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Difference-phase squeezing from amplitude squeezing by means of a beamsplitter

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Abstract. A classical analysis of two equal intensity coherent fields incident on a beamsplitter shows that the difference-phase noise of the output depends only on the noise in the difference of the amplitudes of the input fields and not on their phase noise. This suggests that in the quantum mechanical case squeezing in the amplitudes of the input beams can lead to squeezing in the phase difference of the output beams. We show that this is true. We also find the phase properties of the output when the input consists of two number states with an equal number of photons. The difference phase distribution consists of two narrow peaks, at $\theta = 0$ and $\theta = \pi$. States with small phase difference noise should be useful in the measurement of phase shifts.

1. Introduction

A beamsplitter is perhaps one of the simplest optical devices. They are of interest in themselves and as components of interferometers. In a beamsplitter two input beams are transformed into two output beams. Classically this transformation is quite simple but quantum mechanically the situation is more complicated; the quantum properties of the input and output beams are not the same. With the development of sources of nonclassical light these issues have attracted increased attention. Considerable work has been done on the relation between the quantum properties of the input beams and those of the output beams [1–6]. Some of this work has focused on beamsplitters in interferometers [1], and some has analysed their role in homodyne and heterodyne detection [4, 5]. Campos, Saleh and Teich made a detailed study of the relation between the number distributions of the input and the output states [6]. They found some rather unusual effects. For example, for a 50–50 beamsplitter with number state inputs with equal numbers of photons in each port, the probability of detecting an odd number of photons in either output port vanishes.

Here we want to examine the phase properties of the beamsplitter output. It is the behaviour of the phase difference between the output beams which determines how the beamsplitter will behave as part of an interferometer. In particular, we want to consider the case when the intensities of the input beams are equal. As we shall see, this will allow us to produce output beams with reduced noise in the difference of their phases. Such beams should be useful in the measurement of phase shifts.

A classical analysis shows that for two equal intensity input beams the noise in the phase difference of the output beams depends only on the intensity noise of the input beams and not on their phase noise. This is true as long as the intensity and phase noise are not

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too large. This suggests that in the quantum case intensity squeezing could be converted to phase difference squeezing by a beamsplitter. An analysis with input states which are coherent states squeezed in the amplitude direction shows that this is indeeed true.

We also consider what happens when the intensity noise of the input state is zero but the phase noise is large. This is the case if the input state consists of two number states with equal photon numbers. This situation was first considered by Holland and Burnett, but our results differ somewhat from theirs [7]. We find that the difference phase distribution of the output has two peaks, one at zero and one at π. Each of these peaks exhibits phase squeezing. In fact, the output is a superposition of a state with phase difference zero and one with phase difference π which one can view as a kind of Schrödinger cat.

Finally, we briefly examine the effect of an input state in which the two input beams are correlated. The classical analysis suggests that if the amplitudes of the input beams are equal, and if the amplitude fluctuations are correlated, then the difference phase fluctuations will be small. In the quantum mechanical case, a displaced two-mode squeezed state satisfies these conditions, and we verify that it produces an output state with difference-phase squeezing.

2. Classical analysis

Let us begin our study by considering what happens when classical light is incident on a beamsplitter (figure 1). The complex amplitudes of the input fields will be designated by $\alpha_{1\mathrm{in}} = r_1 \mathrm{e}^{\mathrm{i}\theta_1}$ and $\alpha_{2\mathrm{in}} = r_2 \mathrm{e}^{\mathrm{i}\theta_2}$, while those of the output fields will be denoted by $\alpha_{1\mathrm{out}}$ and $\alpha_{2\mathrm{out}}$. We shall assume that our beamsplitter is such that

$$\begin{pmatrix} \alpha_{1\mathrm{out}} \\ \alpha_{2\mathrm{out}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{1\mathrm{in}} \\ \alpha_{2\mathrm{in}} \end{pmatrix}. \tag{1}$$

We are interested in finding the phase difference, θ_d, between the output beams. This can be done by finding the argument of $\alpha^*_{2\mathrm{out}}\alpha_{1\mathrm{out}}$. We find that

$$\theta_\mathrm{d} = \arg(\alpha^*_{2\mathrm{out}}\alpha_{1\mathrm{out}}) = \arctan\left(\frac{r_1^2 - r_2^2}{2r_1 r_2 \cos(\theta_1 - \theta_2)}\right). \tag{2}$$

