the interaction are adjusted to transfer *completely* the current population of the component state $|k,J-k\rangle\otimes|i\rangle$ to the ''neighboring'' state according to the given type of interaction (see below).

The set of the ''elementary'' unitary transformations $\hat{U}^{(p)}$ is sufficient (even though not unique) to evolve the target state (1.1) into the vacuum and vice versa. To specify them we will analyze the ''deevolution'' of the target state into the vacuum state. We will perform this deevolution via a systematic ''transfer'' of population from state vectors with higher to smaller number of vibrational quanta. That is, we choose the operators $\hat{U}^{(p)}$ so that at a given step of the uni-

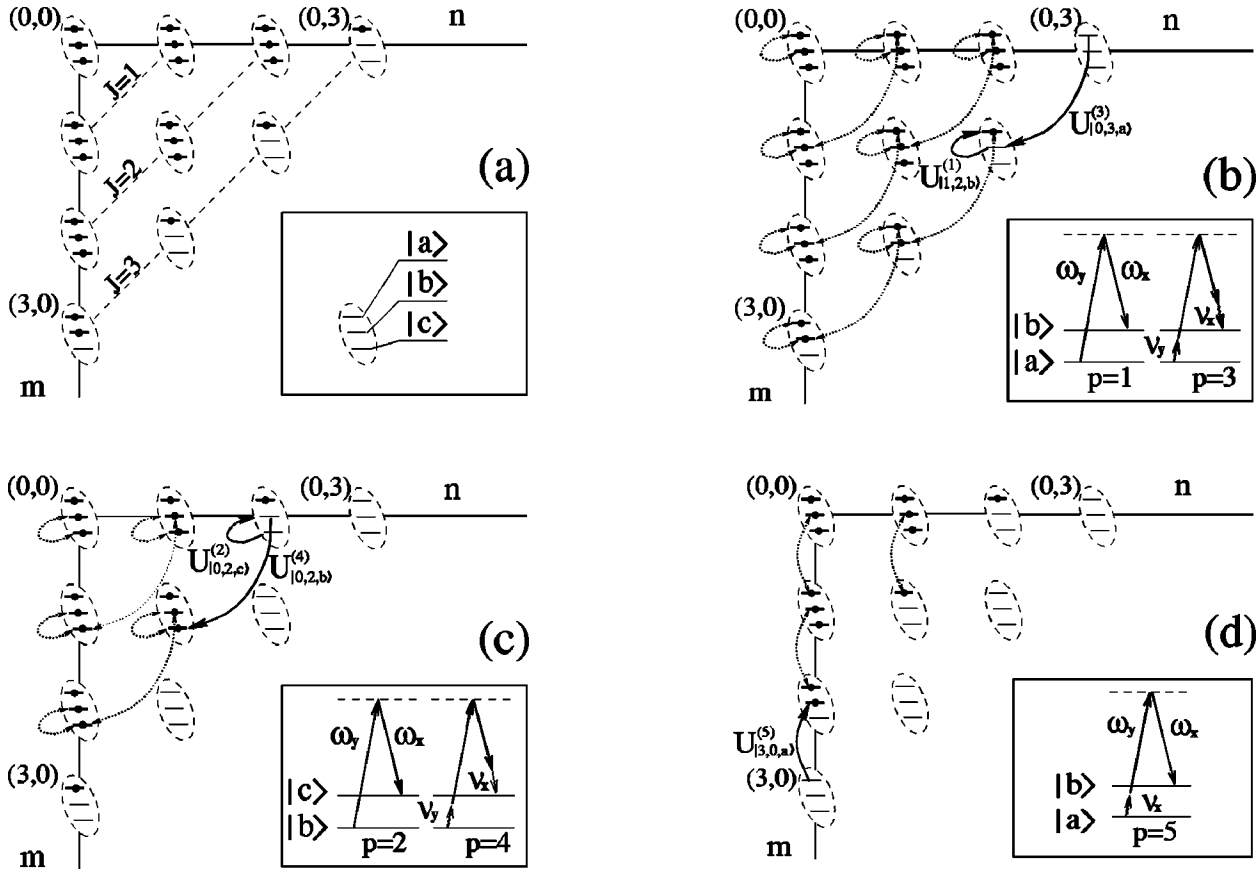

FIG. 1. Recursive ''deevolution'' algorithm: (a) Vectors $|m,n\rangle\otimes|i\rangle$ are represented as points of a lattice m,n with internal levels $i=a,b,c$ shown in ''ovals.'' Components which enter (at this stage of ''deevolution'') the state vector (2.1) are indicated by ●. Dashed lines connect basis vectors from subspaces with constant number of trap quanta J. (b) The action of the operator $\hat{A}_{\tilde{J}}$ for $\tilde{J}=3$. The ''elementary'' transformation $\hat{U}^{(3)}_{|0,\tilde{J},a\rangle}$ transfers *completely* the population of the state $|0,\tilde{J}\rangle\otimes|a\rangle$ to the state $|1,\tilde{J}-1\rangle\otimes|b\rangle$ as indicated by solid arrow. The atomic transition scheme of the corresponding stimulated Raman process $p=3$ is shown in the inset. Next $\hat{U}^{(1)}_{|1,\tilde{J}-1,b\rangle}$ transfers the population from $|1,\tilde{J}-1\rangle\otimes|b\rangle$ to $|1,\tilde{J}-1\rangle\otimes|a\rangle$ by means of the interaction channel $p=1$. The dashed arrows indicate simultaneous transitions which lead to a change of the state vector (2.1). The application of the sequence $\hat{A}_{\tilde{J}}$ ''shrinks'' the whole population from the given subspace $\mathcal{H}_{\tilde{J}}\otimes\mathcal{H}_{\rm in}$ to the component state $|\tilde{J},0\rangle\otimes|a\rangle$. (c) Specific preparation of the subspace $\mathcal{H}_{\tilde{J}-1}\otimes\mathcal{H}_{\rm in}$ by the operator $\hat{B}_{\tilde{J}-1}$. The operator $\hat{U}^{(2)}_{|0,\tilde{J}-1,c\rangle}$ transfers the population of the state $|0,\tilde{J}-1\rangle\otimes|c\rangle$ to $|0,\tilde{J}-1\rangle\otimes|b\rangle$, while $\hat{U}^{(4)}_{|0,\tilde{J}-1,b\rangle}$ transfers the population from $|0,\tilde{J}-1\rangle\otimes|b\rangle$ to $|1,\tilde{J}-2\rangle\otimes|b\rangle$. The inset shows the schemes of the utilized stimulated Raman transitions $p=2,4$. Finally, the sequence $\hat{B}_{\tilde{J}-1}$ leaves on $\mathcal{H}_{\tilde{J}-1}\otimes\mathcal{H}_{\rm in}$ only the component state $|\tilde{J}-1,0\rangle\otimes|b\rangle$ and those states with the internal level $|a\rangle$ contributing to Eq. (2.1). (d) The operator $\hat{C}_{\tilde{J}}=\hat{U}^{(5)}_{|\tilde{J},0,a\rangle}$ transfers the population of the component state $|\tilde{J},0\rangle\otimes|a\rangle$ to the state $|\tilde{J}-1,0\rangle\otimes|b\rangle$. The procedures (b) and (c) are recursively repeated to ''deevolve'' the initial state $|\Psi_{\rm target}\rangle\otimes|a\rangle$ into $|0,0\rangle\otimes|a\rangle$.

tary deevolution a particular probability amplitude $Q_{k,J-k;i}$ is made equal to zero. In Fig. 1 we visualize the ''direction'' of the action of operators $U^{(p)}_{k,J-k,i\rangle}$. Namely, in Fig. 1(a) we represent state vectors $|m,n\rangle\otimes|i\rangle$ as ''ovals'' located at the points of the lattice marked by parameters m,n associated with vibrational state of an ion, while its internal state represented by a simple energy level diagram (population of a given internal level is indicated by a bullet). Dashed lines indicate the subspaces $\mathcal{H}_J\otimes\mathcal{H}_{\rm in}$ labeled by a constant number of trap quanta J.

Let us start the ''deevolution'' procedure of the target state (1.1) into vacuum. After time $\tau=T-t$ the vector describing the vibrational and internal state of the ion is given by Eq. (2.1). We assume that the deevolution is performed in such a way that at this moment the state vector (2.1) is composed only of state vectors from subspaces $\mathcal{H}_J\otimes\mathcal{H}_{\rm in}$ with $J\leqslant\tilde{J}$ ($\leqslant J_{\max}$). Now we apply three sequences of operations described by the operators $\hat{A}_{\tilde{J}}$, $\hat{B}_{\tilde{J}-1}$, and $\hat{C}_{\tilde{J}}$, respectively [see Eq. (2.2)].

