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QUANTUM SYNTHESIS OF 3D VIBRATIONAL STATES OF TRAPPED IONS[1]

B. Hladký[2]
Department of Optics, Comenius University, Mlynská dolina, 84215 Bratislava, Slovakia

G. Drobný[3], **V. Bužek**[4]
Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia

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A universal algorithm for a deterministic preparation of arbitrary three–mode bosonic states is introduced. In particular, we consider preparation of entangled quantum states of a vibrational motion of an ion confined in a 3D trapping potential. The target states are established after a proper sequence of laser stimulated Raman transitions. Stability of the algorithm with respect to a technical noise is discussed and the distance (fidelity) of outputs with respect to target states is studied.

1. Introduction

The deterministic control of preparation of genuine quantum states is one of the desired goals of quantum physics. Recent advances in atomic physics (trapped ions [1, 2]) and quantum optics (cavity QED [3]) have enabled an effective control of dynamics of microscopic quantum systems. These systems can serve for tests of fundamental concepts of quantum mechanics and quantum measurement theory. Other applications can be related to quantum computing [4, 5] and information processing [6].

A great effort has been devoted to the task of preparation of various nonclassical states (e.g., Schrödinger cats, Fock and squeezed states) [7]. Formal introduction of some nonclassical states (e.g., squeezed states) has been based on the Perelomov scheme, i.e., under the action of an appropriate unitary operator on a reference state. This most simple way of quantum-state preparation can be represented as a time evolution of an

[1]Special Issue on Quantum Optics and Quantum Information
[2]E-mail address: hladky@fmph.uniba.sk
[3]E-mail address: drobny@savba.sk
[4]also at Department of Optics, Comenius University, Mlynská dolina, 84215 Bratislava, Slovakia

input state governed by a proper Hamiltonian. However, physical implementation of a required Hamiltonian can face insurmountable difficulties. Quantum-state-engineering methods introduced firstly in the cavity QED (micromasers) have overcome these restrictions by performing conditional measurements on atoms which result in a projection of the cavity field on desired target states [8, 9]. However, due to a specific selection of quantum trajectories these methods suffer from low probabilities of desired outputs.

The process of *deterministic* preparation of an arbitrary one–mode state of electromagnetic field has been described recently by Law and Eberly [10]. In their approach a proper sequence of unitary transformations associated with a sequential switching of two interaction channels transforms the vacuum state of the field to prescribed target wave-function. The experimental preparation of quantum states of 1-D vibrational motion of trapped ions has utilized switching between interaction channels via tuning lasers either to electronic transitions or appropriate vibrational sidebands [1]. Possible generalizations of the 1-D quantum state synthesis to 2-D have been discussed recently [11, 12, 13].

In this paper we consider a deterministic preparation of multi–mode bosonic states. In particular, we consider synthesis of quantum states of a vibrational motion of an ion confined in a 3D trapping potential. The main motivation for utilizing trapped ions is a possibility to neglect decoherence effects due to dissipations. An effective control of coupling between external and internal degrees of freedom via a proper laser irradiation is also advantageous.

2. Quantum state synthesis

Let us consider a preparation of an arbitrary three–mode bosonic state

$$|\Psi_{target}\rangle = \sum_{n_x=0}^{M_x}\sum_{n_y=0}^{M_y}\sum_{n_z=0}^{M_z} Q_{n_x,n_y,n_z}|n_x,n_y,n_z\rangle. \tag{1}$$

In particular, the three modes correspond to vibrational modes of a quantized center–of–mass motion of an ion confined in the 3D trapping potential. The key idea of the quantum state synthesis is a splitting of the whole Hilbert space $\mathcal{H}_{vib}$ of three–mode vibrational states into a hierarchy of two–dimensional subspaces. This enables us to transfer populations between pairs of component Fock states (see below). However, this dynamical "ordering" requires a coupling of vibrational degrees of freedom to an adjoint system. For the trapped ion it is natural to assume the coupling between external (vibrational) and internal (electronic) degrees of freedom. For purposes of our quantum-state-synthesis algorithm we utilize four internal electronic levels $|i\rangle$ $(i = a, b, c, d)$ spanning the Hilbert space $\mathcal{H}_{in}$. In general, during the preparation the vibrational and internal states become entangled and the whole state vector can be written as a superposition of "component" states

$$|\Psi(t)\rangle \quad = \quad \sum_{n_x=0}^{M_x}\sum_{n_y=0}^{M_y}\sum_{n_z=0}^{M_z}\sum_{i=a,b,c,d} Q_{n_x,n_y,n_z;i}(t)|n_x,n_y,n_z\rangle\otimes|i\rangle \tag{2}$$

