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Reconstruction of quantum states from propensities

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Abstract. We focus our attention on the problem of reconstruction of density operators of quantum states from propensities, i.e. generalized quasiprobability density distributions obtained by quantum filtering. We consider propensities obtained by filtering with pure Gaussian as well as non-Gaussian states and we present two examples: when the filter is in a squeezed coherent state and in a Fock state, respectively. We also show that even in the case of filtering with statistical mixtures a complete reconstruction of a density operator of the measured quantum state can be performed.

1. Introduction

It is well known that if all system observables (i.e. the quorum [1]) are measured precisely, then the density operator of a quantum-mechanical system can be completely reconstructed (i.e. the density operator can be determined uniquely based on the available data). In principle, one can consider two different schemes for a complete reconstruction of the density operator of the given quantum-mechanical system (in both schemes we assume an ideal, i.e. unit-efficiency, measurement).

The first type of measurement is such that on each element of the ensemble of the measured states only a *single* observable is measured. In particular, in the case of a quantum-mechanical harmonic oscillator, a result of this kind of measurement is an *infinite* set of distributions $W_{|\Psi\rangle}(x_\theta)$ of the rotated quadrature $\hat{x}_\theta = \hat{q}\cos\theta + \hat{p}\sin\theta$. Each distribution $W_{|\Psi\rangle}(x_\theta)$ can be obtained in a measurement of a *single* observable $\hat{x}_\theta$, in which case a detector (filter) is prepared in an eigenstate $|x_\theta\rangle$ of this observable. It has been shown by Vogel and Risken [2] that from an infinite set (in the case of the harmonic oscillator) of the measured distributions $W_{|\Psi\rangle}(x_\theta)$ for all values of θ such that $[0 < \theta \leqslant \pi]$, the density operator (and the Wigner function) can be reconstructed uniquely via a transformation which was later identified by Raymer *et al* [3] to be the inverse Radon transformation. This scheme for reconstruction of the Wigner function (the so-called *optical homodyne tomography method*) has recently been realized experimentally by Raymer and his co-workers [3]. In these experiments Wigner functions of a coherent state and a squeezed vacuum state have been reconstructed from tomographic data.

In the second type of measurement a *simultaneous* measurement of two non-commuting observables (let us say Q and P) is considered. In this case it is not possible to construct a joint eigenstate of these two operators, and therefore it is inevitable that the simultaneous

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measurement of two non-commuting observables introduces additional noise (of a quantum origin) into the measured data. This noise is associated with Heisenberg's uncertainty relation and it results in a specific 'smoothing' (equivalent to a reduction of resolution) of the original Wigner function of the system under consideration (see [4, 5]). To describe a process of simultaneous measurement of two non-commuting observables, Wódkiewicz [6] (see also [7]) has proposed a formalism based on an operational probability density distribution (also known as the propensity) which takes into account explicitly the action of the measurement device modelled as a 'filter' (quantum ruler). A particular choice of the state of the ruler samples a specific type of accessible information concerning the system, i.e. information about the system is biased by the filtering process. The quantum-mechanical noise induced by filtering results formally in a smoothing of the original Wigner function of the measured state [4, 5, 8], so that the operational probability density distribution can be expressed as a convolution of the original Wigner function and the Wigner function of the filter state. In particular, if the filter is considered to be in its vacuum state then the corresponding operational probability density distribution is equal to the Husimi (Q-) function [4] defined in the modern representation by Kano [8]. The Q-function of optical fields has been measured experimentally by Walker and Carroll [9]. The direct experimental measurement of the operational probability density distribution with the filter in an arbitrary state is feasible in an eight-port experimental set-up of the type used by Noh *et al* [10].

As a consequence of a simultaneous measurement of non-commuting observables, the measured distributions are fuzzy (i.e. they are equal to smoothed Wigner functions). In the present paper we will study how the noise induced by quantum filtering can be 'separated' from the measured data and the density operator (Wigner function) of the measured system can be 'extracted' from propensities.

In this paper we consider a reconstruction on a complete observation level [11], when all system observables are measured, i.e. the data containing complete information about the system are available, but the question is how to invert these data and how to separate them from the noise induced by filtering, so that the information about the measured system in terms of the corresponding density operator (Wigner function) can be obtained. In other words, in this paper we are not going to apply any kind of *a posteriori* estimation of density operators (see for instance [12–14]), but we will study *deterministic* inversion procedures.

2. Generalized coherent-state quasiprobabilities: propensities

Coherent states $|\alpha\rangle$ can be defined as vacuum states $|0\rangle$ displaced by the operator $D(\alpha, \alpha^*)$ [15]. Therefore, it is very natural to define the class of generalized coherent-state quasiprobabilities† with the help of displacements of arbitrary states $|\psi\rangle$. The set of

† The modern form of the coherent-state quasiprobability was introduced in quantum optics in 1965 by Kano [8] and Glauber [16]. Only later was it realized that Husimi [4] had already (in 1940) introduced a class of positive semi-definite quasiprobabilities which contains the coherent-state quasiprobability as a special case (see [17]). Husimi's class of quantum distribution functions corresponds to the subclass of generalized coherent-state quasiprobabilities where coherent states are substituted by the class of states with a minimum uncertainty product. However, these quasiprobabilities are written in such a form that the connection with Glauber's formalism is not obvious (see formulae (5.6) on p 278 in [4]). The states with minimum uncertainty product correspond to the special class of squeezed coherent states for which the squeezing axes are parallel to the coordinate axes [18, 19]. The minimum uncertainty product is not invariant with respect to rotations of the coordinate axes [20–22], but after an appropriate choice of the coordinate axes parallel to the squeezing axes one obtains the minimum uncertainty product for all squeezed coherent states [18, 19, 23–25]. The class of generalized coherent-state quasiprobabilities based on squeezed states and the corresponding generalized Glauber–Sudarshan quasiprobabilities were proposed by Yuen [19] and investigated in 1988 by Haake and Wilkens [26] (see also [27]).

displacement operators $D(\alpha, \alpha^*)$ forms the essential part (without phase factors) of the Heisenberg–Weyl group in quantum mechanics. This was used to define the generalized coherent states by the action of the Heisenberg–Weyl group to arbitrary states $|\psi\rangle$ of the Hilbert space and, more generally, by the action of other Lie groups on states of a Hilbert space [28, 30–32]. In this sense, the action of the set of displacement operators of the Heisenberg–Weyl group on a complete set of states $|\psi\rangle$ of the Hilbert space makes a foliation of the Hilbert space into orbits without intersection where each state belongs exactly to one orbit of generalized coherent states and the class of generalized coherent-state quasiprobabilities corresponds uniquely to the class of orbits with respect to the Heisenberg–Weyl group [33].

Generalized coherent states form an overcomplete set of states. The well known completeness relation for coherent states was first derived by Klauder [34] and later by Sudarshan [35], Glauber [15] and by Cahill and Glauber [36] (see also [27, 28, 37–39]). As in the case of coherent states, the generalized coherent states can be used for a diagonal representation of arbitrary density operators. This leads to generalized Glauber–Sudarshan quasiprobabilities which are in a certain sense dual to the generalized coherent-state quasiprobabilities. The so-called Glauber–Sudarshan quasiprobability was introduced almost simultaneously by Sudarshan [35] and Glauber [15] in 1963 (i.e. before the modern introduction of the coherent-state quasiprobability in 1965 [8, 16]). The most general concept of quasiprobabilities was considered by Agarwal and Wolf [37] and is presented and developed in the monograph by Peřina [40] and in the review articles [17, 41]. The role of the parity operator in this concept was discussed by Bishop and Vourdas [39] and by Czirják and Benedict [42]. The Gaussian class of quasiprobabilities was analysed in detail in [25, 27].

2.1. General formalism

An arbitrary quasiprobability $F(q, p)$ associated with a density operator ϱ can be defined as

$$F(q, p) \equiv \langle \varrho T(q, p) \rangle \qquad \langle \cdots \rangle \equiv \mathrm{Trace}(\cdots) \tag{2.1}$$

with a 'transition' operator $T(q, p)$ of the 'displacement' structure [27, 36–39]:

$$T(q, p) = D(q, p)\, T(0, 0)\, (D(q, p))^\dagger. \tag{2.2}$$

The unitary displacement operator $D(q, p)$ is defined as usual by

$$D(q, p) \equiv \exp\left\{-\frac{\mathrm{i}}{\hbar}(qP - pQ)\right\} \tag{2.3}$$

where Q and P are canonical operators satisfying the canonical commutation relation $[Q, P] = \mathrm{i}\hbar I$. The normalization of the quasiprobability $F(q, p)$ is connected with the phase-space decomposition of the unity operator I by means of the transition operator $T(q, p)$, which is a consequence of its 'displacement' structure and requires a restriction of its trace as follows (see [27, 36, 38, 39]):

$$\begin{aligned} &\int \mathrm{d}q \wedge \mathrm{d}p\, F(q, p) = 1 \Longleftrightarrow \int \mathrm{d}q \wedge \mathrm{d}p\, T(q, p) = I \\ &\langle T(q, p) \rangle = \langle T(0, 0) \rangle = \frac{1}{2\hbar\pi}. \end{aligned} \tag{2.4}$$

If we restrict ourselves to real-valued quasiprobabilities $F(q, p)$ then $T(q, p)$ must be a Hermitian operator and vice versa

$$F(q, p) = (F(q, p))^* \Longleftrightarrow T(q, p) = (T(q, p))^\dagger. \tag{2.5}$$

Such a Hermitian operator with a finite trace possesses the spectral decomposition

$$T(0,0) = \sum_{k=1}^{\infty} \lambda_k |\psi_k\rangle\langle\psi_k| \qquad \langle\psi_k|\psi_l\rangle = \delta_{k,l} \qquad \lambda_k = \lambda_k^*$$
$$\langle T(0,0)\rangle = \sum_{k=1}^{\infty} \lambda_k = \frac{1}{2\hbar\pi} \tag{2.6}$$

with a countably infinite set of, possibly degenerate, real eigenvalues λ_k from which at least one must be positive, which guarantees positivity of the trace of $T(0,0)$. This implies the following general structure of a real-valued (but not necessarily positive semi-definite) quasiprobability:

$$F(q,p) = \sum_{k=1}^{\infty} \lambda_k \langle (q,p), \psi_k|\varrho|(q,p), \psi_k\rangle \tag{2.7}$$

where $|(q,p),\psi_k\rangle$ denotes a generalized coherent state of the Heisenberg–Weyl group corresponding to the definition used by Perelomov [28]

$$|(q,p),\psi\rangle \equiv D(q,p)|\psi\rangle. \tag{2.8}$$

We will call $|(q,p),\psi\rangle$ a displaced $|\psi\rangle$-state, for example, a displaced Fock state $|(q,p),n\rangle$ [29] if $|\psi\rangle$ is a Fock state $|n\rangle$.