It is necessary at this point to discuss which branch of arctan should be taken. The quantity $2r_1r_2\cos(\theta_1 - \theta_2)$ is the real part of $\alpha^*_{2\mathrm{out}}\alpha_{1\mathrm{out}}$. If the real part is positive the argument of $\alpha^*_{2\mathrm{out}}\alpha_{1\mathrm{out}}$ is between $-\pi/2$ and $\pi/2$ while if it is negative the argument is either between $\pi/2$ and π or between $-\pi$ and $-\pi/2$. Therefore, if $\cos(\theta_1 - \theta_2) \geqslant 0$, then $-\pi/2 \leqslant \theta_\mathrm{d} \leqslant \pi/2$, and if $\cos(\theta_1 - \theta_2) < 0$, then either $\pi/2 < \theta_\mathrm{d} \leqslant \pi$ or $-\pi < \theta_\mathrm{d} < -\pi/2$.

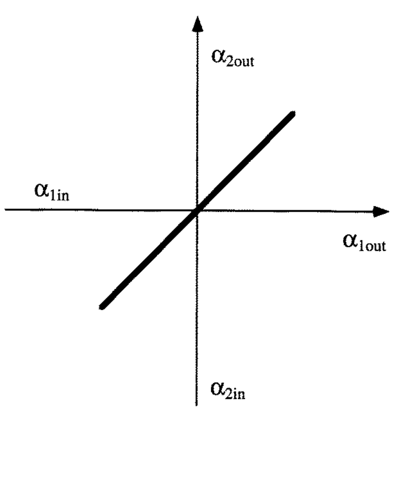

Figure 1. Beamsplitter input and output modes. In the quantum mechanical case the classical mode amplitudes are replaced by annihilation operators.

Now consider the case $r_1 = r_2$. Then θ_d will be either 0 or π depending on whether $\cos(\theta_1 - \theta_2)$ is positive or negative. This offers the possibility of very small phase fluctuations in θ_d even if there are substantial fluctuations in θ_1 and θ_2. In particular, suppose that θ_1 and θ_2 both fluctuate about the value θ_0 and that the fluctuations are such that $\theta_0 - \pi/4 < \theta_j < \theta_0 + \pi/4$, $j = 1, 2$. This implies that $\cos(\theta_1 - \theta_2)$ is positive and implies that, despite large phase fluctuations in θ_1 and θ_2, the value of θ_d is always zero. Therefore, a beamsplitter with input beams of equal intensity can provide output beams with a very well defined difference phase.

Let us do a more detailed analysis incorporating both amplitude and phase fluctuations. Suppose that r_1, r_2, θ_1, and θ_2 all fluctuate and that the fluctuations are small, i.e. small enough so that we need only consider quantities of at most second order in the fluctuations. Assume that r_1 and r_2 have the same mean value, $r = \langle r_1\rangle = \langle r_2\rangle$, and define $\delta r_j = r_j - r$, $j = 1, 2$. Let us also assume, for simplicity, that the mean value of $\theta_1 - \theta_2$ is zero. Making use of equation (2) we find that

$$\langle(\delta\theta_d)^2\rangle = \frac{1}{r^2}\langle(\delta r_1 - \delta r_2)^2\rangle \tag{3}$$

where $\delta\theta_d = \theta_d - \langle\theta_d\rangle$. Note that at this level of approximation the fluctuations in θ_d do not depend on the phase fluctuations but on the intensity fluctuations. Thus, an input with small intensity fluctuations can be used to produce an output with small fluctuations in the difference phase. Note also that an input with correlated amplitude fluctuations can also reduce phase-difference fluctuations. The most extreme case would be if $\delta r_1 = \delta r_2$, which, classically, leads to no difference-phase noise at all. We shall examine a quantum mechanical analogue of this situation in section 6.