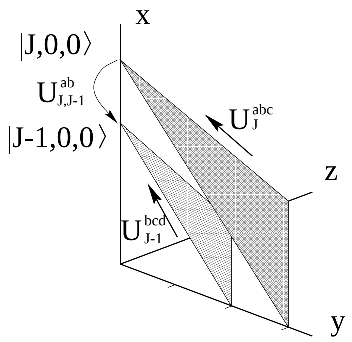

Fig. 1. The scheme of the de-evolution procedure (3): a recursive transfer of excitations to a subspace with a number of trap quanta decreased by one.

$$= \sum_{J=0}^{J_{max}} \sum_{n_x=0}^{J} \sum_{n_y=0}^{J-n_x} \sum_{i=a,b,c,d} Q_{n_x,n_y,J-n_x-n_y;i}(t)|n_x, n_y, J-n_x-n_y\rangle \otimes |i\rangle.$$

The first summation in the second expression is taken over subspaces $\mathcal{H}_J$ of the vibrational Hilbert space $\mathcal{H}_{vib}$. The subspaces $\mathcal{H}_J$ are spanned with component states $|n_x, n_y, n_z\rangle$ with a fixed total number of trap quanta $J = n_x + n_y + n_z$.

Firstly we solve for the given target state (1) an inverse task, i.e. we "de-evolve" it towards the reference vacuum state via a proper sequence of "elementary" unitary operations which are associated with particular laser stimulated Raman processes (see below). The "de-evolution" into vacuum can be schematically written as (see Fig. 1)

$$|0,0,0\rangle \otimes |a\rangle = \left[\hat{U}_0^{abc} \prod_{J=1}^{J_{max}} \hat{U}_{J-1,J}^{ab} \hat{U}_{J-1}^{bcd} \hat{U}_J^{abc}\right] |\Psi_{target}\rangle \otimes |a\rangle. \qquad (3)$$

Here superscripts and subscripts indicate electronic levels and vibrational states from subspace $\mathcal{H}_J$ which are involved in a controlled "de-evolution" of the state vector (2). Namely, $\hat{U}_J^{abc}$ transfers populations of components states $|n_x, n_y, n_y\rangle \otimes |i\rangle$ with $n_x + n_y + n_y = J$ and $i = a, b, c$ to a single component state $|J, 0, 0\rangle \otimes |a\rangle$ which then enters the state vector (2) on the subspace under consideration. Next $\hat{U}_{J-1}^{bcd}$ *completely* transfers populations of components states $|n_x, n_y, n_y\rangle \otimes |i\rangle$ with $n_x + n_y + n_y = J-1$ and $i = b, c, d$ to the state $|J-1, 0, 0\rangle \otimes |b\rangle$. After that $\hat{U}_{J-1,J}^{ab}$ transfers the actual population of the components state $|J, 0, 0\rangle \otimes |a\rangle$ to $|J-1, 0, 0\rangle \otimes |b\rangle$. As result, the sequence $\hat{U}_{J-1,J}^{ab} \hat{U}_{J-1}^{bcd} \hat{U}_J^{abc}$ outlined in Fig. 1 cancels the contribution of component states from the Hilbert subspace $\mathcal{H}_J \otimes \mathcal{H}_{in}$ into the state vector (2). Recursive application of this sequence on subspaces with decreasing J evolves the initial target state into the vacuum.