The action of the operator $\hat{A}_J$ is illustrated in Fig. 1(b). It describes a process in which the first ''elementary'' transformation $\hat{U}^{(3)}_{|0,\tilde{J},a\rangle}$ transfers *completely* the population of the state $|0,\tilde{J}\rangle\otimes|a\rangle$ to the state $|1,\tilde{J}-1\rangle\otimes|b\rangle$ [see the solid arrow in Fig. 1(b); dashed arrows indicate simultaneous transitions which are not controlled at the given stage]. This process can be realized by irradiating the ion with two external laser fields with tunable frequencies ω_x and ω_y in x and y directions. Adjusting the resonance conditions $\omega_y-\omega_x=\omega_b-\omega_a+\nu_x-\nu_y$ ($\equiv\omega_3$) the stimulated Raman transition [see the inset in Fig. 1(b), $p=3$] can be described in the Lamb-Dicke regime by the effective interaction Hamiltonian

(we discuss physical conditions under which this and all subsequent Hamiltonians can be justified in Sec. III):

$$\hat{H}_3 = g_3 \hat{a}_x^\dagger \hat{a}_y |b\rangle\langle a| e^{-i\omega_3 t} + g_3^* \hat{a}_x \hat{a}_y^\dagger |a\rangle\langle b| e^{i\omega_3 t}, \quad (2.3)$$

with the corresponding time evolution operator $\hat{U}^{(3)} = \exp(-i\hat{H}^{(3)}t)$ setting $\hbar = 1$. If the interaction constant $g_3 = |g_3| e^{i\theta_3}$ and the duration of the interaction t are chosen to fulfill the condition

$$ie^{i\theta_3} Q_{k,J-k;a} \cos[|g_3| t\sqrt{(k+1)(J-k)}] + Q_{k+1,J-k-1;b} \sin[|g_3| t\sqrt{(k+1)(J-k)}] = 0, \quad (2.4)$$

then the population of the component state $|k, J-k\rangle \otimes |a\rangle$ is completely transferred to the state $|k+1, J-k-1\rangle \otimes |b\rangle$. [For instance, in the situation described by Fig. 1(b) the probability amplitude $Q_{0,\tilde{J};a}$ becomes equal to zero.] Once this is done, then the transformation $\hat{U}^{(1)}_{|1,\tilde{J}-1,b\rangle}$ is turned on. In this process the population of the component state $|1, \tilde{J}-1\rangle \otimes |b\rangle$ is *completely* transferred to the state $|1, \tilde{J}-1\rangle \otimes |a\rangle$. The corresponding interaction channel is described by the Hamiltonian

$$\hat{H}_1 = g_1 |b\rangle\langle a| e^{-i\omega_1 t} + g_1^* |a\rangle\langle b| e^{i\omega_1 t}. \quad (2.5)$$

Here we assume lasers to be tuned to the electronic transition, i.e., $\omega_y - \omega_x = \omega_b - \omega_a$ $(\equiv \omega_1)$. To cancel the term $Q_{k,J-k;b}$ in the state vector (2.1), the interaction constants $(|g_1|t,\ \theta_1)$ have to be chosen to satisfy the condition

$$Q_{k,J-k;a} \sin(|g_1|t) + ie^{-i\theta_1} Q_{k,J-k;b} \cos(|g_1|t) = 0. \quad (2.6)$$

In particular, in Fig. 1(b) the operator $\hat{U}^{(1)}_{|1,\tilde{J}-1,a\rangle}$ cancels $Q_{1,\tilde{J}-1;b}$ for $\tilde{J}=3$. The successive action of the ''elementary'' transformations $\hat{U}^{(3)}$ and $\hat{U}^{(1)}$ which form the operator $\hat{A}_{\tilde{J}}$ [see Eq. (2.2)] finally ''shrinks'' the population of the subspace $\mathcal{H}_{\tilde{J}} \otimes \mathcal{H}_{\rm in}$ to a single state $|\tilde{J}, 0\rangle \otimes |a\rangle$.