3. Quantum analysis—coherent states

A close connection between the classical and quantum results occurs when the input state to the beamsplitter is a product of a coherent state in mode 1, $|\beta_{1\text{in}}\rangle$, and another coherent state in mode 2, $|\beta_{2\text{in}}\rangle$. The coherent state parameters, $\beta_{1\text{in}}$ and $\beta_{2\text{in}}$, correspond to the classical mode amplitudes. The output state from the beamsplitter is also a product of coherent states, $|\beta_{1\text{out}}\rangle_1 \otimes |\beta_{2\text{out}}\rangle_2$. The output parameters, $\beta_{1\text{out}}$ and $\beta_{2\text{out}}$, are related to the input parameters,$\beta_{1\text{in}}$ and $\beta_{2\text{in}}$, in the same way that the classical output mode amplitudes are related to the classical input amplitudes, i.e. through equation (1) [2, 3].

Let us now justify these statements. Quantum mechanically a beamsplitter is described as a transformation between the in operators of its two modes and the out operators of the same modes [1]. The transformation corresponding to the beamsplitter in the previous section is given by

$$\begin{pmatrix} a_{1\text{out}} \\ a_{2\text{out}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\begin{pmatrix} a_{1\text{in}} \\ a_{2\text{in}} \end{pmatrix} \tag{4}$$

where $a_{j\text{in}}$ and $a_{j\text{out}}$, $j = 1, 2$ are the in and out annihilation operators, respectively. This transformation can also be represented as [1]

$$\begin{pmatrix} a_{1\text{out}} \\ a_{2\text{out}} \end{pmatrix} = U_1^{-1}\begin{pmatrix} a_{1\text{in}} \\ a_{2\text{in}} \end{pmatrix} U_1 \tag{5}$$

where

$$U_1 = \exp[-\mathrm{i}\pi(a_{1\text{in}}^\dagger a_{2\text{in}} + a_{1\text{in}}a_{2\text{in}}^\dagger)/2]. \tag{6}$$

The operator U_1 also transforms input states into output states, i.e. if $|\Psi_{\mathrm{in}}\rangle$ is the input state to the beamsplitter then the output state, $|\Psi_{\mathrm{out}}\rangle$ is

$$|\Psi_{\mathrm{out}}\rangle = U_1|\Psi_{\mathrm{in}}\rangle. \tag{7}$$

For the sake of notational simplicity we shall henceforth drop the subscript 'in' on in operators. Operators without an in or an out subscript will be understood to be in operators.

An input state which is a product of coherent states,

$$|\Psi_{\mathrm{in}}\rangle = |\beta_1\rangle \otimes |\beta_2\rangle \tag{8}$$

can be expressed in the form

$$|\Psi_{\mathrm{in}}\rangle = D_1(\beta_1)D_2(\beta_2)|0\rangle_1 \otimes |0\rangle_2 \tag{9}$$

where $D_1(\beta_1)$ and $D_2(\beta_2)$ are the displacement operators for modes 1 and 2, respectively,

$$D_1(\beta_1) = \exp(\beta_1 a_1^\dagger - \beta_1^* a_1) \tag{10}$$

$$D_2(\beta_2) = \exp(\beta_2 a_2^\dagger - \beta_2^* a_2). \tag{11}$$

The output state is most easily found by applying U_1 to both sides of (9) and making use of the operator transformations in (4) and (5). The result is

$$|\Psi_{\mathrm{out}}\rangle = \left|\frac{\beta_1 - \mathrm{i}\beta_2}{\sqrt{2}}\right\rangle_1 \otimes \left|\frac{\beta_2 - \mathrm{i}\beta_1}{\sqrt{2}}\right\rangle_2 \tag{12}$$

i.e. the product of two coherent states.

In order to discuss the phase properties of $|\Psi_{\mathrm{out}}\rangle$ it is useful to find its phase distribution. This is given by

$$P(\theta_1, \theta_2) = |\langle\theta_1, \theta_2|\Psi_{\mathrm{out}}\rangle|^2 \tag{13}$$

where

$$|\theta_1, \theta_2\rangle = |\theta_1\rangle_1 \otimes |\theta_2\rangle_2 \tag{14}$$

$$= \left[\left(\frac{1}{2\pi}\right)^{1/2} \sum_{n_1=0}^{\infty} \mathrm{e}^{\mathrm{i}n_1\theta_1}|n_1\rangle_1\right] \otimes \left[\left(\frac{1}{2\pi}\right)^{1/2} \sum_{n_2=0}^{\infty} \mathrm{e}^{\mathrm{i}n_2\theta_2}|n_2\rangle_2\right]. \tag{15}$$

