

## Dynamics of open systems governed by the Milburn equation

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Within the framework of the Milburn model of intrinsic decoherence [G. J. Milburn, Phys. Rev. A **44**, 5401 (1991)], which is based on an assumption of an existence of the fundamental time step, we study the dynamics of open systems. We show that evolution of an open system governed by the Milburn equation can be significantly modified compared to the standard Schrödinger quantum mechanics. This modification is most pronounced when an open system is strongly coupled to its environment. [S1050-2947(98)06209-X]

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### I. INTRODUCTION

The problem of decoherence in quantum mechanics still attracts a lot of attention. Several conceptually different approaches have been proposed to explain why quantum coherences cannot be observed on a “macroscopic” level. One explanation of why quantum coherences are deteriorated and difficult to observe is based on a hypothesis that *closed* quantum systems do not evolve unitarily according to the Schrödinger equation, but are governed by more generalized equations that include *intrinsic* decoherence. In particular, Milburn [1] has recently proposed an elegant model of intrinsic decoherence based on a simple modification of a *unitary* Schrödinger evolution. This model is based on an assumption that on sufficiently short time scales the *closed* system evolves by a *random* sequence of unitary phase changes generated by the system Hamiltonian. Based on this assumption Milburn has derived the equation for the time evolution density operators  $\rho(t)$  of closed quantum systems [1],

$$\frac{d}{dt}\rho(t) = \gamma \left\{ \exp \left[ -\frac{i}{\hbar\gamma} H \right] \rho(t) \exp \left[ \frac{i}{\hbar\gamma} H \right] - \rho(t) \right\}, \quad (1.1)$$

where  $\gamma$  is the mean frequency of the unitary (minimum) time step. This equation formally corresponds to the assumption that on a very short time scale  $\tau$  the probability that the system evolves is  $p(\tau) = \gamma\tau$ . In the limit  $\gamma \rightarrow \infty$  (i.e., when the fundamental time step goes to zero), Eq. (1.1) reduces to the ordinary von Neumann equation describing the Schrödinger dynamics of closed quantum systems. If the fundamental time step is nonzero, but still very small, then the Milburn equation (1.1) describes an intrinsic decay of quantum coherences in the *energy* basis. Simultaneously, all constants of the motion associated with the standard Schrödinger dynamics remain constants of the motion in the Milburn model and thus stationary states remain stationary states.

Milburn in his paper has discussed in detail a number of testable consequences of his model of intrinsic decoherence. Nevertheless, he has studied only closed systems governed by Eq. (1.1). In the present paper we analyze dynamics of *open* quantum systems of the type *system + environment* governed by the Milburn equation. To be specific, if  $\mathcal{H}_S$

denotes the Hilbert space of the system  $S$ , and  $\mathcal{H}_E$  the Hilbert space associated with the environment  $E$ , then the formal Hamiltonian of the composite system  $S + E$  is

$$H_{SE} = H_S \otimes 1_E + H_{\text{int}} + 1_S \otimes H_E, \quad (1.2)$$

on  $\mathcal{H}_S \otimes \mathcal{H}_E$ , where  $H_{\text{int}}$  is some interaction term,  $1_S$  and  $1_E$  are unity operators associated with  $\mathcal{H}_S$  and  $\mathcal{H}_E$ , respectively, and  $\otimes$  denotes direct product of two operators. Suppose that at  $t=0$  the state of the composite system is  $\rho_S(0) \otimes \rho_E(0)$ . We assume that the state  $\rho_{SE}(t)$  of the composite system is governed by the Milburn Eq. (1.1). If  $\text{Tr}_E$  denotes the partial trace of the environment, then the *reduced* dynamics  $\mathcal{L}_S(t)$  of the open system  $S$  is defined by

$$\rho_S(t) = \mathcal{L}_S(t) \rho_S(0) = \text{Tr}_E[\rho_{SE}(t)]. \quad (1.3)$$