At this stage we start the process of cancellation of the contribution of component states $|k, \tilde{J}-1-k\rangle \otimes |b\rangle$ and $|k, \tilde{J}-1-k\rangle \otimes |c\rangle$ in the ''neighboring'' subspace $\mathcal{H}_{\tilde{J}-1} \otimes \mathcal{H}_{\rm in}$ [see Fig. 1(c)]. This intermediate procedure is required to prevent a reverse transfer of population from $\mathcal{H}_{\tilde{J}-1} \otimes \mathcal{H}_{\rm in}$ to $\mathcal{H}_{\tilde{J}} \otimes \mathcal{H}_{\rm in}$ [see below and Fig. 1(d)]. For this purpose the operator $\hat{B}_{\tilde{J}-1}$ is constructed from ''elementary'' operations $\hat{U}^{(2)}$ and $\hat{U}^{(4)}$ [see Eq. (2.2)]. Namely, the operator $\hat{U}^{(2)}_{|0,\tilde{J}-1,c\rangle}$ describes the transfer of population of the state $|0, \tilde{J}-1\rangle \otimes |c\rangle$ to $|0, \tilde{J}-1\rangle \otimes |b\rangle$. This transfer can be achieved with the help of laser pulses tuned to the electronic transition between the levels $|b\rangle$ and $|c\rangle$. The corresponding interaction Hamiltonian $\hat{H}^{(2)}$ and the resonance condition are analogous to those for $\hat{H}^{(1)}$, Eq. (2.5) (we have to replace only $b \to c$ and $a \to b$). Further, the operation $\hat{U}^{(4)}_{|0,\tilde{J}-1,b\rangle}$ cancels a contribution of the state $|0, \tilde{J}-1\rangle \otimes |b\rangle$ to the state vector (2.1). This interaction channel is described by the Hamiltonian $\hat{H}^{(4)}$ which is obtained from Eq. (2.3) by the substitution $b \to c$ and $a \to b$. We see that the operator $\hat{B}_{\tilde{J}-1}$ acts like $\hat{A}_{\tilde{J}}$ but instead of the stimulated Raman processes between $a \leftrightarrow b$ the transitions $b \leftrightarrow c$ are utilized. As the result of the action of the operator $\hat{B}_{\tilde{J}-1}$ in the subspace $\mathcal{H}_{\tilde{J}-1} \otimes \mathcal{H}_{\rm in}$ only the component states with the internal level $|a\rangle$ and the state $|\tilde{J}-1, 0\rangle \otimes |b\rangle$ have nonzero amplitudes.

After the action of the operators $\hat{A}_{\tilde{J}}$ and $\hat{B}_{\tilde{J}-1}$ the unitary operator $\hat{C}_{\tilde{J}}$ is utilized to transfer the population of the component state $|\tilde{J}, 0\rangle \otimes |a\rangle$ to the state $|\tilde{J}-1, 0\rangle \otimes |b\rangle$ [see Fig. 1(d) for $\tilde{J}=3$]. The transformation $\hat{U}^{(5)}_{|\tilde{J},0,a\rangle}$ which performs transitions between subspaces with the number of trap quanta differed by one is realized by the process described by a single-mode interaction Hamiltonian

$$\hat{H}_5 = g_5 \hat{a}_x |b\rangle\langle a| e^{-i\omega_5 t} + g_5^* \hat{a}_x^\dagger |a\rangle\langle b| e^{+i\omega_5 t}, \quad (2.7)$$

with the resonance condition $\omega_y - \omega_x = \omega_b - \omega_a - \nu_x$ $(\equiv \omega_5)$. The parameters of this interaction channel are determined by the constraint

$$ie^{i\theta_5} Q_{J,0;a} \cos(|g_5| t\sqrt{J}) + Q_{J-1,0;b} \sin(|g_5| t\sqrt{J}) = 0, \quad (2.8)$$

for $J = \tilde{J}$.

As the result of the action of the operators $\hat{A}_{\tilde{J}}$, $\hat{B}_{\tilde{J}-1}$, and $\hat{C}_{\tilde{J}}$, all coefficients $Q_{k,\tilde{J}-k;i}$ in Eq. (2.1) are equal to zero. Moreover, the situation before [Fig. 1(a)] and after [Fig. 1(d)] the action of these operators is the same, except we just ''moved'' from the subspace $\mathcal{H}_{\tilde{J}} \otimes \mathcal{H}_{\rm in}$ to $\mathcal{H}_{\tilde{J}-1} \otimes \mathcal{H}_{\rm in}$ with the number of trap quanta decreased by one. This means that the procedure can be *recursively* repeated. The given solution of the ''deevolution'' gives immediately the recipe for the creation of the target vibrational state from the two-mode vacuum (with electronic level $|a\rangle$): We have to change properly the phase shift between external laser fields and repeat the sequence in the opposite order, i.e., we apply $\hat{U}^\dagger$ on $|0,0\rangle \otimes |a\rangle$. In this way we can synthesize an arbitrary state. The number of ''elementary'' operations involved in this process is proportional to $2J_{\rm max}^2$, which is important for an experimental realization of the proposed scheme, i.e., the number of necessary operations increases only polynomially with the increase of the size of the Hilbert space in which the target state is embedded.

In the following section a physical implementation of the preparation scheme is discussed in more detail.