Let us assume that $|\beta_1 - \mathrm{i}\beta_2| \gg 1$ and that $|\beta_2 - \mathrm{i}\beta_1| \gg 1$. We can then make use of the formula for the inner product of a large amplitude coherent state, $|\alpha\rangle$, and a phase state $|\theta\rangle$

$$\langle\theta|\alpha\rangle \cong \left(\frac{2|\alpha|^2}{\pi}\right)^{1/4} \mathrm{e}^{\mathrm{i}|\alpha|^2(\phi-\theta)-|\alpha|^2(\phi-\theta)^2} \tag{16}$$

where $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\phi}$ [8].

The application of this formula requires that we find the arguments and magnitudes of $(\beta_1 - \mathrm{i}\beta_2)/\sqrt{2}$ and $(\beta_2 - \mathrm{i}\beta_1)/\sqrt{2}$. If β_1 and β_2 are expressed in polar notation

$$\beta_1 = |\beta_1|\mathrm{e}^{\mathrm{i}\mu_1} \qquad \beta_2 = |\beta_2|\mathrm{e}^{\mathrm{i}\mu_2} \tag{17}$$

then we find that

$$\arg(\beta_1 - \mathrm{i}\beta_2) = \arctan\left(\frac{|\beta_1|\sin\mu_1 - |\beta_2|\cos\mu_2}{|\beta_1|\cos\mu_1 + |\beta_2|\sin\mu_2}\right) \tag{18}$$

$$\arg(\beta_2 - \mathrm{i}\beta_1) = \arctan\left(\frac{|\beta_2|\sin\mu_2 - |\beta_1|\cos\mu_1}{|\beta_2|\cos\mu_2 + |\beta_1|\sin\mu_1}\right). \tag{19}$$

The classical analysis suggests that we look at the case $|\beta_1| = |\beta_2|$. Making this assumption we find from (18) and (19) that if the numerators of the fractions inside the arctan functions have the same sign, then

$$\arg(\beta_1 - \mathrm{i}\beta_2) = \arg(\beta_2 - \mathrm{i}\beta_1). \tag{20}$$

If they have opposite signs, the arguments differ by π. Let us assume that they have the same sign so that equation (20) applies. We shall denote the common phase by θ_0. Turning our attention now to the magnitudes we have

$$\left|\frac{\beta_1 - \mathrm{i}\beta_2}{\sqrt{2}}\right|^2 = r^2[1 - \sin(\mu_1 - \mu_2)] \tag{21}$$

$$\left|\frac{\beta_2 - \mathrm{i}\beta_1}{\sqrt{2}}\right|^2 = r^2[1 + \sin(\mu_1 - \mu_2)] \tag{22}$$

where $r = |\beta_1| = |\beta_2|$.

We are now in a position to find the phase distribution of the output state. We have

$$P(\theta_1, \theta_2) \cong \frac{2}{\pi} r^2 |\cos(\mu_1 - \mu_2)| \exp\{-2r^2[1 - \sin(\mu_1 - \mu_2)](\theta_1 - \theta_0)^2 - 2r^2[1 + \sin(\mu_1 - \mu_2)](\theta_2 - \theta_0)^2\}. \tag{23}$$

From this expression we see that the phase of each mode is peaked about θ_0, and the width of the peaks is of order $1/r$, although the exact widths depend on μ_1 and μ_2. This implies that the difference phase is sharply peaked about zero. This is in agreement with our expectations from the classical argument.

To see this in more detail let us calculate the distribution for θ_d. This can be done by expressing $P(\theta_1, \theta_2)$ in terms of θ_d and the sum angle, $\theta_\mathrm{s} = (\theta_1 + \theta_2)$, and integrating out θ_s. The result is

$$P(\theta_\mathrm{d}) = \frac{r}{\sqrt{\pi}} |\cos(\mu_1 - \mu_2)| \exp\{-r^2 \cos^2(\mu_1 - \mu_2)\theta_\mathrm{d}^2\}. \tag{24}$$

This confirms our result from the preceding paragraph that $P(\theta_\mathrm{d})$ is highly peaked about zero with the width of the peak being of order $1/r$. In addition we see that the peak will be narrowest when $\mu_1 = \mu_2$. Therefore, to achieve the sharpest difference phase in the output the two input coherent states should have complex amplitudes whose magnitudes and phases are equal.