If $|\psi\rangle = |0\rangle$ then from equation (2.1) we obtain the coherent-state quasiprobability $Q(q,p)$ defined as

$$Q(q,p) \equiv \frac{1}{2\hbar\pi}\langle 0|(D(q,p))^\dagger \varrho D(q,p)|0\rangle \equiv \frac{1}{2\hbar\pi}\langle (q,p)|\varrho|(q,p)\rangle \tag{2.9}$$

where $|(q,p)\rangle \equiv |(q,p),0\rangle$ are coherent states parametrized by real canonical variables (q,p). On one hand, the Q-function can be considered as a formal expression for a mean value of the density operator in a coherent state $|(q,p)\rangle$. On the other hand, this function has an operational meaning: it describes a result of filtering with the quantum ruler prepared in the vacuum state $|0\rangle$ [6, 7].

One can define a generalized coherent-state quasiprobability $Q_\psi(q,p)$ by substituting the vacuum state $|0\rangle$ in equation (2.9) by an arbitrary normalized state $|\psi\rangle$, i.e.

$$Q_\psi(q,p) \equiv \frac{1}{2\hbar\pi}\langle\psi|(D(q,p))^\dagger \varrho D(q,p)|\psi\rangle \equiv \frac{1}{2\hbar\pi}\langle (q,p),\psi|\varrho|(q,p),\psi\rangle$$
$$\langle\psi|\psi\rangle = 1. \tag{2.10}$$

This quasiprobability distribution can be viewed as a generalization of the usual coherent-state quasiprobability (i.e. generalization of the Q-function) when one evaluates mean values of the density operator in a displaced state $|\psi\rangle$. Alternatively, this quasidistribution has an operational meaning: it describes a result of filtering when the quantum ruler is in a state $|\psi\rangle$ (i.e. in the case of the eight-port homodyne measurement instead of the vacuum, the state $|\psi\rangle$ is launched into the device). Wódkiewicz [6] has named this operational quasiprobability density distribution a propensity (in what follows we will use both the names, i.e. the generalized coherent-state quasidistribution and the propensity) for the object defined by equation (2.10).

2.2. Properties of propensities

(i) The quasiprobability $Q_\psi(q,p)$ is normalized and is positive semi-definite

$$\int \mathrm{d}q \wedge \mathrm{d}p\, Q_\psi(q,p) = 1 \qquad Q_\psi(q,p) \geqslant 0. \tag{2.11}$$

(ii) One can consider instead of pure states $|\psi\rangle$, mixed states ϱ_f as quantum filters. Then the corresponding propensities are given by the expression

$$F_f(q,p) = \frac{1}{2\overline{h}\pi}\langle \varrho D(q,p)\varrho_f((D(q,p))^\dagger\rangle$$
$$\varrho_f = 2\overline{h}\pi T_f(0,0) = 2\overline{h}\pi \sum_k \lambda_k |\psi_k\rangle\langle\psi_k| \qquad \lambda_k > 0. \tag{2.12}$$

As an example one can consider filtering with thermal states (see section 7).

(iii) One of the consequences of the fundamental laws of quantum mechanics is that filtering with quantum rulers results in additional noise in the measured quasiprobability $Q_\psi(q,p)$. This 'deterioration' of information is intrinsically related to the fact that states $|(q,p),\psi\rangle$ are not orthonormal, i.e. mutually shifted rulers do overlap.

(iv) Wódkiewicz has shown [6] (see also [7]) that the quasiprobability (i.e. the propensity) $Q_\psi(q,p)$ can be expressed as a phase-space 'convolution' (see later) of two Wigner functions, describing the state of interest and the quantum filter. These two Wigner functions can be expressed as

$$W(q,p) \equiv \langle \varrho T_0(q,p)\rangle \qquad \varrho = 2\overline{h}\pi \int \mathrm{d}q\wedge\mathrm{d}p\, W(q,p)T_0(q,p)$$
$$W_{(\psi)}(q,p) \equiv \langle\psi|T_0(q,p)|\psi\rangle \qquad |\psi\rangle\langle\psi| = 2\overline{h}\pi \int \mathrm{d}q\wedge\mathrm{d}p\, W_{(\psi)}(q,p)T_0(q,p) \tag{2.13}$$

where the transition operator $T_0(q,p)$ reads (see [27, 38])

$$T_0(q,p) = \exp\left(-Q\frac{\partial}{\partial q} - P\frac{\partial}{\partial p}\right)\delta(q)\delta(p). \tag{2.14}$$

From the definition of $Q_\psi(q,p)$ given by equation (2.10) and with the help of the expression (5.12) in [27]

$$\langle T_0(q',p')(D(q,p)T_0(q'',p'')(D(q,p))^\dagger\rangle = \langle T_0(q',p')T_0(q''+q,p''+p)\rangle$$
$$= \frac{1}{2\overline{h}\pi}\delta(q'-q''-q)\delta(p'-p''-p) \tag{2.15}$$

we obtain

$$Q_\psi(q,p) = \int \mathrm{d}q'\wedge\mathrm{d}p'\, W_{(\psi)}(q'-q,p'-p)W(q',p'). \tag{2.16}$$

It is obvious that the propensity carries information about both the measured state and the quantum filter.

(v) Marginals of quasiprobabilities $Q_\psi(q,p)$ have been considered in [6, 7]. From the representation in equation (2.16) one easily obtains by integration over p the expression

$$Q_\psi(q) \equiv \int_{-\infty}^{+\infty} \mathrm{d}p\, Q_\psi(q,p) = \int_{-\infty}^{+\infty} \mathrm{d}q'\, \langle q'-q|\psi\rangle\langle\psi|q'-q\rangle\langle q'|\varrho|q'\rangle \tag{2.17}$$

which is associated with the operational distribution in q. This distribution is biased by a particular choice of the filter state.

(vi) We note that with the help of the generalized coherent states $|(q,p),\psi\rangle$ we can define a class of generalized quasiprobabilities $P_\psi(q,p)$ analogous to the Glauber–Sudarshan quasiprobability $P(q,p)$. Using the 'diagonal' representation of the density operator ϱ in the basis of generalized coherent states, we define $P_\psi(q,p)$ as

$$\varrho \equiv \int \mathrm{d}q\wedge\mathrm{d}p\, P_\psi(q,p)|(q,p),\psi\rangle\langle(q,p),\psi| \qquad \int \mathrm{d}q\wedge\mathrm{d}p\, P_\psi(q,p) = 1. \tag{2.18}$$

We note that the two quasiprobability distributions $Q_\psi(q, p)$ and $P_\psi(q, p)$ are related as follows:

$$Q_\psi(q, p) = \frac{1}{2\hbar\pi} \int \mathrm{d}q' \wedge \mathrm{d}p' \, Q_\psi(q' - q, p' - p) P_\psi(q', p'). \tag{2.19}$$

The kernel in this integral relation between $Q_\psi(q, p)$ and $P_\psi(q, p)$ is equal to the generalized coherent-state quasiprobability $Q_\psi(q, p)$ for the density operator $\varrho_f = |\psi\rangle\langle\psi|$.

3. Reconstruction of density operators from propensities

When measured ideally, propensities $Q_\psi(q, p)$ contain complete information about the measured state. This information is 'biased' by the choice of the filter and the noise induced by the filtering. Nevertheless, one can try to 'separate' the filter noise from $Q_\psi(q, p)$ and to reconstruct the density operator of the system *per se*. The formal inverse transformation between the propensity and the density operator can be written as

$$\varrho = 2 \int \mathrm{d}q \wedge \mathrm{d}p \, \exp\left(\frac{q^2 + p^2}{\hbar}\right) \exp\left(Q \frac{\partial}{\partial q} + P \frac{\partial}{\partial p}\right) Q(q, p). \tag{3.1}$$

In general, it is difficult to use this expression. The exponential function $\exp((q^2 + p^2)/\hbar)$ in equation (3.1) can only be considered as an analytic functional and the integrals can be evaluated only by analytic continuation and deformation of the integration contours to the complex planes of each of the variables (q, p) (see section 7).

A more useful inverse transformation between the propensity $Q_\psi(q, p)$ and the corresponding density matrix can be obtained when the phase space is parametrized by a pair of complex numbers (α, α^*). In this case boson operators $(a, a^\dagger)$ are related to the Hermitian operators (Q, P), and (α, α^*) can be expressed in terms of the real canonical variables (q, p) according to

$$\alpha \equiv \frac{q + \mathrm{i}p}{\sqrt{2\hbar}} \qquad \alpha^* \equiv \frac{q - \mathrm{i}p}{\sqrt{2\hbar}} \qquad a \equiv \frac{Q + \mathrm{i}P}{\sqrt{2\hbar}} \qquad a^\dagger \equiv \frac{Q - \mathrm{i}P}{\sqrt{2\hbar}}. \tag{3.2}$$

The integration measures are related as

$$\begin{aligned} &\mathrm{d}q \wedge \mathrm{d}p \, Q_\psi(q, p) \equiv \tfrac{1}{2}\mathrm{i}\, \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \, Q_\psi(\alpha, \alpha^*) \qquad \mathrm{d}q \wedge \mathrm{d}p = 2\hbar \tfrac{1}{2}\mathrm{i}\, \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \\ &|(q, p), \psi\rangle \equiv |\alpha, \psi\rangle. \end{aligned} \tag{3.3}$$

Analogous terms are the related quasiprobabilities $P_\psi(q, p)$ and $P_\psi(\alpha, \alpha^*)$ associated with different parametrizations of the phase space.