III. REALIZATION OF INTERACTION CHANNELS

The Hamiltonians which are eligible for the synthesis of two-mode bosonic states have been discussed recently by several authors [10–13]. Some of the proposals are based on laser-stimulated dipole transitions [10] and phonon-number-dependent interaction via a detuned standing wave [11]. For our purposes we have utilized the stimulated Raman processes discussed in detail by Steinbach, Twamley, and Knight [13]. In this section following Ref. [13] we briefly

derive the Hamiltonians which are used in our algorithm [see Eqs. (2.3), (2.5), and (2.7)] and discuss the range of their applicability. Let us consider a trapped ion confined in a 2D harmonic potential characterized by the trap frequencies ν_x and ν_y in two orthogonal directions x and y. The ion is irradiated along the x and y axes by two external laser fields with frequencies ω_x, ω_y and wave vectors k_x, k_y. The laser fields stimulate Raman transitions between two internal energy levels $|a\rangle$ and $|b\rangle$ via an auxiliary electronic level $|r\rangle$ which is far off resonance. For concreteness, we consider Λ configuration with the upper level $|r\rangle$ as outlined in the inset of Fig. 1(b). The interaction Hamiltonian for the system under consideration can be written in the dipole and rotating-wave approximation (RWA at laser frequencies) in the form

$$\hat{H}_{\rm int}=g_x^* e^{i(k_x\hat{x}-\omega_x t)}|r\rangle\langle b|+g_x e^{-i(k_x\hat{x}-\omega_x t)}|b\rangle\langle r| + g_y^* e^{i(k_y\hat{y}-\omega_y)t}|r\rangle\langle a|+g_y e^{-i(k_y\hat{y}-\omega_y)t}|a\rangle\langle r|. \tag{3.1}$$

The coupling constant g_x (g_y) is proportional to the intensity of the laser field in x (y) direction and the dipole moment of the electronic transition $|b\rangle\leftrightarrow|r\rangle$ $(|a\rangle\leftrightarrow|r\rangle)$. The upper off-resonant level $|r\rangle$ can be adiabatically eliminated provided that $\Delta_x,\Delta_y\gg g_a,g_b,|\Delta_x-\Delta_y|$, where laser detunings for dipole transitions $|b\rangle\leftrightarrow|r\rangle$ and $|a\rangle\leftrightarrow|r\rangle$ are denoted as $\Delta_x=(\omega_r-\omega_b)-\omega_x$ and $\Delta_y=(\omega_r-\omega_a)-\omega_y$, respectively. After adiabatic elimination the effective interaction Hamiltonian for the stimulated Raman transition $|a\rangle\leftrightarrow|b\rangle$ reads [13]

$$\hat{H}_{\rm int}^{(\rm eff)}=g^* e^{-i(\omega_y-\omega_x)t}\hat{D}_x(-i\epsilon_x)\hat{D}_y(i\epsilon_y)|b\rangle\langle a| + g e^{i(\omega_y-\omega_x)t}\hat{D}_x(i\epsilon_x)\hat{D}_y(-i\epsilon_y)|a\rangle\langle b|. \tag{3.2}$$

Here $\hat{D}_q(i\epsilon_q)=e^{i\epsilon_q(\hat{a}_q^\dagger+\hat{a}_q)}=e^{ik_q\hat{q}}$ is the displacement operator $(q=x,y)$; the Lamb-Dicke parameter ϵ_q is defined as $\epsilon_q^2=\hbar^2k_q^2/(2m\hbar\nu_q)$ and the effective interaction constant $g=g_x^*g_y(1/\Delta_x+1/\Delta_y)$. Further, we assume that the energies of the electronic levels $|a\rangle$ and $|b\rangle$ are redefined to include Stark shifts due to the adiabatic elimination of the off-resonant energy level $|r\rangle$ [13].

In the interaction picture the effective interaction Hamiltonian $\tilde{\hat{H}}_{\rm int}^{(\rm eff)}=e^{i\hat{H}_0t}\hat{H}_{\rm int}^{(\rm eff)}e^{-i\hat{H}_0t}$ can be expressed as (the free Hamiltonian $\hat{H}_0$ induces transformations $\hat{a}_q\rightarrow\hat{a}_qe^{-i\nu_qt}$, $|a\rangle\langle b|\rightarrow|a\rangle\langle b|e^{-i(\omega_b-\omega_a)t}$)

$$\tilde{\hat{H}}_{\rm int}^{(\rm eff)}=g^*e^{-(\epsilon_x^2+\epsilon_y^2)/2}\sum_{m,k,l,n}\frac{(-i\epsilon_x)^{k+m}(i\epsilon_y)^{l+n}}{k!l!m!n!} \times e^{-it[\Delta+(k-m)\nu_x+(n-l)\nu_y]}\hat{a}_x^{\dagger m}\hat{a}_x^k\hat{a}_y^{\dagger l}\hat{a}_y^n|b\rangle\langle a|+\text{H.c.} \tag{3.3}$$

The resonant terms in the expansion (3.3) which contribute dominantly to the resulting effective Hamiltonian can be selected by an appropriate choice of laser frequencies. If off-resonant processes are oscillating with sufficiently high frequencies they can be eliminated applying the second RWA at trap frequencies.