Note that the solution of the inverse task (3) offers immediately also the prescription for preparation of the target state from the vacuum as

$$|\Psi_{target}\rangle \otimes |a\rangle = \left[\prod_{J=J_{max}}^{1} \hat{U}_J^{\dagger abc} \hat{U}_{J-1}^{\dagger bcd} \hat{U}_{J-1,J}^{\dagger ab}\right] \hat{U}_0^{\dagger abc} |0,0,0\rangle \otimes |a\rangle. \qquad (4)$$

In what follows we describe in details how to "construct" unitary transformations appearing in Eq.(3) from a set of "elementary" transformations (time evolution operators) associated with laser stimulated Raman processes for a trapped ion. Therefore the preparation sequence (4) corresponds to the action of "elementary" transformations applied in the opposite order and with phases of lasers shifted globally by $\pm\pi$.

3. Manipulation of trapped ions: laser-stimulated Raman processes

In this section we briefly introduce "tools" necessary for our manipulations with vibrational states of trapped ions. For quantum state synthesis of 3D motional states of a trapped ion we utilize laser stimulated Raman processes [14]. Let us consider a trapped ion confined in a 3D harmonic potential characterized by trap frequencies ν_q in orthogonal directions $q = x, y, z$. The ion is irradiated along two axes (y and z for concreteness) by two external laser fields with frequencies ω_y, ω_z and wave vectors k_y, k_z, respectively. The laser fields stimulate Raman transitions between two internal energy levels $|a\rangle$ and $|b\rangle$ via an auxiliary electronic level which is far off the resonance. After the standard dipole and rotating–wave approximations (RWA at laser frequencies) the adiabatic elimination of the auxiliary off-resonant level leads to the effective interaction Hamiltonian [14]

$$\hat{H}_1^{(eff)} = g^* \mathrm{e}^{-i(\omega_z - \omega_x)t} \hat{D}_y(-i\epsilon_y)\hat{D}_z(i\epsilon_z)|b\rangle\langle a| + \mathrm{h.c.} \tag{5}$$

Here $\hat{D}_q(i\epsilon_q) = \mathrm{e}^{i\epsilon_q(\hat{a}_q^\dagger + \hat{a}_q)} = \mathrm{e}^{ik_q\hat{q}}$ is the displacement operator; $\hat{a}_q, \hat{a}_q^\dagger$ are the creation and annihilation operators of the vibrational mode q. The corresponding Lamb–Dicke parameter ϵ_q is defined as $\epsilon_q^2 = \hbar^2 k_q^2/(2m\hbar\nu_q)$. The effective interaction constant $g \sim \mathcal{E}_y^* \mathcal{E}_z/\Delta$ is proportional to complex laser amplitudes $\mathcal{E}_q$ and inversely proportional to detuning Δ of lasers from the auxiliary level. The resonant terms in the Taylor series expansion of Eq.(5) which contribute dominantly to the resulting effective Hamiltonian can be selected by an appropriate choice of laser frequencies. If off–resonant processes are oscillating with sufficiently high frequencies they can be eliminated applying the second RWA at trap frequencies. In particular, tuning lasers to the first vibrational sidebands in order to adjust resonance conditions $\omega_1 \equiv \omega_z - \omega_y = \omega_b - \omega_a + \nu_y - \nu_z$ we find that the retained resonant terms lead to the interaction Hamiltonian [14]

$$\hat{H}_1 \quad = \quad g_1^* \hat{a}_y^\dagger \hat{\mathcal{F}}_{\epsilon_y}[\hat{n}_y]\hat{\mathcal{F}}_{\epsilon_z}[\hat{n}_z]\hat{a}_z|b\rangle\langle a|\mathrm{e}^{-i\omega_1 t} + \mathrm{h.c.}, \tag{6}$$

where

$$\hat{\mathcal{F}}_{\epsilon_q}[\hat{a}_q^\dagger \hat{a}_q] \quad = \quad \exp(-\frac{1}{2}\epsilon_q^2)\sum_k \frac{(-1)^k \epsilon_q^{2k}}{(k+1)!k!}\hat{a}_q^{\dagger k}\hat{a}_q^k. \tag{7}$$

The generalized Rabi frequency $\Omega_{n_y,n_z}^{(1)}$ of the transition between the states $|n_x, n_y, n_z\rangle \otimes |a\rangle$ and $|n_x, n_y+1, n_z-1\rangle \otimes |b\rangle$ is given as $\mathrm{e}^{-\frac{\epsilon_y^2+\epsilon_z^2}{2}} L_{n_y}^1(\epsilon_y^2) L_{n_z-1}^1(\epsilon_z^2)/\sqrt{(n_y+1)n_z}$ where L_n^1 is the associated Laguerre polynomial. In the Lamb–Dicke limit of $\epsilon_q \ll 1$ it reads $\Omega_{n_y,n_z}^{(1)} \to \sqrt{(n_y+1)n_z}$ as $\hat{\mathcal{F}}_{\epsilon_q}$ is close to the unity operator.