By definition  $\mathcal{L}_S(t)$  is linear, so that the state of the system  $S$  is always mapped onto a state. In what follows we will study how this mapping depends on the value of the parameter  $\gamma$ . To make our discussion physical we will model the system of interest as a two-level atom and the environment as an electromagnetic field. We will study a coupling of this two-level atom to a single-mode field as well as to a multimode field (we will present results of numerical calculations with the atom coupled to 500 modes of the electromagnetic field). The paper is organized as follows. In Sec. II we present a formal solution of the Milburn equation (1.1) for Hamiltonians  $H_{SE}$  with discrete spectra. This solution is then used in Sec. III for a description of the dynamics of a single two-level atom coupled to a single-mode electromagnetic field. In Sec. IV we study the “decay” of a two-level atom coupled to a multimode electromagnetic field. The paper is concluded with a discussion.

### II. GENERAL SOLUTION OF THE MILBURN EQUATION FOR COMPOSITE SYSTEMS

We rewrite the Milburn equation (1.1) in the basis of eigenvectors  $|i\rangle_{SE}$  of the total Hamiltonian (1.2),

$$\frac{d}{dt}\rho_{ij}(t) = \gamma \left\{ \exp \left[ -\frac{i}{\hbar\gamma} (\mathcal{E}_i - \mathcal{E}_j) \right] - 1 \right\} \rho_{ij}(t), \quad (2.1)$$

where  $\rho_{ij}(t) \equiv \langle i | \rho(t) | j \rangle$  and  $\mathcal{E}_i$  are eigenvalues of  $H_{SE}$ . The solution of this equation reads

$$\rho_{ij}(t) = \rho_{ij}(0) \exp\left(\left\{\exp\left[-\frac{i}{\hbar\gamma}(\mathcal{E}_i - \mathcal{E}_j)\right] - 1\right\}\gamma t\right), \quad (2.2)$$

where the matrix elements  $\rho_{ij}(0)$  describe the initial state of the composite system  $SE$ . This initial state can be also expressed in terms of eigenstates  $|\mu\rangle_{SE} = |\alpha\rangle_S |\beta\rangle_E$  of the ‘‘free Hamiltonian’’  $H_S \otimes 1_E + 1_S \otimes H_E$ , such that  $H_S |\alpha\rangle_S = \alpha |\alpha\rangle_S$  and  $H_E |\beta\rangle_E = \beta |\beta\rangle_E$ . Taking into account that  $|i\rangle = \sum_{\mu} c_{\mu}^i |\mu\rangle_{SE} \equiv \sum_{\alpha, \beta} c_{\alpha, \beta}^i |\alpha\rangle_S |\beta\rangle_E$ , we have  $\rho_{ij}(0) = \sum_{\mu, \nu} \rho_{\mu\nu}(0) (c_{\mu}^i)^* c_{\nu}^j$  and we can rewrite the solution (2.2) in the basis  $\{|\mu\rangle_{SE}\}$  as

$$\begin{aligned} \rho_{\mu\nu}(t) &= \sum_{i,j} c_{\mu}^i (c_{\nu}^j)^* \exp\left(\left\{\exp\left[-\frac{i}{\hbar\gamma}(\mathcal{E}_i - \mathcal{E}_j)\right] - 1\right\}\gamma t\right) \\ &\quad \times \sum_{\mu', \nu'} \rho_{\mu' \nu'}(0) (c_{\mu'}^i)^* c_{\nu'}^j. \end{aligned} \quad (2.3)$$

Finally, by performing a trace over the environment, we obtain the solution for the density operator of the system under consideration in the basis  $\{|\alpha\rangle_S\}$ .