In particular, tuning lasers to the first red sidebands $\Delta\equiv\Delta_x-\Delta_y=\nu_x-\nu_y$ with *incommensurate* trap frequencies only the resonant terms with $k-m=1$ and $l-n=1$ are retained in the expansion (3.3):

$$\tilde{\hat{H}}_{\rm int}^{(3)}=g^*\epsilon_x\epsilon_y\hat{a}_x^\dagger\hat{\mathcal{F}}(\hat{a}_x^\dagger\hat{a}_x,\hat{a}_y^\dagger\hat{a}_y)\hat{a}_y|b\rangle\langle a|+\text{H.c.}, \tag{3.4}$$

where

$$\hat{\mathcal{F}}=e^{-(\epsilon_x^2+\epsilon_y^2)/2}\sum_{k,l}\frac{(-1)^{k+l}\epsilon_x^{2k}\epsilon_y^{2l}}{(k+1)!k!(l+1)!l!}\hat{a}_x^{\dagger k}\hat{a}_x^k\hat{a}_y^{\dagger l}\hat{a}_y^l. \tag{3.5}$$

In the Lamb-Dicke regime $\epsilon_x,\epsilon_y\ll1$ the operator $\hat{\mathcal{F}}$ is close to the unity operator and the Hamiltonian (3.4) which is written in the interaction picture acquires in the Schrödinger picture exactly the form of the two-mode Hamiltonian $\hat{H}_3$, Eq. (2.3). With proper laser tunings we can design within the Lamb-Dicke limit all the remaining interaction Hamiltonians required for the quantum-state synthesis described in Sec. II. In particular, retaining only resonant terms in Eq. (3.3) for the laser-stimulated Raman process with $\Delta=0$ the interaction Hamiltonian $\hat{H}_1$, Eq. (2.5), is obtained in the Schrödinger picture. The process with $\Delta=-\nu_x$ is described by the one-mode interaction Hamiltonian $\hat{H}_5$, Eq. (2.7). Let us also notice that beyond the Lamb-Dicke limit this process is analogous to a nonlinear Jaynes-Cummings dynamics discussed by Vogel and de Matos Filho [7].

Considered approximations impose limitations on the applicability of the interaction Hamiltonians. Driven electronic transition (2.5) and one-mode interaction (2.7) have been considered already for the 1D quantum-state synthesis [8]. A new tool in our approach represents the two-mode interaction Hamiltonian $\hat{H}_3$, Eq. (2.3). Comprehensive analysis of the limitations for Eq. (2.3) was done in Ref. [13]. It turns out that the most subtle point is the second RWA at trap frequencies [see Eqs. (3.3), (3.4)] which imposes restrictions [13]

$$|g|\epsilon_x\epsilon_y\max(N_{\max},M_{\max})\ll\min(\nu_x,\nu_y),\quad \frac{\max(\nu_x,\nu_y)}{\min(\nu_x,\nu_y)}\geqslant5. \tag{3.6}$$

The trap anisotropy, i.e., *incommensurate* trap frequencies, is required to avoid additional resonances in Eq. (3.3). Numerical simulations in Ref. [13] demonstrated that it is experimentally feasible to operate the considered Hamiltonians within the Lamb-Dicke limit.

It is worth noticing that already the nonlinear form (3.4) corresponding to the two-mode interaction Hamiltonian (2.3) outside of the Lamb-Dicke regime allows us to adopt the proposed algorithm for quantum-state synthesis. The main difference consists in the form of generalized Rabi frequencies. In particular, the matrix element of the interaction Hamiltonian (3.4) in the Lamb-Dicke regime $\epsilon_q\ll1$ [i.e., Eq. (2.3) in the interaction picture] reads $\langle b,n-1,m+1|\tilde{\hat{H}}_{\rm int}^{(3)}|m,n,a\rangle=g_3\sqrt{(m+1)n}$ while beyond the Lamb-Dicke regime the corresponding matrix element is given as $g_3e^{-(\epsilon_x^2+\epsilon_y^2)/2}L_m^1(\epsilon_x^2)L_{n-1}^1(\epsilon_y^2)/\sqrt{(m+1)n}$ where L_m^1 is the associated Laguerre polynomial. Inserting the ''nonlinear''