3.1. Reconstruction of density operator from the Q-function

The formula for the reconstruction of the density operator ϱ from the usual coherent-state quasiprobability $Q(\alpha, \alpha^*)$ (i.e. from the Q-function) can be written in the 'normal-ordered' form [27, 38] as

$$\varrho = \pi \left\{ \exp\left(a^\dagger \frac{\partial}{\partial \alpha^*}\right) \exp\left(a \frac{\partial}{\partial \alpha}\right) Q(\alpha, \alpha^*) \right\}_{(\alpha = \alpha^* = 0)}. \tag{3.4}$$

After performing the Taylor series expansion we find (see also appendix A for the definition of the operator $a_{k,l}$)

$$\varrho = \pi \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{a^{\dagger k} a^l}{k! l!} \left\{ \frac{\partial^{k+l}}{\partial \alpha^{*k} \partial \alpha^l} Q(\alpha, \alpha^*) \right\}_{(\alpha = \alpha^* = 0)} \equiv \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho a_{k,l} \rangle a^{\dagger k} a^l. \tag{3.5}$$

This formula or the preceding equation (3.4) illustrate the fact that, in principle, the function $Q(\alpha, \alpha^*)$, in an arbitrarily small vicinity of the point $\alpha = \alpha^* = 0$, contains complete information about the system. In other words, it is not necessary to measure the Q-function in the whole (infinite) phase space. An equally good reconstruction might emerge from a scheme which would allow us to measure the value of the Q-function and *all* its derivatives at the origin of the phase space. This also follows from the fact that the density operator ϱ is determined by a countably infinite set of matrix elements, for example, $\langle m|\varrho|n\rangle$ in the Fock-state representation ($m, n = 0, 1, \ldots, \infty$), whereas $Q(\alpha, \alpha^*)$ is a function of the continuous variables α and α^*. In equation (3.5), the abbreviation $a_{k,l}$ is introduced for a set of operators. In appendix A we show how the density operator ϱ can be reconstructed from the normally ordered moments $\langle \varrho a^{\dagger k} a^l\rangle$.

For completeness of the discussion we note that one can easily obtain an expression which relates the Q-function and the density operator in the Fock basis

$$\varrho = \pi \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{m!n!}} \left\{ \frac{\partial^{m+n}}{\partial\alpha^m \partial\alpha^{*n}} \exp(\alpha\alpha^*) Q(\alpha, \alpha^*) \right\}_{(\alpha=\alpha^*=0)}. \tag{3.6}$$

3.2. Reconstruction of the density operator from arbitrary propensities

Let us perform the Fock-state expansions of the filter state $|\psi\rangle$. If we use equation (5.4) in [38] which relates the generalized projection operators for the displaced Fock states with the projector $|\alpha\rangle\langle\alpha|$ we then find

$$\begin{aligned} Q_\psi(\alpha, \alpha^*) &\equiv \frac{1}{\pi} \langle\psi|(D(\alpha, \alpha^*))^\dagger \varrho D(\alpha, \alpha^*)|\psi\rangle \\ &= \frac{1}{\pi} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle\psi|n\rangle\langle\alpha, n|\varrho|\alpha, m\rangle\langle m|\psi\rangle \\ &\equiv L_\psi\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\right) Q(\alpha, \alpha^*) \end{aligned} \tag{3.7}$$

where

$$L_\psi\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\right) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle m|\psi\rangle\langle\psi|n\rangle \sqrt{\frac{n!}{m!}} \frac{\partial^{m-n}}{\partial\alpha^{m-n}} L_n^{m-n}\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) \tag{3.8}$$

and $L_n^\nu(z)$ are associated Laguerre polynomials.

Now, it becomes clear that one has to determine the inverse of the operator in equation (3.7) to obtain the Q-function from the propensity $Q_\psi(\alpha, \alpha^*)$, i.e.

$$Q(\alpha, \alpha^*) = \left(L_\psi\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\right)\right)^{-1} Q_\psi(\alpha, \alpha^*). \tag{3.9}$$

The reconstruction relation between the arbitrary propensity and the density operator of the state under consideration reads

$$\varrho = \pi \left\{ \left(L_\psi\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\right)\right)^{-1} \exp\left(a^\dagger \frac{\partial}{\partial\alpha^*}\right) \exp\left(a \frac{\partial}{\partial\alpha}\right) Q_\psi(\alpha, \alpha^*) \right\}_{(\alpha=\alpha^*=0)}. \tag{3.10}$$

Analogously, we obtain the expression for the generalized Glauber–Sudarshan quasiprobability

$$P_\psi(\alpha,\alpha^*) = \left(L_\psi\left(-\frac{\partial}{\partial\alpha}, -\frac{\partial}{\partial\alpha^*}\right)\right)^{-1} P(\alpha,\alpha^*)$$
$$= \left\langle \varrho \exp\left(-a^\dagger \frac{\partial}{\partial\alpha^*}\right) \exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle \left(L_\psi\left(-\frac{\partial}{\partial\alpha}, -\frac{\partial}{\partial\alpha^*}\right)\right)^{-1} \delta(\alpha,\alpha^*). \qquad (3.11)$$

The formal inverse transform for the density operator ϱ from $P_\psi(\alpha,\alpha^*)$ is given by

$$\varrho = \int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*\, P_\psi(\alpha,\alpha^*) D(\alpha,\alpha^*)|\psi\rangle\langle\psi|(D(\alpha,\alpha^*))^\dagger \qquad (3.12)$$

which obviously follows from the definition of $P_\psi(\alpha,\alpha^*)$. Consequently, when we insert the last expression for ϱ into the definition of the generalized coherent-state quasiprobability (propensity) we find

$$Q_\psi(\alpha,\alpha^*) = \frac{1}{\pi}\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}\beta\wedge\mathrm{d}\beta^* \langle\psi|(D(\beta-\alpha,\beta^*-\alpha^*))^\dagger|\psi\rangle$$
$$\times\langle\psi|D(\beta-\alpha,\beta^*-\alpha^*)|\psi\rangle P_\psi(\beta,\beta^*). \qquad (3.13)$$

One can also relate the propensity $Q_\psi(\alpha,\alpha^*)$ with the Glauber–Sudarshan quasiprobability $P(\alpha,\alpha^*)$:

$$Q_\psi(\alpha,\alpha^*) = \int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}\beta\wedge\mathrm{d}\beta^*\, Q_{(\psi)}(\beta-\alpha,\beta^*-\alpha^*) P(\beta,\beta^*) \qquad (3.14)$$

where $Q_{(\psi)}(\alpha,\alpha^*)$ denotes the 'usual' coherent-state quasiprobability of the filter state $|\psi\rangle$. The corresponding 'dual' expression is

$$Q_\psi(\alpha,\alpha^*) = \int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}\beta\wedge\mathrm{d}\beta^*\, P_{(\psi)}(\beta-\alpha,\beta^*-\alpha^*) Q(\beta,\beta^*) \qquad (3.15)$$

where $P_{(\psi)}(\alpha,\alpha^*)$ denotes the Glauber–Sudarshan quasiprobability of the filter state $|\psi\rangle$. We note that equations (3.14) and (3.15) have a form which is analogous to the Wódkiewicz expression (2.19) for the propensity

$$Q_\psi(\alpha,\alpha^*) = \int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}\beta\wedge\mathrm{d}\beta^*\, W_{(\psi)}(\beta-\alpha,\beta^*-\alpha^*) W(\beta,\beta^*) \qquad (3.16)$$

rewritten in terms of complex variables, where the density operators of the filter state and of the considered state are both expressed by their Wigner quasiprobabilities $W_{(\psi)}(\alpha,\alpha^*)$ and $W(\alpha,\alpha^*)$, respectively. Equations (3.14)–(3.16) can be used for a calculation of generalized coherent-state quasiprobabilities of given density operators. These equations may also be represented as convolutions, for example,

$$Q_\psi(\alpha,\alpha^*) = W_{(\psi)}(-\alpha,-\alpha^*) * W(\alpha,\alpha^*) \qquad (3.17)$$

however, with inverted signs of the arguments in the Wigner quasiprobability of the filter state.

3.3. Comments

(i) To give an operational meaning to equation (3.8) we have to prove the existence and the uniqueness of the inverse of the operator L_ψ and to find its explicit form. Sometimes it is possible to perform summations in equation (3.7) and to find this operator and its inverse in a closed form. In addition to the trivial case of filtering with the vacuum state when $L_\psi = 1$, we derive the explicit expression for L_ψ^{-1} when the filter state is in a squeezed vacuum state (see section 5).