4. Two-squeezed-state input

Let us now consider the situation when identical displaced squeezed states are sent into the input ports. Each state is squeezed in the direction of the displacement so that the amplitude noise of each state is decreased and the phase noise is increased. According to our classical argument this input state should produce an output whose difference phase is squeezed. We would like to verify that this is so.

One approach is to calculate the difference phase distribution for the output state. This is rather complicated so we shall take a different approach which is valid if the squeezing is not too large. Consider the operator

$$V = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\phi_1} a_1 + \mathrm{e}^{\mathrm{i}\phi_2} a_2 + \mathrm{e}^{-\mathrm{i}\phi_1} a_1^\dagger + \mathrm{e}^{-\mathrm{i}\phi_2} a_2^\dagger) \tag{25}$$

which corresponds to the classical quantity

$$V_\mathrm{cl} = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\phi_1} \alpha_1 + \mathrm{e}^{\mathrm{i}\phi_2} \alpha_2 + \mathrm{e}^{-\mathrm{i}\phi_1} \alpha_1^* + \mathrm{e}^{-\mathrm{i}\phi_2} \alpha_2^*) \tag{26}$$

where $\alpha_1 = r_1 e^{i\theta_1}$ and $\alpha_2 = r_2 e^{i\theta_2}$. As in section 2 we assume that r_1, r_2, θ_1, and θ_2 fluctuate, and that

$$\langle r_1 \rangle = \langle r_2 \rangle = r \qquad \langle \theta_1 - \theta_2 \rangle = 0. \tag{27}$$

Setting $\phi_1 = \frac{1}{2}\pi - \langle \theta_1 \rangle$ and $\phi_2 = \frac{3}{2}\pi - \langle \theta_2 \rangle$ and keeping quantities of up to second order in δr_j and $\delta\theta_j$, $j = 1, 2$, we find

$$(\delta V_{\mathrm{cl}})^2 = r^2 \langle (\delta\theta_1 - \delta\theta_2)^2 \rangle \tag{28}$$

i.e. we can identify $(\delta V_{\mathrm{cl}})^2$ with $r^2 \langle (\delta\theta_{\mathrm{d}})^2 \rangle$. For displaced squeezed states, with identical displacements, $r e^{i\theta}$, we can set

$$\langle (\delta\theta_{\mathrm{d}})^2 \rangle = \frac{1}{r^2} (\Delta V)^2 \tag{29}$$

with $\phi_1 = \frac{1}{2}\pi - \theta$ and $\phi_2 = \frac{3}{2}\pi - \theta$. Therefore, in order to find $\langle (\delta\theta_{\mathrm{d}})^2 \rangle$ for our output state we need to apply the beamsplitter transformation to the input state and calculate $(\Delta V)^2$.

For the input state in each mode we choose a displaced squeezed state. In order to produce this state we start with a squeezed vacuum state, which is squeezed in the real direction, and then displace it along the positive real axis of the complex amplitude plane (figure 1). Explicitly the state is

$$|\Psi_{\mathrm{in}}\rangle = D_1(r) D_2(r) S_1(-\mu) S_2(-\mu) |0\rangle_1 \otimes |0\rangle_2 \tag{30}$$

where r and μ are real and greater than zero. The squeeze operator, $S_j(-\mu)$, for $j = 1, 2$ is given by

$$S_j(-\mu) = \exp\{-\mu[(a_j^\dagger)^2 - a_j^2]/2\}. \tag{31}$$

We next apply the beamsplitter transformation, U_1, to $|\Psi_{\mathrm{in}}\rangle$. Use of equations (6) and (30) yields

$$\begin{aligned} |\Psi_{\mathrm{out}}\rangle &= U_1 |\Psi_{\mathrm{in}}\rangle \\ &= D_1(r e^{-i\pi/4}) D_2(r e^{-i\pi/4}) \exp[i\mu(a_1^\dagger a_2^\dagger + a_1 a_2)] |0\rangle_1 \otimes |0\rangle_2. \end{aligned} \tag{32}$$