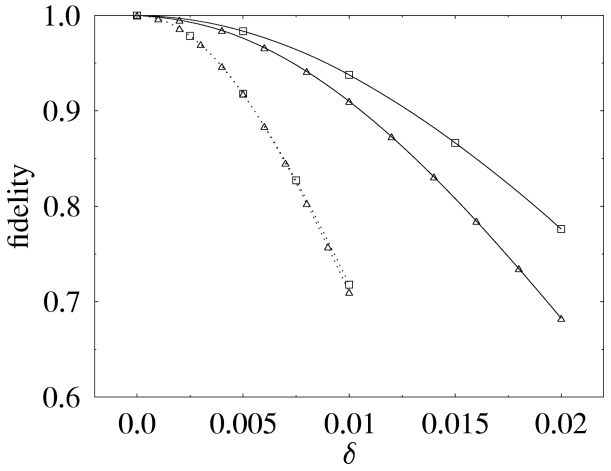

FIG. 2. We plot fidelity given by Eq. (4.1) as a function of the range of fluctuations δ. We consider two different target states, two-mode catlike state $|\Psi_{\rm cat}\rangle$ (denoted by $\triangle$) and two-mode correlated state $|\Psi_{\rm corr}\rangle$ (denoted by $\square$) with $\alpha=2$. We have considered two cases when $M_{\rm max}=12$ (solid lines) and $M_{\rm max}=20$ (dashed lines). Comparing two solid lines we see that the smaller the number of nonzero amplitudes Q_{mn} in the target state (1.1) the larger is the fidelity for a given value of δ.

Rabi frequencies into Eqs. (2.4), (2.6), and (2.8) which determine the choice of the interaction constants and switching times, the same sequence of elementary transformations (2.2) could be applied for preparation of a given state even outside of the Lamb-Dicke regime. On the other hand, the conditions for the applicability of the second RWA at trap frequencies to obtain Eq. (3.4) from Eq. (3.3) are not so transparent as restrictions for the RWA within the Lamb-Dicke regime (3.6). This problem goes beyond the scope of the present paper in which we demonstrate the potential of our algorithm in the Lamb-Dicke regime.

IV. STABILITY OF SYNTHESIS

Our algorithm works ideally when there are essentially no dissipations in the system. It is the reason why we have considered effectively dissipation-free dynamics of trapped ions. Nevertheless, we have to stress that the ideal synthesis of quantum motional states assumes also perfect control of interaction constants and switching times of particular interaction channels. Namely, the values of $g_p t$ found via the ''deevolution'' procedure [i.e., solutions of Eqs. (2.4), (2.6), and (2.8)] have to be controlled as precisely as possible. In practice one cannot avoid some level of ''technical'' noise, for example, due to imperfect timing of switching between interaction channels. Therefore in what follows we will study the stability of the presented algorithm with respect to the ''technical'' noise. This noise is simulated as random fluctuations of the ideal values $g_p t$. In particular, fluctuations are equally distributed around the ideal (complex) values $g_p t$ [i.e., solutions of Eqs. (2.4), (2.6), and (2.8)] within a fixed interval $(1+i)\delta$. In general, the output state vectors $|\Psi_\delta\rangle$ of the composed system which are prepared in the presence of ''technical'' fluctuations take the form (2.1), i.e., the vibrational and internal degrees of freedom are not disentangled. Figure 2 shows the fidelity f of the ''imperfect'' output states $|\Psi_\delta\rangle$ with respect to the desired state vector $|\Psi_{\rm target}\rangle\otimes|a\rangle$. The fidelity is defined as the averaged squared scalar product of particular realizations $|\Psi_\delta\rangle$ with $|\Psi_{\rm target}\rangle\otimes|a\rangle$, i.e.,

$$f=\langle\langle|\langle\Psi_\delta|\Psi_{\rm target}\rangle|a\rangle|^2\rangle\rangle_\delta. \tag{4.1}$$

In our simulations we have performed averaging over 100 runs of state-synthesis sequences. In these runs each value $q_p t$ associated with a given elementary operation $\hat{U}^{(p)}$ acquires a random fluctuation within the interval $(1+i)\delta$. We have considered two different target states, two-mode catlike state $|\Psi_{\rm cat}\rangle=\mathcal{N}_{\rm cat}(|\alpha\rangle|\alpha\rangle+|-\alpha\rangle|-\alpha\rangle)$ and two-mode correlated state $|\Psi_{\rm corr}\rangle=e^{-|\alpha|^2/2}\Sigma_m(\alpha^m/\sqrt{m!})|m,m\rangle$, with $\alpha=2$. For this value of the amplitude these two states have approximately the same mean number of vibrational quanta $\overline{n}=\overline{n_x}+\overline{n_y}\simeq 8.0$. We have considered two cases when $M_{\rm max}=12$ and $M_{\rm max}=20$ (here $N_{\rm max}=M_{\rm max}$).