With the proper laser tunings we can design also other interaction Hamiltonians required for the quantum state synthesis. Namely we consider counterpropagating laser beams in the x direction which classically drive the electronic transition $|a\rangle \leftrightarrow |b\rangle$ via a resonant laser-stimulated Raman process. The resonance condition for laser frequencies $\omega_2 \equiv \omega_{x2} - \omega_{x1} = \omega_b - \omega_a$ leads to the interaction Hamiltonian

$$\hat{H}_2 \quad = \quad g_2^* \hat{\mathcal{F}}_{\epsilon'_x} |b\rangle\langle a| \mathrm{e}^{-i\omega_2 t} + g_2 \hat{\mathcal{F}}_{\epsilon'_x} |a\rangle\langle b| \mathrm{e}^{i\omega_2 t}. \tag{8}$$

The effective Lamb-Dicke parameter ϵ'_x with $k' = k_{x2} - k_{x1}$ is much smaller compared with ϵ_x and thus for simplicity it is assumed to be close to zero. Consequently $\hat{\mathcal{F}}_{\epsilon'_x}$ can be approximated by the unity operator. The analogous process with the laser frequencies such that $\omega_9 \equiv \omega_{x2} - \omega_{x1} = \omega_b - \omega_a - \nu_x$ is described by the one–mode interaction Hamiltonian $\hat{H}_3$

$$\hat{H}_9 \quad = \quad g_9^* \hat{\mathcal{F}}_{\epsilon'_x} \hat{a}_x |b\rangle\langle a| \mathrm{e}^{-i\omega_9 t} + g_9 \hat{\mathcal{F}}_{\epsilon'_x} \hat{a}_x^\dagger |a\rangle\langle b| \mathrm{e}^{i\omega_9 t} \tag{9}$$

which is known as the nonlinear Jaynes-Cummings model [15].

4. "De-evolution" procedure as sequence of "elementary" transformations

Let us now assume that at some stage of the "de-evolution" procedure we operate on the subspace $\mathcal{H}_J \otimes \mathcal{H}_{abc}$ and contributions of component states from subspaces with higher total number of trap quanta J as well as from the subspace $\mathcal{H}_J \otimes \mathcal{H}_d$ into the state vector (2) have been already canceled. (The subspaces of $\mathcal{H}_{in}$ are spanned by indicated electronic levels.) Firstly, we explicitly construct the operator $\hat{U}_J^{abc}$ from Eq.(4) which transfers *completely* populations of component states from the subspace under consideration to the single component state $|J,0,0\rangle \otimes |a\rangle$. The operator $\hat{U}_J^{abc}$ can be expressed as

$$\begin{aligned}
\hat{U}_J^{abc} &= \hat{A}_{J,J} \prod_{n_x=0}^{J-1} \hat{C}_{J,n_x} \hat{B}_{J,n_x+1} \hat{A}_{J,n_x}, \qquad \hat{C}_{J,n_x} = \hat{U}^{(5)}_{|n_x,J-n_x,0;b\rangle}, \qquad (10)\\
\hat{A}_{J,n_x} &= \hat{U}^{(2)}_{|n_x,J-n_x,0;b\rangle} \hat{U}^{(1)}_{|n_x,J-1-n_x,1;a\rangle} \cdots \hat{U}^{(2)}_{|n_x,1,J-1-n_x;b\rangle} \hat{U}^{(1)}_{|n_x,0,J-n_x;a\rangle},\\
\hat{B}_{J,n_x} &= \left[\hat{U}^{(4)}_{|n_x,J-n_x,0;c\rangle} \hat{U}^{(3)}_{|n_x,J-1-n_x,1;b\rangle} \cdots \hat{U}^{(4)}_{|n_x,1,J-1-n_x;c\rangle} \hat{U}^{(3)}_{|n_x,0,J-n_x;b\rangle} \right] \hat{U}^{(4)}_{|n_x,0,J-n_x;c\rangle}.
\end{aligned}$$