We now apply equation (29) to this state. We first note that the angle of the displacement is $-\pi/4$ so that $\phi_1 = 3\pi/4$ and $\phi_2 = -\pi/4$. With these choices, $(\Delta V)^2$ for $|\Psi_{\mathrm{out}}\rangle$ becomes

$$(\Delta V)^2 = \tfrac{1}{2} e^{-2\mu} \tag{33}$$

and

$$\langle (\delta\theta_{\mathrm{d}})^2 \rangle = \frac{1}{2r^2} e^{-2\mu}. \tag{34}$$

From equation (34) we clearly see that as μ increases the variance in the difference phase decreases. In fact, because $\mu = 0$ corresponds to coherent states, for $\mu > 0$ there is less noise in the difference phase than there is for coherent states, and we have difference phase squeezing. Therefore,amplitude squeezing in the input can lead to phase difference squeezing in the output.

5. Number state inputs

A product of number states with an equal number of photons in each mode is a state with no intensity fluctuations and equal intensities in both modes. A superficial examination of our classical argument would suggest that such an input state would produce an output state with a very well defined difference phase. A more careful analysis leads to somewhat different conclusions. Because the phase of a number state is arbitrary the factor $\cos(\theta_1 - \theta_2)$ in (2) can be either positive or negative with number state inputs. This suggests that, according to the arguments in section 2, the phase of the output state can be either 0 or π. In fact, we shall find that the output state is a superposition of a state with a phase of 0 and another with a phase of π.

In order to perform this calculation it is useful to introduce the analogy between beamsplitters and rotations which has been introduced by a number of authors [1, 6]. This is done by using the mode operators a_1 and a_2 to construct the Schwinger representation of the angular momentum operators

$$J_1 = (a_1 a_2^\dagger + a_1^\dagger a_2)/2 \qquad J_2 = \mathrm{i}(a_1 a_2^\dagger - a_1^\dagger a_2)/2 \qquad J_3 = (a_1^\dagger a_1 - a_2^\dagger a_2)/2. \tag{35}$$

The operator U_1 which describes the action of our beamsplitter can be expressed as

$$U_1 = \mathrm{e}^{-\mathrm{i}\pi J_1/2}. \tag{36}$$

The number state $|n_1\rangle \otimes |n_2\rangle$ corresponds to the angular momentum state $|j, m\rangle$ where

$$j = (n_1 + n_2)/2 \qquad m = (n_1 - n_2)/2. \tag{37}$$

This state is an eigenstate of the total angular momentum $\vec{J}^2$ with eigenvalue $j(j+1)$, and of J_3 with eigenvalue m.

Because we are interested in states for which $n_1 = n_2$ we shall consider the input state to be $|j, 0\rangle$ where j is an integer. The action of the beamsplitter on this state is given by

$$U_1|j, 0\rangle = \sum_{m'=-j}^{j} D^j_{m',0}|j, m'\rangle \tag{38}$$

where $D^j_{m_1,m_2}$ is the matrix for the rotation $\exp(-\mathrm{i}\pi J_1/2)$ in the representation of $SU(2)$ corresponding to j. This rotation can also be expressed as

$$U_1|j, 0\rangle = \sum_{m'=-j}^{j} \mathrm{e}^{\mathrm{i}m'\pi/2} d^j_{m',0}(\pi/2)|j, m'\rangle \tag{39}$$

where $d^j_{m_1,m_2}(\pi/2)$ is the Wigner rotation matrix for the representation labelled by j. For integer values of j we have the relation

$$d^j_{m,0}(\theta) = \sqrt{\frac{4\pi}{2j+1}} Y_{j,m}(\theta, 0) \tag{40}$$

where $Y_{j,m}(\theta, \phi)$ is a spherical harmonic. Setting $\theta = \pi/2$ and making use of the definition of $Y_{j,m}$ in terms of associated Legendre functions, $P_j^m(x)$, we have [9]

$$\begin{aligned} d^j_{m,0}(\pi/2) &= \sqrt{\frac{(j-m)!}{(j+m)!}}\; P_j^m(0) \\ &= 2^{-j}\cos[\pi(j+m)/2] \frac{\sqrt{(j+m)!(j-m)!}}{((j+m)/2)!((j-m)/2)!}. \end{aligned} \tag{41}$$