(ii) We note that marginal distributions of the generalized coherent-state quasiprobabilities (propensities) can be obtained by going back from the (α, α^*)-representation to the phase-space representation in terms of the real canonical variables (q, p). Using the equation

$$\frac{\partial}{\partial\alpha} = \sqrt{\frac{\hbar}{2}}\left(\frac{\partial}{\partial q} - \mathrm{i}\frac{\partial}{\partial p}\right) \qquad \frac{\partial}{\partial\alpha^*} = \sqrt{\frac{\hbar}{2}}\left(\frac{\partial}{\partial q} + \mathrm{i}\frac{\partial}{\partial p}\right) \tag{3.18}$$

we obtain the expression for the marginal distribution associated with the propensity $Q_\psi(q, p)$ given in terms of the marginal distribution related to the ordinary Q-function:

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathrm{d}p\, Q_\psi(q, p) &= L_{1,\psi}\left(\frac{\partial}{\partial q}\right) \int_{-\infty}^{+\infty} \mathrm{d}p\, Q(q, p) \\ \int_{-\infty}^{+\infty} \mathrm{d}q\, Q_\psi(q, p) &= L_{2,\psi}\left(\frac{\partial}{\partial p}\right) \int_{-\infty}^{+\infty} \mathrm{d}q\, Q(q, p) \end{aligned} \tag{3.19}$$

where the two differential operators $L_{1,\psi}$ and $L_{2,\psi}$ are defined as

$$\begin{aligned} L_{1,\psi}\left(\frac{\partial}{\partial q}\right) &\equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\langle m|\psi\rangle\langle\psi|n\rangle}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\sqrt{\frac{\hbar}{2}}\frac{\partial}{\partial q}\right)^{m+n-2j} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle m|\psi\rangle\langle\psi|n\rangle \sqrt{\frac{n!}{m!}} \left(\sqrt{\frac{\hbar}{2}}\frac{\partial}{\partial q}\right)^{m-n} L_n^{m-n}\left(-\frac{\hbar}{2}\frac{\partial^2}{\partial q^2}\right) \end{aligned} \tag{3.20}$$

and

$$L_{2,\psi}\left(\frac{\partial}{\partial p}\right) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\langle m|\psi\rangle\langle\psi|n\rangle}{\sqrt{m!n!}} \mathrm{i}^{n-m} \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\sqrt{\frac{\hbar}{2}}\frac{\partial}{\partial p}\right)^{m+n-2j}. \tag{3.21}$$

(iii) Some of the formulae in this section can be easily extended to more general cases of quasiprobabilities if one substitutes the density operator $|\psi\rangle\langle\psi|$ of the pure filter state $|\psi\rangle$ by more general density operators

$$|\psi\rangle\langle\psi| \to \varrho_f \to \pi T(\alpha = 0, \alpha^* = 0) = 2\hbar\pi T(q = 0, p = 0). \tag{3.22}$$

In other words, when filtering by statistical mixtures, we can still make use of equation (3.7) which relates $Q_\psi(\alpha, \alpha^*)$ and $Q(\alpha, \alpha^*)$. Analogously equation (3.19) can be used.

4. Filtering with Fock states

Let us assume that the quantum ruler is prepared in a Fock state $|1\rangle$. The corresponding propensity is then defined as

$$Q_1(\alpha, \alpha^*) \equiv \frac{1}{\pi}\langle\alpha, 1|\varrho|\alpha, 1\rangle. \tag{4.1}$$

Using equations (3.7) and (3.8) we express this propensity via the Q-function as

$$Q_1(\alpha, \alpha^*) = L_1\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)Q(\alpha, \alpha^*) \equiv \left(1 + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)Q(\alpha, \alpha^*). \tag{4.2}$$

From the general solution of the homogeneous equation (two-dimensional Helmholtz equation)

$$\begin{aligned}&\left(1+\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)f(\alpha,\alpha^*)=0\\&f(\alpha,\alpha^*)\equiv\int r\,\mathrm{d}r\wedge\mathrm{d}\varphi\,f(r,\varphi)\exp\left(r\mathrm{e}^{\mathrm{i}\varphi}\alpha-r^{-1}\mathrm{e}^{-\mathrm{i}\varphi}\alpha^*\right)\end{aligned}\tag{4.3}$$

with an arbitrary 2π-periodic function $f(r,\varphi)$ of φ, it follows that among the eigenvalues of the differential operator L_1 in equation (4.2) is the eigenvalue equal to zero. Consequently, the inverse of this operator is not uniquely defined. Nevertheless, one can find an explicit solution for this inverse operator. To do so we determine a Green function $G(\alpha,\alpha^*)$ defined as

$$\left(1+\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)G(\alpha,\alpha^*)=\delta(\alpha,\alpha^*).\tag{4.4}$$

A particular solution of equation (4.4) (see, e.g., [43]) reads

$$G(\alpha,\alpha^*)=Y_0\big(2\sqrt{\alpha\alpha^*}\big)\tag{4.5}$$

where $Y_0(x)$ is the Neumann function with index 0 (here we adopt the notation introduced in [44]). We note that the corresponding Bessel function $J_0(x)$ is a particular solution of the following homogeneous equation:

$$\left(1+\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)J_0\big(2\sqrt{\alpha\alpha^*}\big)=0\qquad f(r,\varphi)=\frac{1}{2\pi}\delta(r-1).\tag{4.6}$$

From equations (4.2) and (4.4) together with (4.5) it follows that

$$\begin{aligned}Q(\alpha,\alpha^*)&=Y_0\big(2\sqrt{\alpha\alpha^*}\big)*Q_1(\alpha,\alpha^*)\\&=\int\tfrac{1}{2}\mathrm{i}\,\mathrm{d}\beta\wedge\mathrm{d}\beta^*\,Y_0\Big(2\sqrt{(\alpha-\beta)(\alpha^*-\beta^*)}\Big)Q_1(\beta,\beta^*).\end{aligned}\tag{4.7}$$

Using equations (3.4) we now find a reconstruction (inversion) formula for the density operator ϱ from $Q_1(\alpha,\alpha^*)$

$$\varrho=\pi\left\{Y_0\big(2\sqrt{\alpha\alpha^*}\big)*\exp\left(a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(a\frac{\partial}{\partial\alpha}\right)Q_1(\alpha,\alpha^*)\right\}_{(\alpha=\alpha^*=0)}.\tag{4.8}$$

With the help of the definition of the Glauber–Sudarshan quasiprobability $P(\alpha,\alpha^*)$

$$P(\alpha,\alpha^*)=\left\langle\varrho\exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle\delta(\alpha,\alpha^*)\tag{4.9}$$

and the results presented above we find the generalized Glauber–Sudarshan quasiprobability $P_1(\alpha,\alpha^*)$,

$$P_1(\alpha,\alpha^*)=\left\langle\varrho\exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle Y_0\big(2\sqrt{\alpha\alpha^*}\big).\tag{4.10}$$

4.1. Comments

(i) Due to the fact that the Green function (see equations (4.3)–(4.5)) is not uniquely defined, the corresponding generalized Glauber–Sudarshan quasiprobability $P_1(\alpha,\alpha^*)$ is also not defined uniquely. This means that one can utilize some other Green function, for example, the Hankel functions $-\mathrm{i}H_0^{(1)}\big(2\sqrt{\alpha\alpha^*}\big)$ or $\mathrm{i}H_0^{(2)}\big(2\sqrt{\alpha\alpha^*}\big)$ instead of the Neumann function

$Y_0\big(2\sqrt{\alpha\alpha^*}\big)$. We stress that in spite of this ambiguity in the choice of the Green function, the reconstructed density ϱ operator is uniquely determined from the propensity $Q_1(\alpha^*, \alpha)$.

(ii) When the quantum system is filtered by higher-excited Fock states $|n\rangle$ ($n > 1$) then in order to reconstruct the density operator of the measured system, one has to determine the inverse of a differential operator of order $2n$ (which is due to the presence of the Laguerre polynomial of order n in equations (3.7)–(3.9)). Obviously, in this case one can, in principle, derive an inversion transformation between the propensity and the corresponding density operator, but we do not see the way to obtain the solution in a compact analytical form. Consequently, it would be difficult to invert data obtained in a simultaneous measurement of the conjugate observables based on the filtering with an arbitrary Fock state.

5. Filtering with squeezed states

Let us assume that the quantum ruler is prepared in a squeezed state. In this case the propensity given by equation (2.10) can be understood as a mean value of the density operator in a squeezed coherent state. We therefore start this section with a brief description of squeezed and squeezed coherent states. For technical convenience in what follows we will consider nonunitary squeezed states defined as (see, for instance, [23–25, 46])

$$\begin{aligned} |0;\zeta\rangle &\equiv \exp\left(-\tfrac{1}{2}\zeta a^{\dagger 2}\right)|0\rangle = \sum_{m=0}^{\infty}\frac{(-1)^m\sqrt{(2m)!}}{2^m m!}\zeta^m|2m\rangle \\ \langle 0;\zeta|0;\zeta\rangle &= \frac{1}{\sqrt{1-\zeta\zeta^*}} \end{aligned} \tag{5.1}$$

where ζ is a complex squeezing parameter such that $|\zeta| < 1$. In the unitary approach one can determine the normalized squeezed vacuum states $|\zeta'\rangle$ by the action of a unitary squeezing operator $S(\zeta'^*, 0, \zeta')$ onto the vacuum state $|0\rangle$ as follows:

$$\begin{aligned} &|\zeta'\rangle = S(\zeta'^*, 0, \zeta')|0\rangle \qquad \langle\zeta'|\zeta'\rangle = 1 \\ &S(\xi,\eta,\zeta) \equiv \exp\left\{\tfrac{1}{2}\xi a^2 + \mathrm{i}\tfrac{1}{2}\eta(aa^\dagger + a^\dagger a) - \tfrac{1}{2}\zeta a^{\dagger 2}\right\} \\ &S(\zeta'^*, 0, \zeta') = \exp\left(-\frac{\zeta'}{2}\frac{\tanh|\zeta'|}{|\zeta'|}a^{\dagger 2}\right)\left(\frac{1}{\cosh|\zeta'|}\right)^{\frac{1}{2}(aa^\dagger + a^\dagger a)}\exp\left(\frac{\zeta'^*}{2}\frac{\tanh|\zeta'|}{|\zeta'|}a^2\right) \end{aligned} \tag{5.2}$$

where the complex parameters ζ and ζ' are related as (cf [27])

$$\begin{aligned} &\zeta = \zeta'\frac{\tanh|\zeta'|}{|\zeta'|} \qquad \zeta' = \zeta\frac{\operatorname{arctanh}|\zeta|}{|\zeta|} \\ &\to \tanh|\zeta'| = |\zeta| \qquad \cosh|\zeta'| = \frac{1}{\sqrt{1-|\zeta|^2}} \qquad \frac{\zeta'}{|\zeta'|} = \frac{\zeta}{|\zeta|}. \end{aligned} \tag{5.3}$$

The equivalence (up to a normalization constant) between the action of the unitary squeezing operator $S(\zeta'^*, 0, \zeta')$ and the nonunitary squeezing operator $S(0, 0, \zeta)$ on the Fock state $|0\rangle$ can also be extended to the Fock state $|1\rangle$. Unfortunately, it does not work for higher Fock states (the reason being that Fock states $|n\rangle$ ($n \geqslant 2$) are not annihilated by the operator a^2).