Figure 2 clearly indicates the fact that the larger the value of $M_{\rm max}$ the more pronounced is the role of fluctuations (compare solid and dashed lines which correspond to $M_{\rm max}=12$ and $M_{\rm max}=20$, respectively). This observation is easy to explain: The total number of operations in our algorithm is proportional to $8M_{\rm max}^2$ (for $M_{\rm max}=N_{\rm max}$) which means that the case $M_{\rm max}=20$ requires almost three times more operations compared to the case with $M_{\rm max}=12$. The noise is accumulated as a function of elementary operations, therefore to improve the fidelity of the preparation process it is important to choose $M_{\rm max}$ carefully. To be specific, for a given ε we have to choose the *minimal* values of $M_{\rm max}$ and $N_{\rm max}$ such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|Q_{mn}|^2-\sum_{m=0}^{M_{\rm max}}\sum_{n=0}^{N_{\rm max}}|Q_{mn}|^2\leqslant\varepsilon. \tag{4.2}$$

We can also use Fig. 2 to illustrate the fact that for a given value of $M_{\rm max}$ the fidelity of the preparation may depend on the target state. To be specific, we have found that the smaller the number of nonzero amplitudes Q_{mn} in the target state (1.1) the higher the fidelity is for a given value of the range of fluctuations δ. This behavior can be rather surprising as for synthesis of two states ''localized'' within the same region of the vibrational ''lattice'' m,n we need a comparable number of elementary operations. Nevertheless, each nonzero Q_{mn} after the synthesis is biased by more or less the same error and, consequently, the states with smaller number of nonzero amplitudes Q_{mn} are less sensitive to fluctuations. As already stated, this observation concerns only states ''localized'' within the same region of the vibrational lattice m,n and with a comparable mean number of vibrational quanta.

Finally, we briefly compare our algorithm with the one proposed by Gardiner, Cirac, and Zoller [10] in which the number of operations in the preparation sequence is growing *exponentially* as $2M_{\rm max}\times 2^{M_{\rm max}}$. Therefore fluctuations for $M_{\rm max}$ large enough cause an insurmountable problem. There is also another problem with this procedure. Namely, Gardiner, Cirac, and Zoller have utilized only two internal atomic levels ($|a\rangle$, $|b\rangle$) which results in the fact that their algorithm is based on manipulations with vibrational states which are out of the original Hilbert space specified by the

cutoffs $M_{\max}$ and $N_{\max}$. In other words, manipulations with highly excited vibrational states are required for construction of states with relatively small number of vibrational quanta. This means that not only the number of operations is exponentially growing but also the number of vibrational quanta during the preparation procedure may transiently exponentially increase.

We stress that our algorithm is associated with manipulations only within the original subspace of the Hilbert space specified by $M_{\max}$ and $N_{\max}$. Moreover, the number of operations grows only polynomially as $8M_{\max}^2$. The dimension of the two-mode Fock subspace from which the component states are does not increase during the preparation procedure. This great reduction of number of operations is due to the fact that we have employed the third atomic level $|c\rangle$ in the preparation procedure. Very recently a new 2D preparation scheme was introduced by Kneer and Law [11]. The standing-wave laser field in y direction induces a photon-number-dependent interaction enabling us thus to use 1D schemes selectively for particular subspaces with constant number of trap quanta in y direction. The number of operations scales also polynomially as $2M_{\max}^2$ with only two electronic levels involved.

V. CONCLUSION

In this paper we have presented a universal algorithm for an efficient deterministic preparation of an arbitrary two-mode bosonic state. We have adapted this algorithm as a computer program [14]. The proposed method can be generalized to 3D trapping potential and three-mode vibrational states. In this case one would need four internal electronic levels and nine interaction channels coupled to the 3D vibrational field. It can be shown that the number of operations required for quantum-state synthesis scales polynomially ($\sim M_{\max}^3$). Further generalization to multimode fields is possible [15].

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