### III. MILBURN’S DYNAMICS OF TWO-LEVEL ATOM

Let us assume that a two-level atom (the system) interacts with a single-mode cavity electromagnetic field (environment) that is initially in the vacuum state. The atom-field dynamics in the dipole and the rotating-wave approximations are governed by the Jaynes-Cummings Hamiltonian [2]

$$H = \frac{1}{2} \hbar \omega_A (1 + \sigma_z) + \hbar \omega a^\dagger a + \hbar G (\sigma_+ a + \sigma_- a^\dagger), \quad (3.1)$$

where  $\sigma_{\pm}$  and  $\sigma_z$  are Pauli matrices and  $a$  and  $a^\dagger$  are the field-mode annihilation and creation operators, respectively.  $\omega_A$  is the atomic transition frequency and  $\omega$  is the field mode frequency, while  $G$  is the atom-mode coupling constant. In what follows we will consider that the field is initially in the vacuum state, i.e.,  $|\Phi\rangle_E = |0\rangle$  and the atom in the superposition  $|\Psi\rangle = \alpha |e\rangle + \beta |g\rangle$  of the upper ( $|e\rangle$ ) and the lower ( $|g\rangle$ ) states.

Taking into account that the excitation number is an integral of motion of the Jaynes-Cummings dynamics, it is clear that with the given initial state the dynamics is restricted to the subspace of the total Hilbert space spanned by three orthonormal vectors  $\{|e; 0\rangle, |g; 1\rangle, |g; 0\rangle\}$ . In this basis we diagonalize the Hamiltonian (3.1) and we can write its eigenenergies in the form (see [2])

$$\mathcal{E}_{\pm} = \frac{1}{2} \hbar (\omega_A + \omega) \pm \hbar \sqrt{\left(\frac{\omega_A - \omega}{2}\right)^2 + G^2}, \quad (3.2)$$

and  $\mathcal{E}_3 = 0$ . The corresponding eigenstates can be written as

$$|\Phi_{\pm}\rangle = e_{\pm} |e; 0\rangle + g_{\pm} |g; 1\rangle, \quad (3.3)$$

where the coefficients  $e_{\pm}$  are given by

$$\begin{aligned} e_{\pm} &= \frac{G}{\sqrt{(\mathcal{E}_{\pm}/\hbar - \omega_A)^2 + G^2}}, \\ g_{\pm} &= \frac{\mathcal{E}_{\pm}/\hbar - \omega_A}{\sqrt{(\mathcal{E}_{\pm}/\hbar - \omega_A)^2 + G^2}}, \end{aligned} \quad (3.4)$$

while  $|\Phi_3\rangle = |g; 0\rangle$ . Now, using the results of Sec. II we evaluate the atom-field density operator  $\rho_{SE}(t)$ , which is the solution of the Milburn equation with the Hamiltonian (3.1). After tracing over the field variables we obtain the atomic density operator from which we evaluate the probability  $P_{ee}(t)$  that at time  $t$  the atom is in the excited state

$$\begin{aligned} P_{ee}(t) &= |\alpha|^2 \left[1 + 2|e_+|^2 |e_-|^2\right. \\ &\quad \times \left(\exp\left[\left[\cos\left(\frac{\mathcal{E}_+ - \mathcal{E}_-}{\hbar\gamma}\right) - 1\right]\gamma t\right]\right. \\ &\quad \left.\left. \times \cos\left[\sin\left(\frac{\mathcal{E}_+ - \mathcal{E}_-}{\hbar\gamma}\right)\gamma t\right] - 1\right]\right]. \end{aligned} \quad (3.5)$$

This probability is associated with the diagonal elements of the atomic density operator in the basis  $\{|e\rangle, |g\rangle\}$ . The off-diagonal elements describe atomic quantum coherences. In particular, from the general solution we find that the absolute value of these nondiagonal elements  $P_{eg}(t) = |\langle g | \rho_A(t) | e \rangle|$  reads

$$P_{eg}(t) = \sqrt{A_+^2 + A_-^2 + 2A_+ A_- \cos(\phi_+ - \phi_-)}, \quad (3.6)$$

where

$$A_{\pm} = \alpha \beta^* |e_{\pm}|^2 \exp\left[\left[\cos\left(\frac{\mathcal{E}_{\pm} - \mathcal{E}_3}{\hbar\gamma}\right) - 1\right]\gamma t\right] \quad (3.7)$$

and

$$\phi_{\pm} = \gamma t \sin\left(\frac{\mathcal{E}_{\pm} - \mathcal{E}_3}{\hbar\gamma}\right). \quad (3.8)$$