The displacement of the squeezed vacuum states yields

$$D(\alpha,\alpha^*)|0;\zeta\rangle = \exp\left\{-\tfrac{1}{2}(\alpha+\zeta\alpha^*)\alpha^*\right\}|\alpha+\zeta\alpha^*;\zeta\rangle \tag{5.4}$$

where we have changed the order of action of the displacement and squeezing operator and have used the notation $|\beta;\zeta\rangle$ introduced in [23]

$$|\beta;\zeta\rangle \equiv \exp\left(\beta a^\dagger - \tfrac{1}{2}\zeta a^{\dagger 2}\right)|0\rangle = \sum_{n=0}^{\infty}\frac{\left(\sqrt{2\zeta}\right)^n}{2^n\sqrt{n!}}H_n\left(\frac{\beta}{\sqrt{2\zeta}}\right)|n\rangle. \tag{5.5}$$

Now, we define the normalized squeezed-coherent-state quasiprobability $Q_\zeta(\alpha,\alpha^*)$ in the following way (cf also [26])

$$Q_\zeta(\alpha,\alpha^*) \equiv \frac{1}{\pi}\frac{\langle 0;\zeta|\big(D(\alpha,\alpha^*)\big)^\dagger \varrho D(\alpha,\alpha^*)|0;\zeta\rangle}{\langle 0;\zeta|0;\zeta\rangle} = \frac{\sqrt{1-\zeta\zeta^*}}{\pi}\int \tfrac{1}{2}\mathrm{i}\,\mathrm{d}\beta\wedge\mathrm{d}\beta^*\,\langle 0;\zeta|\beta-\alpha\rangle\langle\beta-\alpha|0;\zeta\rangle P(\beta,\beta^*). \tag{5.6}$$

Here we have used the representation of the density operator ϱ by the Glauber–Sudarshan quasiprobability $P(\beta,\beta^*)$. Taking into account $\langle\alpha|a^\dagger = \alpha^*\langle\alpha|$ we find for the scalar product $\langle\alpha|0;\zeta\rangle$ the expression

$$\langle\alpha|0;\zeta\rangle = \exp\left\{-\tfrac{1}{2}(\alpha+\zeta\alpha^*)\alpha^*\right\} \tag{5.7}$$

and using equation (3.14) (see also [27]) we find

$$Q_\zeta(\alpha,\alpha^*) = \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}P(\alpha,\alpha^*). \tag{5.8}$$

Using the results of previous sections we can express the propensity $Q_\zeta(\alpha,\alpha^*)$ via the Q-function of the same state (cf equation (3.15)):

$$Q_\zeta(\alpha,\alpha^*) = \frac{1}{\pi}\sqrt{-\frac{1-\zeta\zeta^*}{\zeta\zeta^*}}\exp\left\{\alpha\alpha^* + \frac{\alpha^2}{2\zeta} + \frac{\alpha^{*2}}{2\zeta^*}\right\} * Q(\alpha,\alpha^*) = \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\zeta\zeta^*\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}Q(\alpha,\alpha^*). \tag{5.9}$$

Comparing this equation with equations (3.7) and (5.1) we find the following identity (which we prove in appendix B):

$$L_\zeta\left(\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*}\right) = \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\zeta\zeta^*\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\} = \sqrt{1-\zeta\zeta^*}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}\zeta^m\zeta^{*n}}{2^{m+n}m!n!} \times \sum_{j=0}^{\{2m,2n\}}\frac{(2m)!(2n)!}{j!(2m-j)!(2n-j)!}\frac{\partial^{2(m+n-j)}}{\partial\alpha^{2m-j}\partial\alpha^{*2n-j}}. \tag{5.10}$$

From equations (3.11), (3.9) and (5.9) we find the connection between $P_\zeta(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$,

$$P_\zeta(\alpha,\alpha^*) = \frac{1}{\pi}\sqrt{-\frac{1-\zeta\zeta^*}{\zeta\zeta^*}}\exp\left\{-\left(\alpha\alpha^* + \frac{\alpha^2}{2\zeta} + \frac{\alpha^{*2}}{2\zeta^*}\right)\right\} * P(\alpha,\alpha^*) = \exp\left\{-\frac{1}{1-\zeta\zeta^*}\left(\zeta\zeta^*\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}P(\alpha,\alpha^*). \tag{5.11}$$

Analogously we can derive two other formulae connecting $P_\zeta(\alpha,\alpha^*)$ with the Wigner and the Q-functions of the state under consideration. A complete list of all connections of $Q_\zeta(\alpha,\alpha^*)$ and $P_\zeta(\alpha,\alpha^*)$ to the usual quasiprobabilities will be given in appendix C. The direct connection between $Q_\zeta(\alpha,\alpha^*)$ and $P_\zeta(\alpha,\alpha^*)$ can be obtained when we combine the convolution in equation (5.8) with the inverse convolution in equation (5.11), that is

$$
\begin{aligned}
Q_\zeta(\alpha,\alpha^*) &= \frac{1}{\pi}\exp\left\{-\frac{(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\} * P_\zeta(\alpha,\alpha^*) \\
&= \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\frac{\partial}{\partial\alpha}-\zeta^*\frac{\partial}{\partial\alpha^*}\right)\left(\frac{\partial}{\partial\alpha^*}-\zeta\frac{\partial}{\partial\alpha}\right)\right\}P_\zeta(\alpha,\alpha^*). \qquad (5.12)
\end{aligned}
$$

In the case of convolutions with normalized Gaussian functions one can determine the inverse convolutions as convolutions with normalized Gaussian functions with a changed sign in the exponent. Therefore, the quasiprobabilities $Q(\alpha,\alpha^*)$, $W(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$ can be expressed via the propensity $Q_\zeta(\alpha,\alpha^*)$ or the generalized Glauber–Sudarshan quasiprobability $P_\zeta(\alpha,\alpha^*)$.

An explicit formula for the calculation of the generalized Glauber–Sudarshan quasiprobability $P_\zeta(\alpha,\alpha^*)$ from the given density operator ϱ can be obtained from equations (3.11) and (5.11) as follows:

$$
\begin{aligned}
P_\zeta(\alpha,\alpha^*) &= \left\langle \varrho\ \exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle \\
&\quad\times\exp\left\{-\frac{1}{1-\zeta\zeta^*}\left(\zeta\zeta^*\frac{\partial^2}{\partial\alpha\partial\alpha^*}-\frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^2}-\frac{\zeta}{2}\frac{\partial}{\partial\alpha^{*2}}\right)\right\}\delta(\alpha,\alpha^*) \\
&= \left\langle \varrho\ \exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle \\
&\quad\times\frac{1}{\pi}\sqrt{-\frac{1-\zeta\zeta^*}{\zeta\zeta^*}}\exp\left\{-\left(\alpha\alpha^*+\frac{\alpha^2}{2\zeta}+\frac{\alpha^{*2}}{2\zeta^*}\right)\right\}. \qquad (5.13)
\end{aligned}
$$

With the help of the convolution (5.12) we obtain from equation (5.13) the expression

$$
\begin{aligned}
Q_\zeta(\alpha,\alpha^*) &= \left\langle \varrho\ \exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle \\
&\quad\times\frac{\sqrt{1-\zeta\zeta^*}}{\pi}\exp\left\{-\left(\alpha\alpha^*+\tfrac{1}{2}\zeta^*\alpha^2+\tfrac{1}{2}\zeta\alpha^{*2}\right)\right\}. \qquad (5.14)
\end{aligned}
$$

Equations (5.13) and (5.14) are expressed in the normal-ordered form. It is not difficult to rewrite these equation with other orderings of the creation and annihilation operators.

6. Squeezed-state propensities with squeezed filters

As an example of a propensity with the squeezed filter, in this section we evaluate the propensity of a squeezed state filtered by the squeezed quantum rulers with the same complex parameter ζ. In this case one obtains the corresponding squeezed-state propensities by linear symplectic argument transformations of the corresponding usual quasiprobabilities for the unsqueezed initial state. We will consider the squeezed vacuum state $|0;\zeta\rangle_{\rm norm}$. In this case the displacement with the unitary displacement operator $D(\beta,\beta^*)$ results in a simple transformation of phase-space variables $\alpha\to\alpha-\beta,\ \alpha^*\to\alpha^*-\beta^*$. Using the general formalism presented above we easily find the propensity for the squeezed vacuum filtered by the same squeezed vacuum as

$$
Q_\zeta(\alpha,\alpha^*) = \frac{1}{\pi}\exp\left\{-\frac{(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\}. \qquad (6.1)
$$

We recall that the Wigner quasiprobability for the state $|0;\zeta\rangle_{\rm norm}$ is given by the expression

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\exp\left\{-\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\}. \tag{6.2}$$

Here we also present the generalized Glauber–Sudarshan quasiprobability $P_\zeta(\alpha,\alpha^*)$ for the states $|0;\zeta\rangle_{\rm norm}$

$$P_\zeta(\alpha,\alpha^*) = \delta\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},\frac{\alpha^*+\zeta^*\alpha}{\sqrt{1-\zeta\zeta^*}}\right) = \delta(\alpha,\alpha^*). \tag{6.3}$$