Here an "elementary" unitary transformation $\hat{U}^{(p)}_{|n_x,n_y,n_z;i\rangle} = \exp(-i\tau \hat{H}_p)$ is equal to the evolution operator corresponding to a particular interaction Hamiltonian $\hat{H}_p$ applied for time τ. Subscript indicates a component state $|n_x, n_y, n_z\rangle \otimes |i\rangle$ the population of which is completely transferred to another component state according to the applied interaction "channel" p.

Actions of the "elementary" transformations which form the operator $\hat{U}_J^{abc}$ are outlined in Fig. 2a. In particular, under the action of $\hat{U}^{(1)}_{|n_x,J-n_x,0;a\rangle}$ [see Eq.(10)] the *complete* transfer of the population of the component state $|n_x, 0, J - n_x\rangle \otimes |a\rangle$ to the component state $|n_x, 1, J-1-n_x\rangle \otimes |b\rangle$ is achieved providing that the interaction time

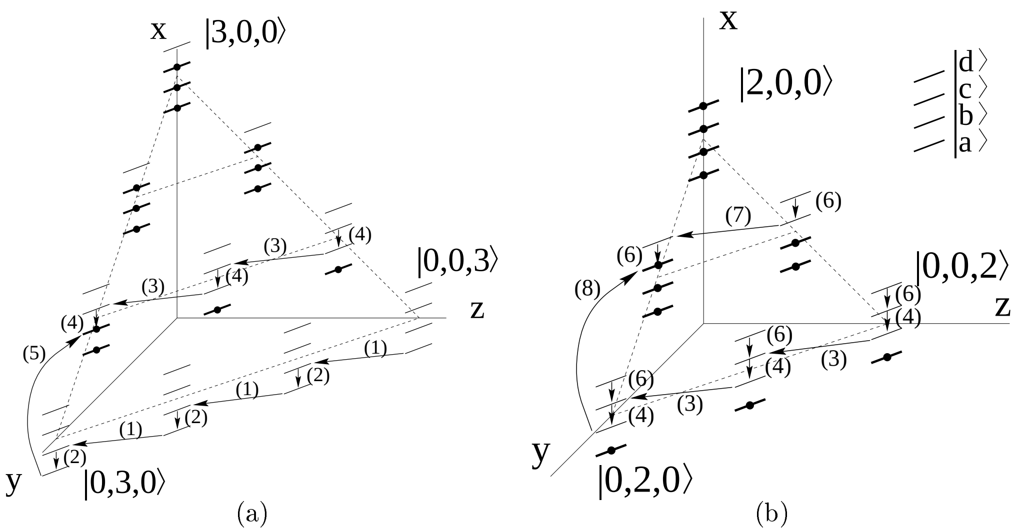

Fig. 2. The scheme of the de-evolution procedure (3). (a) Transfer of populations within the subspace with a given number of trap quanta J under action of $\hat{U}_J^{abc}$. (b) The "neighboring"subspace with the number of trap quanta $J-1$ under action of $\hat{U}_{J-1}^{bcd}$. Labels of arrows indicate the "direction" of action of the elementary transformations $\hat{U}^{(p)}$.

τ and parameters $|g_1|, \theta_1$ of the Hamiltonian $\hat{H}_1$ (6) are chosen to fulfill for $n_y = 0$ the condition ($n_z = J - n_x - n_y$):

$$\mathrm{ie}^{-\mathrm{i}\theta_1} Q_{n_x,n_y,n_z;a} \cos[|g_1|\tau\Omega^{(1)}_{n_y,n_z}] + Q_{n_x,n_y+1,n_z-1;b} \sin[|g_1|\tau\Omega^{(1)}_{n_y,n_z}] = 0. \tag{11}$$