We start our analysis of the atom dynamics with the case when the atom does not interact with the cavity field, i.e.,  $G=0$ . From the general solution (3.5) for the population of the excited state  $P_{ee}(t)$  we find that it is constant, i.e.,  $P_{ee}(t) = |\alpha|^2$ , which illustrates the fact that the diagonal elements of the density operator of the closed system in the energy basis are not affected by Milburn’s modification of the quantum dynamics. On the other hand, for the absolute value of off-diagonal matrix elements  $P_{eg}(t)$  in the limit  $G \rightarrow 0$  we find

$$P_{eg}(t) = |\alpha \beta^*| \exp\left[\left[\cos\left(\frac{\omega_A}{\gamma}\right) - 1\right]\gamma t\right]. \quad (3.9)$$

In the limit  $\omega/\gamma \ll 1$  this expression can be approximated as  $P_{eg}(t) \approx |\alpha \beta^*| \exp[-\omega^2 t / (2\gamma)]$ , which clearly describes the effect of intrinsic decoherence [3]. From Eq. (3.9) it is also seen that with the decrease of the parameter  $\gamma$  the decoherence is more pronounced. The effect becomes most

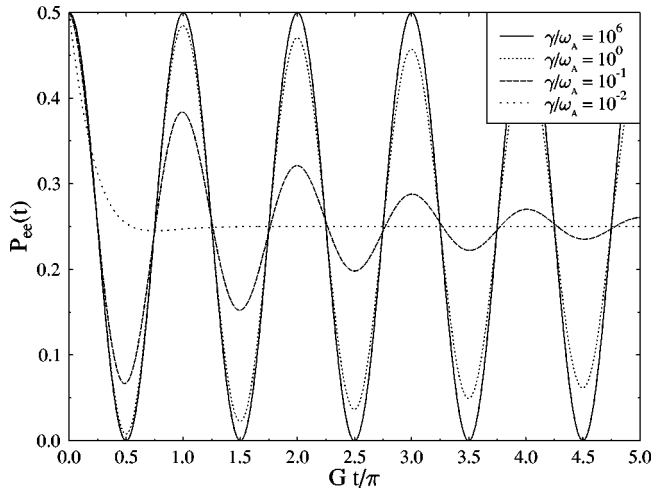


FIG. 1. The population of the excited atomic level  $P_{ee}(t)$  given by Eq. (3.5) as a function of the scaled time  $Gt/\pi$ . The field mode is initially in the vacuum state and the atom is in the state  $|\Psi(0)\rangle = 2^{-1/2}(|e\rangle + |g\rangle)$ . We assume  $\omega = \omega_A$  and  $G = \omega_A/100$ .

transparent for  $\gamma/\omega_A \approx 0.43$ . With the further decrease of  $\gamma$  the decoherence becomes smaller and for  $\gamma/\omega_A = 1/(2\pi)$  the oscillatory behavior of the atomic coherence becomes frozen. Then for even smaller values of  $\gamma$  the function  $P_{eg}(t)$  again exhibits intrinsic decoherence. Nevertheless, it is not as pronounced as for  $\gamma/\omega_A \approx 0.43$ . In the limit  $\gamma \rightarrow 0$  the evolution of the atom is totally “frozen,” which is associated with the introduction of the minimum time step, that is, one cannot produce an atomic oscillator with a period shorter than the minimum time step (see the discussion below and Fig. 2).