It is of the same form as the usual Glauber–Sudarshan quasiprobability $P(\alpha,\alpha^*)$ for the vacuum state $|0\rangle$. We see that equations (6.1)–(6.3) can also be obtained from the quasiprobabilities $Q(\alpha,\alpha^*)$, $W(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$ for the vacuum state $|0\rangle$ with the help of the following substitution of the phase-space variables:

$$\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix} \to \begin{pmatrix}\alpha'\\ \alpha'^*\end{pmatrix} = \frac{1}{\sqrt{1-\zeta\zeta^*}}\begin{pmatrix}1, & \zeta\\ \zeta^*, & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix} \tag{6.4}$$

which is true in the case of the usual quasiprobabilities only for the Wigner quasiprobability $W(\alpha,\alpha^*)$. The determinant of the linear transformation in equation (6.4) is equal to unity. Therefore equation (6.4) describes a homogeneous linear canonical (symplectic) transformation. This corresponds to a linear transformation of the boson operators $(a,a^\dagger)$

$$\begin{aligned}(S(\zeta'^*,0,\zeta'))^\dagger(a,a^\dagger)S(\zeta'^*,0,\zeta') &= (a,a^\dagger)\begin{pmatrix}\cosh|\zeta'| & , & -\zeta'^*\frac{\sinh|\zeta'|}{|\zeta'|}\\ -\zeta'\frac{\sinh|\zeta'|}{|\zeta'|} & , & \cosh|\zeta'|\end{pmatrix}\\ &= \left(\frac{a-\zeta a^\dagger}{\sqrt{1-\zeta\zeta^*}},\frac{a^\dagger-\zeta^* a}{\sqrt{1-\zeta\zeta^*}}\right) \qquad S(\zeta'^*,0,\zeta') \equiv \exp\left(\tfrac{1}{2}\zeta'^*a^2-\tfrac{1}{2}\zeta' a^{\dagger 2}\right)\end{aligned} \tag{6.5}$$

where we have used the connection between ζ and ζ' given in equation (5.3). From here we find the following unitary transformation of displacement operators:

$$(S(\zeta'^*,0,\zeta'))^\dagger D(\alpha,\alpha^*)S(\zeta'^*,0,\zeta') = D\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},\frac{\alpha^*+\zeta^*\alpha}{\sqrt{1-\zeta\zeta^*}}\right) \tag{6.6}$$

which is equal to a symplectic transformation (6.4) of the phase-space variables. For a density operator ϱ obtained from a density operator ϱ_0 by squeezing with the unitary squeezing operator $S(\zeta'^*,0,\zeta')$ according to

$$\varrho = S(\zeta'^*,0,\zeta')\varrho_0(S(\zeta'^*,0,\zeta'))^\dagger \tag{6.7}$$

one obtains for the generalized coherent-state quasiprobability $Q_\zeta(\alpha,\alpha^*)$

$$\begin{aligned}Q_\zeta(\alpha,\alpha^*) &= \frac{1}{\pi}\langle 0|(S(\zeta'^*,0,\zeta'))^\dagger(D(\alpha,\alpha^*))^\dagger\varrho D(\alpha,\alpha^*)S(\zeta'^*,0,\zeta')|0\rangle\\ &= \frac{1}{\pi}\langle 0|\left(D\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},\frac{\alpha^*+\zeta^*\alpha}{\sqrt{1-\zeta\zeta^*}}\right)\right)^\dagger\varrho_0 D\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},\frac{\alpha^*+\zeta^*\alpha}{\sqrt{1-\zeta\zeta^*}}\right)|0\rangle\\ &= \frac{1}{\pi}\left\langle\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}}\right|\varrho_0\left|\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}}\right\rangle.\end{aligned} \tag{6.8}$$

Thus we have proved that the generalized coherent-state quasiprobability $Q_\zeta(\alpha,\alpha^*)$ for the density operator ϱ can be obtained from the coherent-state quasiprobability $Q(\alpha,\alpha)$ for the density operator ϱ_0 by the argument substitutions according to equation (6.4).

6.1. s-parametrized quasiprobabilities

It is well known that the linear interpolation between the quasiprobabilities $Q(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$ leads to the class of s-ordered quasiprobabilities. Therefore we will now consider quasiprobabilities obtained via a linear interpolation (in terms of the Fourier transforms) between $Q_\zeta(\alpha,\alpha^*)$ and $P_\zeta(\alpha,\alpha^*)$ with the Wigner quasiprobability in its centre. Using the formalism presented in [25, 27, 38] (here we set the parameter $r \equiv r_3 = -s$) we define the generalized class of quasiprobabilities $F_{\zeta,(0,0,r)}(\alpha,\alpha^*)$ as

$$\begin{aligned} F_{\zeta,(0,0,r)}(\alpha,\alpha^*) \equiv {} & \frac{2}{(1+r)\pi}\sum_{n=0}^{\infty}\left(-\frac{1-r}{1+r}\right)^n \\ & \times\langle n|(S(\zeta'^*,0,\zeta'))^\dagger(D(\alpha,\alpha^*))^\dagger\varrho D(\alpha,\alpha^*)S(\zeta'^*,0,\zeta')|n\rangle. \end{aligned} \tag{6.9}$$

Alternatively, if we take into account equation (6.7) we can write

$$F_{\zeta,(0,0,r)}(\alpha,\alpha^*) = \frac{2}{(1+r)\pi}\sum_{n=0}^{\infty}\left(-\frac{1-r}{1+r}\right)^n\left\langle\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},n\middle|\varrho_0\middle|\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},n\right\rangle \tag{6.10}$$

where $|\beta,n\rangle$ denotes the displaced Fock states. In particular, one obtains for the normalized squeezed vacuum states $|0;\zeta\rangle_{\mathrm{norm}}$

$$F_{\zeta,(0,0,r)}(\alpha,\alpha^*) = \frac{2}{(1+r)\pi}\exp\left\{-\frac{2}{1+r}\frac{(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\}. \tag{6.11}$$

We can now easily check that for $r = 1, 0, -1$ $F_{\zeta,(0,0,r)}(\alpha,\alpha^*)$ equals the quasiprobabilities given by equations (6.1)–(6.3), respectively.

7. Filtering with thermal states

Let us now consider that the quantum ruler is prepared in a thermal state described by the density operator ϱ_f [15]

$$\begin{aligned} &\varrho_f = \frac{1}{1+\bar N}\left(\frac{\bar N}{1+\bar N}\right)^N = \frac{1}{1+\bar N}\sum_{n=0}^{\infty}\left(\frac{\bar N}{1+\bar N}\right)^n|n\rangle\langle n| \\ &\bar N \equiv \langle\varrho_f N\rangle = \frac{1}{\exp(\hbar\omega/kT)-1} \qquad \frac{\bar N}{1+\bar N} = \exp\left(-\frac{\hbar\omega}{kT}\right) \qquad N \equiv a^\dagger a \end{aligned} \tag{7.1}$$

where ω is the frequency of the harmonic oscillator mode, T is the temperature, k is Boltzmann's constant and N is the number operator. From equations (3.7) and (3.8) with the help of the substitution $|\psi\rangle\langle\psi| \to \varrho_f$ we find for the quasiprobability $F_f(\alpha,\alpha^*)$ the expression

$$\begin{aligned} F_f(\alpha,\alpha^*) &\equiv \frac{1}{\pi}\langle\varrho D(\alpha,\alpha^*)\varrho_f(D(\alpha,\alpha^*))^\dagger\rangle \\ &= \sum_{n=0}^{\infty}\langle n|\varrho_f|n\rangle L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)Q(\alpha,\alpha^*) \\ &= \frac{1}{1+\bar N}\sum_{n=0}^{\infty}\left(\frac{\bar N}{1+\bar N}\right)^n L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)Q(\alpha,\alpha^*). \end{aligned} \tag{7.2}$$

Using the generating function of the Laguerre polynomials $L_n(z)$ we can rewrite equation (7.2) as

$$F_f(\alpha,\alpha^*) = \exp\left(\bar{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)Q(\alpha,\alpha^*)$$
$$= \frac{1}{\bar{N}\pi}\exp\left(-\frac{\alpha\alpha^*}{\bar{N}}\right) * Q(\alpha,\alpha^*). \tag{7.3}$$

This means that the quasiprobability $F_f(\alpha,\alpha^*)$ associated with the thermal-state filtering can be represented as a convolution of the coherent-state quasiprobability $Q(\alpha,\alpha^*)$ with a Gaussian function. The quasiprobability $F_f(\alpha,\alpha^*)$ can also be expressed via the Wigner function of the measured state

$$F_f(\alpha,\alpha^*) = \exp\left\{\left(\bar{N}+\tfrac{1}{2}\right)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}W(\alpha,\alpha^*)$$
$$= \frac{1}{\left(\bar{N}+\frac{1}{2}\right)\pi}\exp\left(-\frac{\alpha\alpha^*}{\bar{N}+\frac{1}{2}}\right) * W(\alpha,\alpha^*) \tag{7.4}$$

and via the Glauber–Sudarshan quasiprobability of the measured state

$$F_f(\alpha,\alpha^*) = \exp\left\{(\bar{N}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}P(\alpha,\alpha^*)$$
$$= \frac{1}{(\bar{N}+1)\pi}\exp\left(-\frac{\alpha\alpha^*}{\bar{N}+1}\right) * P(\alpha,\alpha^*). \tag{7.5}$$

It is obvious that at zero temperature (i.e. when $\bar{N}=0$) the quasiprobability $F_f(\alpha,\alpha^*)$ equals the coherent-state quasiprobability $Q(\alpha,\alpha^*)$.

We note that the quasiprobability $F_f(\alpha,\alpha^*)$ obtained as a result of filtering with a thermal state can be expressed as a properly weighted sum of propensities associated with filtering with Fock states $|n\rangle$, i.e.

$$F_f(\alpha,\alpha^*) = \frac{1}{(1+\bar{N})\pi}\sum_{n=0}^{\infty}\left(\frac{\bar{N}}{1+\bar{N}}\right)^n\langle\alpha,n|\varrho|\alpha,n\rangle. \tag{7.6}$$

When we compare the expression (7.6) for $F_f(\alpha,\alpha^*)$ with the class of s-ordered quasiprobabilities $F_{(0,0,r_3)}(\alpha,\alpha^*)$ which possesses the diagonal representation (see [25, 27], $s=-r_3$)

$$F_{(0,0,r_3)}(\alpha,\alpha^*) = \frac{2}{(1+r_3)\pi}\sum_{n=0}^{\infty}\left(-\frac{1-r_3}{1+r_3}\right)^n\langle\alpha,n|\varrho|\alpha,n\rangle \tag{7.7}$$

we find that $F_f(\alpha,\alpha^*)$ and $F_{(0,0,r_3)}(\alpha,\alpha^*)$ are equal when

$$r_3 = 1+2\bar{N} \geqslant 1 \qquad (\bar{N}\geqslant 0). \tag{7.8}$$

This means that one can obtain quasiprobabilities $F_f(\alpha,\alpha^*)$ corresponding to the filtering with thermal states as a formal extension of the class of s-ordered quasiprobabilities to values $r_3 \geqslant 1$.