Next, the transfer of the population from the component state $|n_x, 1, J-1-n_x\rangle \otimes |b\rangle$ to the component state $|n_x, 1, J-1-n_x\rangle \otimes |a\rangle$ is achieved under the action of the interaction Hamiltonian $\hat{H}_2$ (8). The following choice of the interaction parameters $|g_2|\tau$ and θ_2 is required for $n_y = 1$

$$\mathrm{ie}^{\mathrm{i}\theta_2} Q_{n_x,n_y,J-n_x-n_y;b} \cos(|g_2|\tau) + Q_{n_x,n_y,J-n_x-n_y;a} \sin(|g_2|\tau) = 0. \tag{12}$$

The repeated sequence of these "elementary" transformations forms the operator $\hat{A}_{J,n_x}$ [see Eq.(10)] which transfers the population of the states $|n_x, n_y, J-n_y-n_x\rangle \otimes |i\rangle$ with $n_y = 0, \ldots, J-n_x$ and $i = a, b$ to the component state $|n_x, J-n_x, 0\rangle \otimes |a\rangle$.

Further, under the action of the operator $\hat{B}_{J,n_x+1}$ populations of states $|n_x + 1, n_y, J-n_y-(n_x+1)\rangle \otimes |i\rangle$ with $n_y = 0, \ldots, J-(n_x+1)$ and $i = b, c$ are transferred to the component state $|n_x+1, J-(n_x+1), 0\rangle \otimes |b\rangle$. It is seen from Eq.(10) that the form of $\hat{B}_{J,n_x+1}$ (acting on "neighboring lines" in Fig.2a with number of x-quanta increased by one) is quite analogous to $\hat{A}_{J,n_x}$ but instead of levels $|a\rangle, |b\rangle$ the electronic levels $|b\rangle, |c\rangle$ are involved in the transfer of populations. The corresponding Hamiltonians $\hat{H}_3$, $\hat{H}_4$ and the transfer conditions are obtained from Eqs.(6),(11) and (8),(12) by the substitutions: $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$.

As the next step of the "de-evolution" sequence (10) the operator $\hat{C}_{J,n_x} = \hat{U}^{(5)}_{|n_x,J-n_x,0;b\rangle}$ is applied to the state vector (2). It transfers the whole population of $|n_x, J-n_x, 0\rangle \otimes |a\rangle$ to $|n_x+1, J-(n_x+1), 0\rangle \otimes |b\rangle$. The corresponding interaction Hamiltonian $\hat{H}_5$ is obtained from $\hat{H}_1$ after replacements $y \to x$, $z \to y$. In other words, two lasers stimulating the Raman transition operate in x, y directions instead of y, z. Note that the action of the operator $\hat{B}_{J,n_x+1}$ was necessary to avoid a transfer of population backwards to component states with smaller n_x on the given subspace $\mathcal{H}_J \otimes \mathcal{H}_{abc}$.

Altogether, the rôle of the operator $\hat{U}^{abc}_J$ (10) is to transfer populations of all component states from $\mathcal{H}_J \otimes \mathcal{H}_{abc}$ to the only component state $|J,0,0\rangle \otimes |a\rangle$. The next important step in the "de-evolution" procedure (3) is related to the operator $\hat{U}^{bcd}_{J-1}$ which transfers populations of all component states from the "neighboring" subspace $\mathcal{H}_{J-1} \otimes \mathcal{H}_{bcd}$ to its only component state $|J-1,0,0\rangle \otimes |b\rangle$. The operator $\hat{U}^{bcd}_J$ can be built up from a set of "elementary" transformations in a close analogy with the unitary transformation $\hat{U}^{abc}_J$ by utilizing the electronic levels $i = b, c, d$ instead of $i = a, b, c$. The action of $\hat{U}^{bcd}_{J-1}$ represented by a sequence of "elementary" transformations is outlined in Fig. 2b. The interaction Hamiltonians $\hat{H}_6$ and $\hat{H}_7$ associated with the indicated "elementary" transformations are obtained from $\hat{H}_4$ and $\hat{H}_3$, respectively, by the substitutions $|c\rangle \to |d\rangle$ and $|b\rangle \to |c\rangle$. The Hamiltonian $\hat{H}_8$ arises from $\hat{H}_5$ after substitutions $|b\rangle \to |c\rangle$ and $|a\rangle \to |b\rangle$.