Let us define

$$\begin{aligned} c_m &= 2^{-j} \frac{\sqrt{(j+m)!(j-m)!}}{((j+m)/2)!((j-m)/2)!} \\ &\cong \sqrt{\frac{2}{\pi}} \frac{1}{(j^2 - m^2)^{1/4}} \end{aligned} \tag{42}$$

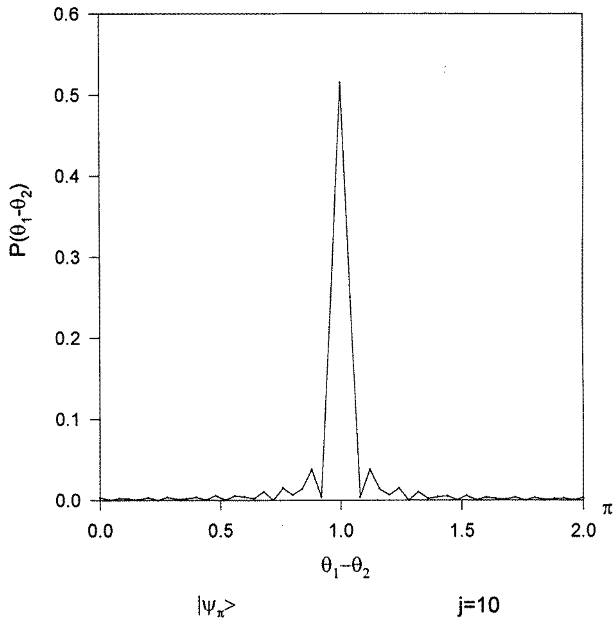

Figure 2. Difference-phase distribution for the state $|\psi_\pi\rangle$ for $j = 10$. Note that the distribution is sharply peaked.

where the approximate form, which is valid for m not too close to j, comes from applying the Stirling approximation to the factorials. Finally, we have for the output state

$$U_1|j,0\rangle = \sum_{m'=-j}^{j} \mathrm{i}^{m'} \cos[\pi(j+m')/2] c_{m'}|j,m'\rangle. \tag{43}$$

To be specific let us consider the case when j is even. The case j odd is similar. If j is even, the cosine factor in (43) is zero for m' odd. This means that the output state can be expressed as

$$U_1|j,0\rangle = \frac{1}{N_0}(|\psi_0\rangle + |\psi_\pi\rangle) \tag{44}$$

where

$$|\psi_0\rangle = \frac{N_0}{2}\cos(\pi j/2) \sum_{m'=-j}^{j} c_{m'}|j,m'\rangle$$

$$|\psi_\pi\rangle = \frac{N_0}{2}\cos(\pi j/2) \sum_{m'=-j}^{j} (-1)^{m'} c_{m'}|j,m'\rangle \tag{45}$$

and N_0 is a constant which guarantees that $|\psi_0\rangle$ and $|\psi_\pi\rangle$ are normalized to one. The state $|\psi_0\rangle$ has a phase distribution which has a peak at $\theta_{\mathrm{d}} = 0$ and $|\psi_\pi\rangle$ has a distribution with a peak at $\theta_{\mathrm{d}} = \pi$. The difference phase distribution of $|\psi_\pi\rangle$ is plotted in figure 2 for the case $j = 10$.

The rotational width in the difference phase of each of these states can be estimated from the expression in (45). The rotational width is derived from examining the overlap between a state and a rotated version of itself [10–12]. It gives a good indication of the utility of a state for measuring phase shifts. In the case of the difference phase we consider the function

$$F(\theta_{\mathrm{d}}) = |\langle\psi|\mathrm{e}^{-\mathrm{i}\theta_{\mathrm{d}}J_3}\psi\rangle|^2. \tag{46}$$

This function has a maximum value of one for $\theta_{\mathrm{d}} = 0$ and near this peak its behaviour is given by

$$F(\theta_{\mathrm{d}}) \cong 1 - (\Delta J_3)^2\theta_{\mathrm{d}}^2. \tag{47}$$