We turn our attention back to the dynamics of the composite atom-field system. For simplicity we assume that the field mode is on resonance with the atomic transition frequency  $\omega_A = \omega$ . This allows us to see how the dynamics of an open system is modified by the Milburn equation. In Figs. 1 and 2 we plot the time evolution of  $P_{ee}(t)$  and  $P_{eg}(t)$  given by Eqs. (3.5) and (3.6), respectively. Here we first consider weak coupling, i.e.,  $G = \omega_A/100$ . The atom is initially prepared in the superposition state  $|\Psi(0)\rangle$

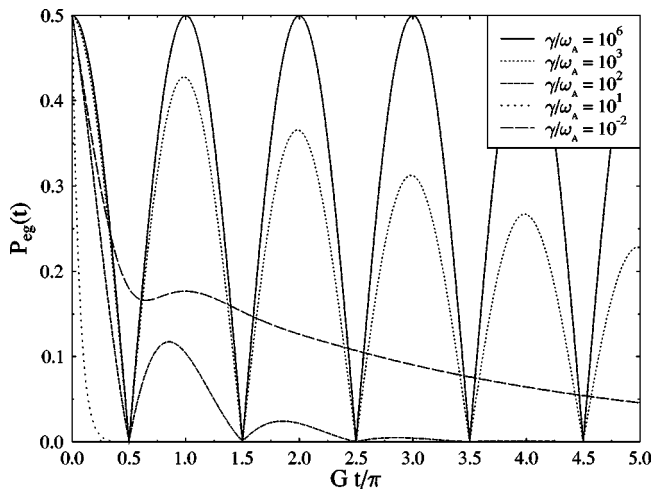


FIG. 2. The time evolution of the atomic coherence  $P_{eg}(t)$  given by Eq. (3.6) as a function of the scaled time  $Gt/\pi$ . All conditions are the same as in Fig. 1.

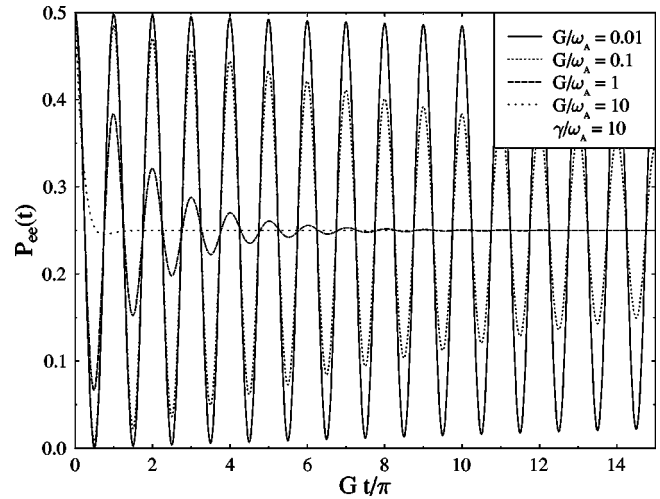


FIG. 3. The population of the excited atomic level  $P_{ee}(t)$  given by Eq. (3.5) as a function of the scaled time  $Gt/\pi$  for various values of the coupling constant  $G$  and fixed value of  $\gamma/\omega_A = 10$ . All other conditions are the same as in Fig. 1.