The next step which connects the subspaces shown in Figs. 2a,b has been already sketched in Fig. 1. The operator $\hat{U}^{ab}_{J-1,J} = \hat{U}^{(9)}_{|J,0,0;a\rangle}$ associated with the interaction Hamiltonian (9) transfers the population of the component state $|J,0,0\rangle \otimes |a\rangle$ to $|J-1,0,0\rangle \otimes |b\rangle$. In other words, the operator $\hat{U}^{ab}_{J-1,J}$ moves us from the subspace $\mathcal{H}_J \otimes \mathcal{H}_{abc}$ to the subspace $\mathcal{H}_{J-1} \otimes \mathcal{H}_{abc}$. Note that contribution from $\mathcal{H}_{J-1} \otimes \mathcal{H}_d$ into the state vector (2) has been canceled by the previous action of $\hat{U}^{bcd}_{J-1}$. It means that we can apply the whole "de-evolution" procedure (3) recursively decreasing the total number of trap quanta.

5. Remarks on imperfect preparation

The *deterministic* preparation procedure as described above is based on the unitary time evolution (4) of the reference state to the target state. The solution for the inverse problem (3) can be found only in the absence of dissipations. However, one important source of noise (even for almost dissipation–free dynamics of trapped ions) is an imperfect control of interaction constants (which depend on laser amplitudes and phases) and switching times required for transfer of populations of component states. To estimate the influence of this "technical noise" now we introduce random fluctuations around the ideal interaction constants and switching times defined by (4). In particular, we assume that fluctuations are equally distributed around the ideal values of interaction lengths $|g_p|\tau$ [e.g., solutions of Eqs.(11,12)] within a given interval δ. Also phases fluctuate around their ideal values θ_p within a fixed interval δ_θ. The preparation sequence (4) with non-ideal interaction constants and switching times then results in some output state $|\Psi_\delta\rangle$ of the form (2). Its internal and external degrees of freedom are in general entangled. The deviation from the desired target state $|\Psi_{target}\rangle \otimes |a\rangle$ can

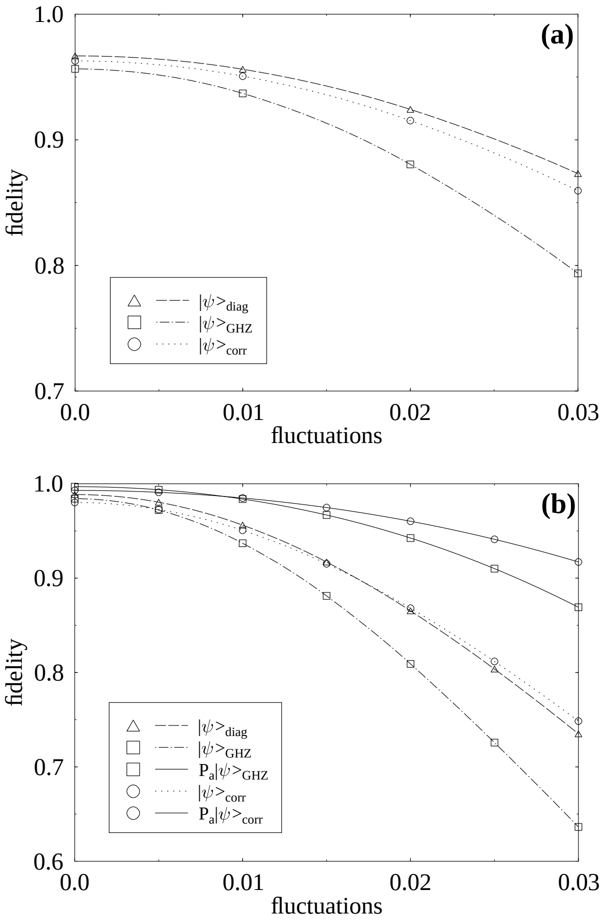

Fig. 3. Fidelities of the outputs to the indicated target states in the presence of a technical noise. (a) Varying phase fluctuations δ_θ with fixed fluctuations $\delta = 0.01$ of the interaction lengths. (b) Varying fluctuations δ of interaction lengths with fixed phase fluctuations $\delta_\theta = 0.01$.