7.1. Reconstruction of density operators from 'noisy' propensities

It is not surprising that one can completely reconstruct the density operator of the measured state out of the propensity obtained from filtering with *pure-state* filters. Obviously, when the filter is prepared in a highly nonclassical state, then the reconstruction procedure can be very difficult technically and rather inefficient practically. Nevertheless, in principle, it is possible. On the other hand, when the quantum ruler is prepared in a statistical mixture, then information contained in the measured propensity is also biased by *classical* noise. So the question is whether the inversion from the propensity to the density operator of the measured state can be performed *uniquely*. In what follows we show that this is possible. We will consider one particular example when the filter is in a thermal state.

The reconstruction of the density operator ϱ from the quasiprobability $F_f(\alpha, \alpha^*)$ can be made in two steps. Firstly, we eliminate the classical noise from the propensity $F_f(\alpha, \alpha^*)$, i.e. we perform a transformation $F_f(\alpha, \alpha^*) \to Q(\alpha, \alpha^*)$. Once this is done, then using results of previous sections (see equations (3.4) and (3.5)) we can obtain the density operator of the measured system.

The first step of this procedure is based on the formal inversion of equation (7.3), i.e.

$$\begin{aligned} Q(\alpha, \alpha^*) &= \exp\left(-\bar{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)F_f(\alpha, \alpha^*) \\ &= \frac{1}{\sqrt{\bar{N}^2}\pi}\exp\left(\frac{\alpha\alpha^*}{\bar{N}}\right) * F_f(\alpha, \alpha^*). \end{aligned} \tag{7.9}$$

Unfortunately, this is a convolution of $F_f(\alpha, \alpha^*)$ with a Gaussian function having a positive-definite quadratic form in the exponent. This simply means that the integral (7.9) cannot be performed in the sense of the usual integral representation of the convolution. To overcome the problem we have to define this convolution according to the theory of generalized functions [43, 45]. That is, we have to treat $\exp(\alpha\alpha^*/\bar{N})$ as a generalized function, i.e. a linear analytic functional for which the real variables $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ can be separately extended to complex variables. Simultaneously, we transform the two integrations over the real axes into integrations over the imaginary axes of the new independent complex variables (the sign of the square root in equation (7.9) depends on the chosen direction on integration paths).

Alternatively, the convolution (7.9) can be performed with the help of Fourier transforms of the right- and left-hand side of this equation. In this case we obtain in the right-hand side of the new equation the product of the Fourier transform of $F_f(\alpha, \alpha^*)$ and the Fourier transform of the Gaussian function with positive exponent (which has to be considered in a sense of the Fourier transformations of generalized functions, which in this particular case is equal to a Gaussian function with a positive-definite quadratic form in the exponent). The inversion of the Fourier transform of the product of the Fourier transforms in the right-hand side of equation (7.9) can be performed due to the rapid decrease of the Fourier transform of $F_f(\alpha, \alpha^*)$ at infinity. Finally, the convolution (7.9) can be evaluated with the help of the Taylor series expansion of the integral operator $\exp(-\bar{N}\,\partial^2/(\partial\alpha\partial\alpha^*))$ which is the Fourier transform of the Gaussian function in (7.9) with the variables substituted by corresponding partial derivatives

$$Q(\alpha, \alpha^*) = \sum_{n=0}^{\infty}\frac{(-\bar{N})^n}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}F_f(\alpha, \alpha^*). \tag{7.10}$$

As an example let us consider that the measured system is prepared in a coherent state $|\beta\rangle$. If the filter is in a thermal state, then the corresponding propensity reads

$$F_f(\alpha, \alpha^*) = \frac{1}{(\bar{N}+1)\pi}\exp\left\{-\frac{(\alpha-\beta)(\alpha^*-\beta^*)}{\bar{N}+1}\right\}. \tag{7.11}$$

To obtain the Q-function from this 'noisy' propensity we utilize equation (7.10) and we find†

$$
\begin{aligned}
Q(\alpha,\alpha^*) &= \sum_{n=0}^{\infty}\frac{(-\bar{N})^n}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\frac{1}{(\bar{N}+1)\pi}\exp\left\{-\frac{(\alpha-\beta)(\alpha^*-\beta^*)}{\bar{N}+1}\right\} \\
&= \frac{1}{(\bar{N}+1)\pi}\exp\left\{-\frac{(\alpha-\beta)(\alpha^*-\beta^*)}{\bar{N}+1}\right\} \\
&\quad\times\sum_{n=0}^{\infty}\frac{(-\bar{N})^n}{n!}\left(\frac{\partial}{\partial\alpha}-\frac{\alpha^*-\beta^*}{\bar{N}+1}\right)^n\left(-\frac{\alpha-\beta}{\bar{N}+1}\right)^n \\
&= \frac{1}{(\bar{N}+1)\pi}\exp\left\{-\frac{(\alpha-\beta)(\alpha^*-\beta^*)}{\bar{N}+1}\right\} \\
&\quad\times(\bar{N}+1)\exp\left\{-\frac{\bar{N}(\alpha-\beta)(\alpha^*-\beta^*)}{\bar{N}+1}\right\} \\
&= \frac{1}{\pi}\exp\{-(\alpha-\beta)(\alpha^*-\beta^*)\}.
\end{aligned}
\tag{7.12}
$$

Thus we show that the Q-function (as well as the corresponding density operator) of a coherent state $|\beta\rangle$ can be *uniquely* reconstructed from propensities obtained from filtering with the help of thermal filters.

To conclude this section we make two comments:

(i) The convolution in equation (7.3) is unique in the sense that there exists a one-to-one mapping between quasiprobabilities $Q(\alpha,\alpha^*)$ and propensities $F_f(\alpha,\alpha^*)$. This can be understood as a consequence of the fact that the equation $\exp(\bar{N}\,\partial^2/(\partial\alpha\partial\alpha^*))f(\alpha,\alpha^*)=0$ only has the trivial solution $f(\alpha,\alpha^*)=0$. Operationally, this allows us to create a (complete) list of direct transformations from specific functions $Q(\alpha,\alpha^*)$ to their 'noisy' counterparts $F_f(\alpha,\alpha^*)$ and vice versa.

(ii) Problems and difficulties in the evaluation of convolutions with Gaussian functions having positive-definite quadratic forms in the exponent are present in every case when one has to perform the transition from one s-ordered quasiprobability to another less smoothed s-ordered quasiprobability (for example, from $Q(\alpha,\alpha^*)$ to $W(\alpha,\alpha^*)$ or from $W(\alpha,\alpha^*)$ to $P(\alpha,\alpha^*)$). Convolutions with normalized Gaussian functions form Abelian Lie groups which in the two-dimensional case lead to a three-parameter group of Gaussian functions centred at the coordinate origin with the delta function $\delta(\alpha,\alpha^*)$ as the unit element [27]. More generally, they form some algebra of convolutions [43].

8. Conclusions

In this paper we have focused our attention on the problem of reconstruction of density operators from propensities obtained in eight-port homodyne detection with arbitrary quantum rulers. We have also introduced the corresponding generalized Glauber–Sudarshan quasiprobabilities associated with these propensities. We have shown that the transition from

† The sum over n in this equation is evaluated in the following way. First, the binomial formula is applied to the differential operator and then the differentiations are accomplished. The double sum which arises is reordered. The first summation leads to the sum of the Taylor series of $k!(1-z)^{-(k+1)}$ with $z=\bar{N}/(\bar{N}+1)$ which is convergent for arbitrary $|z|<1$ and therefore for arbitrary $\bar{N}$. The remaining summation leads to the sum of the Taylor series of the exponential function which is an entire function and therefore is uniformly convergent in arbitrary compact regions of the real axis or complex plane.

the usual coherent-state quasiprobability (i.e. the Q-function) to the generalized coherent-state quasiprobability is associated with the problem of the inversion of certain differential or integral operators. However, as we have shown, in general, it is difficult to perform this inversion explicitly. For instance, we have not found a closed expression for the inverted operators under consideration in the case when Fock states $|n\rangle$ ($n \geqslant 2$) are used as quantum rulers. On the other hand, if quantum rulers belong to the class of Gaussian states (for instance, squeezed vacuum states $|0;\zeta\rangle_{\text{norm}}$ and their displacements) it is possible to give all necessary formulae in a closed explicit form. Finally, we have shown that propensities based on filtering with quantum rulers prepared in statistical-mixture states can be inverted and the noise induced by filtering can be separated from measured data, so the density operator of the measured state can be reconstructed uniquely.