$= 2^{-1/2}(|e\rangle + |g\rangle)$ . From Fig. 1 we see that for large values of  $\gamma$  (i.e., a very small fundamental time step) the atom exhibits the usual vacuum Rabi oscillations as predicted by the standard Schrödinger equation. Nevertheless, with the decrease of  $\gamma$  not only the intrinsic decoherence of the initial atom coherence becomes transparent (see Fig. 2), but also the atomic population becomes affected by modification of the dynamics. As seen from Fig. 1, for small values of  $G$  the frequency of Rabi oscillations does not depend on  $\gamma$ , but the amplitude of these oscillations become smaller. For  $\gamma$  small enough (in our picture  $\gamma/\omega_A = 0.01$ , i.e., in this case  $\gamma = G$ ) the Rabi oscillations become completely suppressed and the atom “collapses” very rapidly to the mixture  $\rho = \frac{1}{4}|e\rangle\langle e| + \frac{3}{4}|g\rangle\langle g|$ . Obviously, this is a completely different situation compared to the Schrödinger picture. This means that the Milburn dynamics leads to changes in populations in energy eigenstates of an open system (i.e., a sub-system of the composite system). In fact, the larger the coupling  $G$  between the subsystems, the larger is the deviation from the Schrödinger dynamics. This property of the Milburn dynamics is clearly seen from Fig. 3, in which we plot the time evolution of the function  $P_{ee}(t)$  for the fixed value of  $\gamma$  (we take  $\gamma/\omega_A = 10$ ) and for various values of  $G$ .

We note that in the limit of very strong coupling  $G$  the Milburn dynamics can be significantly different from the Schrödinger evolution even for very large values of  $\gamma$  (small fundamental time steps). To be specific, let us introduce a parameter  $\xi = 2G/\gamma$ . If  $\xi \neq 2\pi n$ , then in the strong-coupling limit  $G \rightarrow \infty$ , when simultaneously  $\gamma \rightarrow \infty$  (so that  $\xi = \text{const}$ ) the function  $P_{ee}(t)$  given by Eq. (3.5) describes an “instantaneous” collapse to the stationary value  $|\alpha|^2/2$ . On the contrary, the Schrödinger dynamics describes periodic Rabi oscillations (obviously, the frequency of these oscillations is very large in the large-coupling limit).

We conclude this section with a brief remark on “freezing” of the Milburn dynamics. From Eq. (3.5) we see that for  $G/\gamma = \pi n$  the atomic-level population  $P_{ee}(t)$  is frozen (here we assume that  $\omega = \omega_A$ ). If in addition  $\omega_A/\gamma = \pi n$ , then also the atomic coherences do not evolve, i.e.,

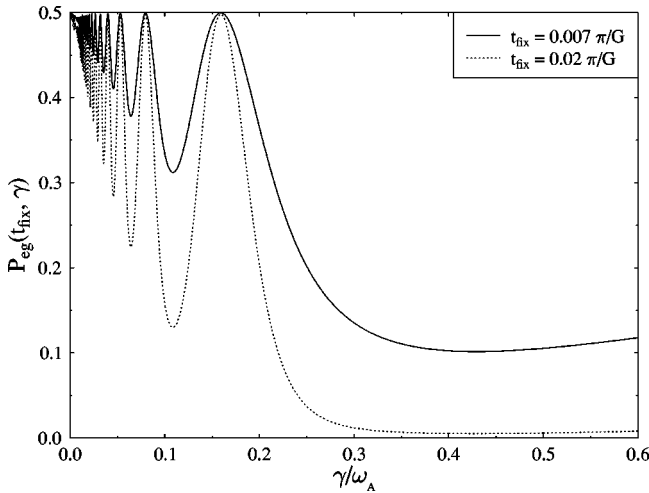


FIG. 4. The atomic coherence  $P_{eg}(t)$  given by Eq. (3.6) as a function of  $\gamma/\omega_A$  for two fixed values of time  $t_{\text{fix}}$ . All conditions are the same as in Fig. 1.

$P_{eg}(t) = \text{const.}$  For illustration we plot in Fig. 4 the time evolution of  $P_{eg}(t)$  for two fixed values of time as a function of  $\gamma/\omega_A$ . We see that the decoherence is strongest for  $\gamma/\omega_A \approx 0.43$ , which is the same as for an isolated two-level atom (see above). The evolution is frozen for  $\gamma/\omega_A = 1/(2\pi n)$ . In the limit  $\gamma \rightarrow 0$ , the system does not evolve at all.