be quantified using an average fidelity of particular imperfect realizations $|\Psi_\delta\rangle$ to the target state:

$$f_\delta = \langle\!\langle |\langle \Psi_\delta | \Psi_{target}\rangle |a\rangle|^2 \rangle\!\rangle_\delta, \tag{13}$$

where $\langle\!\langle \ldots \rangle\!\rangle_\delta$ stands for the average over imperfect realizations with a given range of fluctuations δ. We have performed simulations of noisy quantum state synthesis for three specific target states. We have considered the three-mode "Greenberger–Horne–Zeilinger" state $|\Psi_{GHZ}\rangle = \mathcal{N}(|\alpha,\alpha,\alpha\rangle + |-\alpha,-\alpha,-\alpha\rangle)$ with coherent components; three–mode correlated state $|\Psi_{corr}\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n,n,n\rangle$, with $\alpha = 1$; and the superposition state $|\Psi_{diag}\rangle = \frac{1}{\sqrt{5}}\sum_{n=0}^{4}|n,n,n\rangle$. The cutoff in Fock basis is chosen as $J_{max} = 10$. The corresponding average fidelities (over 100 noisy preparations) are shown in Fig. 3a for an increasing interval of phase fluctuations δ_θ around the ideal phases θ_p. The interaction lengths fluctuate around ideal values $|g_p|\tau$ within a fixed interval $\delta = 0.01$. The Lamb–Dicke parameters are chosen as $\epsilon_x = 0.3$, $\epsilon_y = 0.1$, $\epsilon_z = 0.2$, $\epsilon'_x = 0.1$. In Fig. 3b we show averaged fidelities for increasing fluctuations δ of the interaction lengths, now with fixed phase fluctuations $\delta_\theta = 0.01$.

The figures clearly demonstrate that the preparation procedure works well even in the presence of relatively small fluctuations. In general, the overall error is accumulated by each "elementary" transformation. The total number of "elementary" transformations scales polynomially with respect to the dimensionality of the Hilbert space. In our particular case the total number of operations is proportional to J^3_{max}. We have simulated the noisy preparation procedure also for higher values of the Lamb–Dicke parameters (up to $\epsilon_q = 0.4$). The dependence of fidelities on amplitudes of fluctuations is very similar to that one shown in Fig. 3. In particular, it turns out that the fidelities are less sensitive to (absolute) fluctuations δ of the interaction lengths when increasing the Lamb–Dicke parameters. This is due to an increase of the ideal values $|g_p|\tau$ of the interactions lengths required for the quantum-state synthesis.

One possibility to improve fidelities of outputs is to utilize conditional measurements of the internal electronic levels. The preparation procedure starts and ends with the ion in the same internal state $|a\rangle$. So after preparation sequence we can verify whether the ion occupies the desired internal state. It can be done by driving dipole transitions from other internal levels to proper auxiliary levels and observing fluorescence signals. No signal means that the undisturbed ion is in the right electronic state $|a\rangle$. Wrong preparation sequences (with the test signal) are thrown away. In other words, outputs are projected on $\hat{P}_a = |a\rangle\langle a|$. The improvement of fidelities due to the conditional selection of outputs is shown in Fig. 3b. Corresponding decrease of probabilities to find on the check the right final internal state is not so dramatic and state-dependent as known from conditional methods for micromasers. For example, the fidelity is lifted from 75% to 92% (64% $\to$ 87%) for the state $|\Psi_{corr}\rangle$ ($|\Psi_{GHZ}\rangle$) with fluctuations $\delta = 0.03$ with the efficiency 82% (74%).

6. Summary

We have investigated preparation of entangled quantum states of the vibrational motion of the ion confined in the 3D trapping potential. We introduced the universal algorithm for deterministic preparation of quantum states which is based on a sequence of laser stimulated Raman transitions. In the presence of a technical noise fidelities of outputs with respect to desired target states can be significantly improved when conditional measurements on the electronic levels of the ion are performed. The method can be adopted also for other three–mode bosonic systems.

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