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Appendix A. Reconstruction of density operators from normally ordered moments

In equation (3.5) we defined the set of operators $a_{k,l}$ ($k, l = 0, \ldots, \infty$) as

$$a_{k,l} \equiv \frac{1}{k!l!}\left\{\frac{\partial^{k+l}}{\partial\alpha^{*k}\partial\alpha^{l}}|\alpha\rangle\langle\alpha|\right\}_{(\alpha=\alpha^*=0)}. \tag{A.1}$$

In this appendix we derive the Fock-state representation of these operators $a_{k,l}$. Rewriting equation (A.1) in the Fock basis we find

$$\begin{aligned} a_{k,l} &= \frac{1}{k!l!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{m!n!}}\left\{\frac{\partial^{k+l}}{\partial\alpha^{*k}\partial\alpha^{l}}\exp(-\alpha\alpha^*)\alpha^{*m}\alpha^{n}\right\}_{(\alpha=\alpha^*=0)} \\ &= \frac{1}{k!l!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{m!n!}}\left\{\exp(-\alpha\alpha^*)\left(\frac{\partial}{\partial\alpha^*}-\alpha\right)^{k}\left(\frac{\partial}{\partial\alpha}-\alpha^*\right)^{l}\alpha^{*m}\alpha^{n}\right\}_{(\alpha=\alpha^*=0)} \\ &= \sum_{j=0}^{\{k,l\}}\frac{(-1)^j|l-j\rangle\langle k-j|}{j!\sqrt{(k-j)!(l-j)!}} \end{aligned} \tag{A.2}$$

where $\{k,l\}$ denotes the smaller of the two integers k and l. In a normally ordered representation one obtains from equation (A.2) the expression

$$a_{k,l} = \frac{(-1)^{k+l}}{k!l!}\sum_{s=[k,l]}^{\infty}\frac{(-1)^s s!}{(s-k)!(s-l)!}a^{\dagger\, s-k}a^{s-l} \tag{A.3}$$

where $[k,l]$ denotes the larger of the two integers k and l. We find that the operators $a_{k,l}$ are orthogonal to the operators $a^{\dagger k}a^{l}$ according to

$$\langle a^{\dagger k}a^{l}a_{m,n}\rangle = \delta_{k,m}\delta_{l,n}. \tag{A.4}$$

So we can write the normally ordered expansion of the density operator in equation (3.5) as

$$\varrho = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle\varrho a_{k,l}\rangle a^{\dagger k}a^{l}. \tag{A.5}$$

With the help of the orthogonality relation (A.4) we also obtain the dual relation

$$\varrho = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} \langle a^{\dagger k} a^l \varrho \rangle. \tag{A.6}$$

This equation can be interpreted as a reconstruction formula for the density operator ϱ from its normally ordered moments $\langle a^{\dagger k} a^l \varrho \rangle$ [47].

Appendix B. Proof of a summation identity

We prove the identity in equation (5.11) by using the Mehler formula [44]

$$\sum_{k=0}^{\infty} \frac{z^k}{2^k k!} H_k(x) H_k(y) = \frac{1}{\sqrt{1-z^2}} \exp\left\{ \frac{2xyz - (x^2+y^2)z^2}{1-z^2} \right\}. \tag{B.1}$$

First, we perform the following chain of transformations:

$$\begin{aligned}
&\frac{1}{\sqrt{1-z^2}} \exp\left\{ \frac{z^2}{1-z^2} \big(2xyz - (x^2+y^2)\big) \right\} \\
&\quad = \frac{1}{\sqrt{1-z^2}} \exp\left\{ \frac{2xyz - (x^2+y^2)z^2}{1-z^2} \right\} \exp(-2xyz) \\
&\quad = \sum_{k=0}^{\infty} \frac{z^k}{2^k k!} H_k(x) H_k(y) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (2xyz)^l \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{2^{2(m+n)} m! n!} \\
&\qquad \times \sum_{k=0}^{\{2m,2n\}} \sum_{l=0}^{\infty} \frac{(-1)^l \, k!}{l! (k-2n)! (k-2m)!} x^{k+l-2n} y^{k+l-2m} (2z)^{k+l} \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{2^{2(m+n)} m! n!} \sum_{j=0}^{\{2m,2n\}} x^{2m-j} y^{2n-j} (2z)^{2(m+n)-j} \\
&\qquad \times \sum_{l=0}^{\{2m-j,2n-j\}} \frac{(-1)^l \big(2(m+n) - j - l\big)!}{l! (2m-j-l)! (2n-j-l)!} \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} z^{m+n}}{2^{m+n} m! n!} \\
&\qquad \times \sum_{j=0}^{\{2m,2n\}} \frac{(2m)!(2n)!}{j!(2m-j)!(2n-j)!} \big(\sqrt{2z}x\big)^{2m-j} \big(\sqrt{2z}y\big)^{2n-j}
\end{aligned} \tag{B.2}$$

where after a change of the order of the summations in the fourth step we performed the index substitution $j + k + l = 2(m+n)$ and then used the summation formula

$$\sum_{l=0}^{\{r-j,s-j\}} \frac{(-1)^l (r+s-j-l)!}{l!(r-j-l)!(s-j-l)!} = \frac{r!s!}{j!(r-j)!(s-j)!} \tag{B.3}$$

in the special case when $r = 2m$ and $s = 2n$. This summation formula is true for $r = s = 0$ with arbitrary j and can be easily proved then for arbitrary (r, s) and j by complete induction $r \to r+1$ and $s \to s+1$. The polynomial written by the summation over j in equation (B.2) can also be expressed by a Laguerre polynomial and we obtain a new identity for Laguerre polynomials. With the substitutions

$$x \equiv \frac{1}{\sqrt{2\zeta^*}}\frac{\partial}{\partial\alpha} \qquad y \equiv \frac{1}{\sqrt{2\zeta}}\frac{\partial}{\partial\alpha^*} \qquad z \equiv \sqrt{\zeta\zeta^*} \tag{B.4}$$

we transform this identity into equation (5.10).

Appendix C. Connections between squeezed-state propensities

We give here a complete list of the connections of the squeezed-state propensities to the usual quasiprobabilities $P(\alpha, \alpha^*)$, $W(\alpha, \alpha^*)$ and $Q(\alpha, \alpha^*)$. This can be obtained by using the convolutions of normalized Gaussian functions with the additive complex parameters (r_1, r_2, r_3) introduced in [27].

(i) Connection of $Q_\zeta(\alpha, \alpha^*)$ to usual quasiprobabilities:

(a) Connection to $Q(\alpha, \alpha^*)$

$$\begin{aligned} Q_\zeta(\alpha, \alpha^*) &= \frac{1}{\pi}\sqrt{-\frac{1-\zeta\zeta^*}{\zeta\zeta^*}}\exp\left\{\alpha\alpha^* + \frac{\alpha^2}{2\zeta} + \frac{\alpha^{*2}}{2\zeta^*}\right\} * Q(\alpha, \alpha^*) \\ &= \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\zeta\zeta^*\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}Q(\alpha, \alpha^*). \end{aligned} \tag{C.1}$$

(b) Connection to $W(\alpha, \alpha^*)$

$$\begin{aligned} Q_\zeta(\alpha, \alpha^*) &= \frac{2}{\pi}\exp\left\{-\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\} * W(\alpha, \alpha^*) \\ &= \exp\left\{\frac{1}{2(1-\zeta\zeta^*)}\left(\frac{\partial}{\partial\alpha} - \zeta^*\frac{\partial}{\partial\alpha^*}\right)\left(\frac{\partial}{\partial\alpha^*} - \zeta\frac{\partial}{\partial\alpha}\right)\right\}W(\alpha, \alpha^*). \end{aligned} \tag{C.2}$$

(c) Connection to $P(\alpha, \alpha^*)$

$$\begin{aligned} Q_\zeta(\alpha, \alpha^*) &= \frac{\sqrt{1-\zeta\zeta^*}}{\pi}\exp\left\{-\left(\alpha\alpha^* + \tfrac{1}{2}\zeta^*\alpha^2 + \tfrac{1}{2}\zeta\alpha^{*2}\right)\right\} * P(\alpha, \alpha^*) \\ &= \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}P(\alpha, \alpha^*). \end{aligned} \tag{C.3}$$

(ii) Connection of $P_\zeta(\alpha, \alpha^*)$ to usual quasiprobabilities:

(a) Connection to $Q(\alpha, \alpha^*)$

$$\begin{aligned} P_\zeta(\alpha, \alpha^*) &= \frac{\sqrt{1-\zeta\zeta^*}}{\pi}\exp\left\{\alpha\alpha^* + \tfrac{1}{2}\zeta^*\alpha^2 + \tfrac{1}{2}\zeta\alpha^{*2}\right\} * Q(\alpha, \alpha^*) \\ &= \exp\left\{-\frac{1}{1-\zeta\zeta^*}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}Q(\alpha, \alpha^*). \end{aligned} \tag{C.4}$$

(b) Connection to $W(\alpha, \alpha^*)$

$$\begin{aligned} P_\zeta(\alpha, \alpha^*) &= \frac{2}{\pi}\exp\left\{\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\} * W(\alpha, \alpha^*) \\ &= \exp\left\{-\frac{1}{2(1-\zeta\zeta^*)}\left(\frac{\partial}{\partial\alpha} - \zeta^*\frac{\partial}{\partial\alpha^*}\right)\left(\frac{\partial}{\partial\alpha^*} - \zeta\frac{\partial}{\partial\alpha}\right)\right\}W(\alpha, \alpha^*). \end{aligned} \tag{C.5}$$

(c) Connection to $P(\alpha, \alpha^*)$

$$P_\zeta(\alpha,\alpha^*) = \frac{1}{\pi}\sqrt{-\frac{1-\zeta\zeta^*}{\zeta\zeta^*}}\exp\left\{-\left(\alpha\alpha^* + \frac{\alpha^2}{2\zeta} + \frac{\alpha^{*2}}{2\zeta^*}\right)\right\} * P(\alpha,\alpha^*)$$
$$= \exp\left\{-\frac{1}{1-\zeta\zeta^*}\left(\zeta\zeta^*\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\zeta}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{\zeta^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}P(\alpha,\alpha^*). \tag{C.6}$$

(iii) Direct connection between $P_\zeta(\alpha, \alpha^*)$ and $Q_\zeta(\alpha, \alpha^*)$

$$Q_\zeta(\alpha,\alpha^*) = \frac{1}{\pi}\exp\left\{-\frac{(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\} * P_\zeta(\alpha,\alpha^*)$$
$$= \exp\left\{\frac{1}{1-\zeta\zeta^*}\left(\frac{\partial}{\partial\alpha} - \zeta^*\frac{\partial}{\partial\alpha^*}\right)\left(\frac{\partial}{\partial\alpha^*} - \zeta\frac{\partial}{\partial\alpha}\right)\right\}P_\zeta(\alpha,\alpha^*). \tag{C.7}$$

The inversion of all these convolutions can be obtained by changing the sign in the exponents of the Gaussian functions in these convolutions. By setting $\zeta = 0$ one can take from (C.1)–(C.7) the connections between the usual quasiprobabilities $Q(\alpha, \alpha^*)$, $W(\alpha, \alpha^*)$ and $P(\alpha, \alpha^*)$.

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