#### IV. MODIFICATION OF THE WEISSKOPF-WIGNER DECAY

In what follows we want to generalize the results of the preceding section and we will study the dynamics of the atom coupled to the multimode vacuum of the electromagnetic field. The corresponding Hamiltonian in the dipole and the rotating-wave approximations reads [4]

$$H = \frac{1}{2} \hbar \omega_A (1 + \sigma_z) + \sum_{\lambda} \hbar \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda} + \sum_{\lambda} \hbar (G_{\lambda} \sigma_+ a_{\lambda} + G_{\lambda}^* \sigma_- a_{\lambda}^{\dagger}), \quad (4.1)$$

where summations run over the field modes. We will consider the field to be initially in the vacuum state and the atom in the superposition state  $|\Psi\rangle = \alpha|e\rangle + \beta|g\rangle$ .

Combining the results of Sec. II with the numerical diagonalization of the Hamiltonian (4.1) we have analyzed the decay of the two-level atom under consideration into the multimode vacuum of the electromagnetic field. We have considered 500 modes of the field, such that in the scaled units  $\omega_A = 1$  and the cutoff frequency is  $\omega_{\text{cutoff}} = 8$ . The results of our numerical simulations are presented in Fig. 5, which shows the time evolution of the population of the upper atom level in the weak-coupling limit ( $G = \omega_A/80$ ). We see that for a very small fundamental time step ( $\gamma/\omega_A = 10^6$ ) the Milburn dynamics leads to the exponential decay of the atomic population  $P_{ee}(t)$ , which is exactly the same as within the framework of the Weisskopf-Wigner theory [4] (see solid line in Fig. 5). With the increase of the

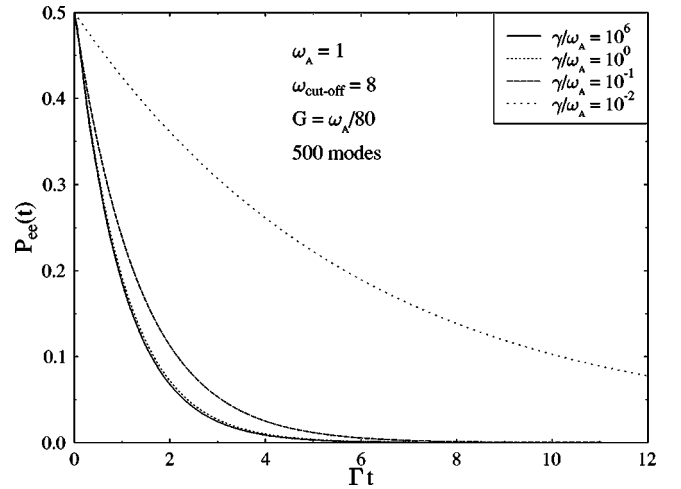


FIG. 5. The population of the excited atomic level  $P_{ee}(t)$  as a function of the scaled time  $\Gamma t$ . We assume  $N = 500$  field modes and all of them are initially prepared in the vacuum state. The atomic-transition frequency  $\omega_A$  is chosen in scaled units to be equal to unity and the cutoff frequency  $\omega_{\text{cutoff}} = 8$ . In this figure we assume the weak-coupling limit with constant  $G_{\lambda}$  equal to  $G = \omega_A/80$  for all  $\lambda$ 's. The decay rate given by Weisskopf-Wigner theory [4] reads  $\Gamma = 2\pi G^2 d$  where for the density of modes we have  $d = N/\omega_{\text{cutoff}}$ . We assume the atom initially in the state  $|\Psi(0)\rangle = 2^{-1/2}(|e\rangle + |g\rangle)$ .

fundamental time step we observe suppression of the decay, which is caused by the nonunitary evolution of the system on a short time scale (here ‘‘short’’ is specified by the value of  $\gamma$ ). This freezing of the time evolution is analogous to the quantum Zeno effect [5]. The difference is that the nonunitarity in the quantum Zeno effect is caused by a sequence of frequent measurements while in our case the nonunitarity of the evolution is caused by the existence of the fundamental time step. In the Milburn model the physical background for this fundamental time step is not analyzed. Anyway, in the particular case of an atom discussed in this paper one may find an analogy between the Milburn evolution and the dynamics of the atom when the atomic collisions are taken into account. These collisions can be considered to be responsible for a nonunitary evolution of the atom and the frequency of collisions is then directly related to the time step considered by Milburn. On the other hand, there might exist another more fundamental reason (e.g., a structure of the space-time is not continuous), which would justify the Milburn equation (if it is correct). Anyway, it is not the aim of our paper to illuminate this fundamental question. Our task is to show that the Milburn dynamics of open systems might be completely different from the Schrödinger dynamics and that maybe this difference can be observed.

To complete the picture of the decay of a two-level atom, we present in Fig. 6 the time evolution of the function  $P_{eg}(t)$  representing atomic coherences. We see that quantum coherences in the Milburn model of the atomic decay depend on the value of  $\gamma$ . In particular, with the decrease of  $\gamma$  coherences are deteriorated faster than in the standard Weisskopf-Wigner model. The fastest intrinsic decoherence in the weak-coupling limit can be observed for  $\gamma/\omega_A \approx 0.43$ . Nevertheless, further decrease of  $\gamma$  does not lead to faster deterioration of quantum coherences. On the contrary, for  $\gamma$

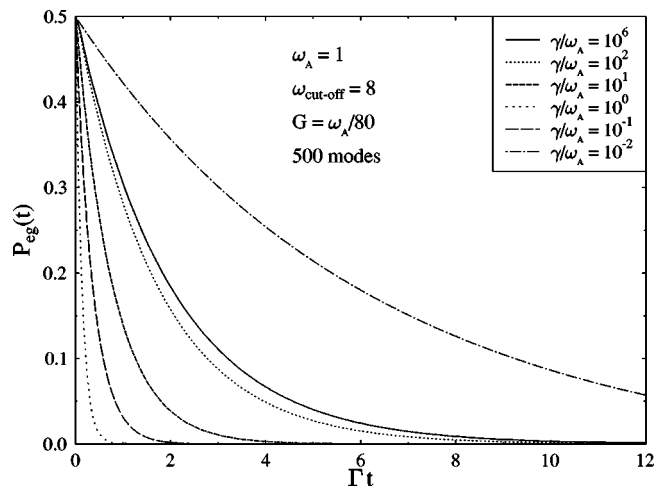


FIG. 6. The time evolution of the atomic coherence  $P_{eg}(t)$  as a function of the scaled time  $\Gamma t$ . All conditions are the same as in Fig. 5.

small enough,  $P_{eg}(t)$  decays slower than in the Weisskopf-Wigner model and in the limit  $\gamma \rightarrow 0$  the dynamics is completely frozen.

We have also studied dynamics of the two-level atom interacting with the multimode vacuum ( $N=500$ ) in the strong-coupling limit. Here the problem is that the standard Weisskopf-Wigner model with a large but finite number of

modes for large coupling constants does not lead to an exponential decay of the atom. In particular, for a certain choice of the system parameters the population of the atomic upper level will periodically oscillate, while within the framework of the Milburn model the dynamics of the atom can be partially frozen. Because of numerical problems we have not been able to analyze this regime in detail.

## V. DISCUSSION AND CONCLUSIONS

In the paper we have shown that in the Milburn model the dynamics of open systems can be significantly modified in a sense that populations of eigenstates of the subsystem Hamiltonian do evolve differently compared to the Schrödinger dynamics. Within the framework of the model describing an interaction of a single two-level atom and a single-mode electromagnetic field, we have shown that in the strong-coupling limit Rabi oscillations can be completely suppressed even for very short fundamental time steps while the system rapidly “collapses” into a statistical mixture. On the other hand, in the weak-coupling limit and the large enough fundamental time step we can observe freezing of the atomic subsystem.

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