Consequently, we can obtain an estimate of the width of the maximum, which is the rotational width of the state, by finding $1/(\Delta J_3)$. For $|\psi_0\rangle$ we have

$$\langle\psi_0|J_3|\psi_0\rangle = \frac{N_0^2}{4}\sum_{m=-j}^{j}|c_m|^2 m = 0 \tag{48}$$

$$\langle\psi_0|J_3^2|\psi_0\rangle = \frac{N_0^2}{4}\sum_{m=-j}^{j}|c_m|^2 m^2$$

$$\cong \frac{1}{\pi}\int_{-j}^{j}\mathrm{d}m\,\frac{m^2}{(j^2-m^2)^{1/2}} = \frac{j^2}{2} \tag{49}$$

where in (49) we have replaced the sum by an integral and used the approximate form for c_m given in (42). We see, then, that the rotational width of $|\psi_0\rangle$ is approximately

$$\frac{1}{\Delta J_3} \cong \frac{\sqrt{2}}{j} \tag{50}$$

i.e. the rotational width goes like the inverse of the number of photons in the state. This is a very narrow width. A state consisting of a coherent state in each mode, both of which have the same average number of photons, has a width in its difference phase which goes like the inverse of the square root of the number of photons in the state. Therefore, our number state input produces an output state which is the superposition of two states each of which has a very well defined difference phase.

6. Correlated inputs

As was mentioned in section 2 another method of reducing phase-difference fluctuations at the output of a beamsplitter is to introduce correlations into the amplitudes of the two input beams. Here we shall consider a specific example of this. For our input state we choose a displaced two-mode squeezed state

$$|\Psi_{\mathrm{in}}\rangle = D_1(r)D_2(r)S_{12}(\xi)|0\rangle_1 \otimes |0\rangle_2 \tag{51}$$

where $S_{12}(\xi) = \exp(\xi a_1^\dagger a_2^\dagger - \xi^* a_1 a_2)$, and we shall assume that ξ is real and positive. We first need to show that this state has small fluctuations in the difference of the amplitudes of the two modes, and that, when sent through a beamsplitter, it produces an output whose difference-phase noise is squeezed.

Going back to our classical analysis we note that if the two modes have the same mean amplitude, i.e. $\langle r_1\rangle = \langle r_2\rangle = r$, then

$$r_1^2 - r_2^2 = 2r(\delta r_1 - \delta r_2) \tag{52}$$

so that

$$\langle(\delta r_1 - \delta r_2)^2\rangle = \frac{1}{4r^2}\langle(r_1^2 - r_2^2)^2\rangle. \tag{53}$$

The classical quantities r_1^2 and r_2^2 correspond to the quantum mechanical operators $a_1^\dagger a_1$ and $a_2^\dagger a_2$, respectively. Therefore, for a quantum state whose mean amplitude is much larger than the size of its fluctuations, we can set

$$\langle(\delta r_1 - \delta r_2)^2\rangle = \frac{1}{r^2}\langle J_3^2\rangle. \tag{54}$$

If we evaluate the right-hand side for $|\Psi_{\mathrm{in}}\rangle$, we find

$$\langle(\delta r_1 - \delta r_2)^2\rangle = \tfrac{1}{2}\mathrm{e}^{-2\xi} \tag{55}$$

so that the fluctuations in the difference of the amplitudes are squeezed.

We now send $|\Psi_{\mathrm{in}}\rangle$ into the beamsplitter and find the fluctuations in the difference phase at the output using the same method as in section 4. The state emerging from the beamsplitter is

$$|\Psi_{\mathrm{out}}\rangle = D_1(r\mathrm{e}^{-\mathrm{i}\pi/4})D_2(r\mathrm{e}^{\mathrm{i}\pi/4})S_1(-\mathrm{i}\xi)S_2(-\mathrm{i}\xi)|0\rangle_1 \otimes |0\rangle_2 \tag{56}$$

where $S_j(z) = \exp[(z(a_j^\dagger)^2 - z^* a_j^2)/2]$, and $j = 1, 2$, is the single-mode squeezing operator. Our result for $|\Psi_{\mathrm{out}}\rangle$ implies that the choice of angles in computing V is $\phi_1 = 3\pi/4$ and $\phi_2 = -\pi/4$. For the difference-phase fluctuations of our output state we find

$$\langle(\delta\theta_{\mathrm{d}})^2\rangle = \frac{1}{2r^2}\mathrm{e}^{-2\xi} \tag{57}$$

i.e. they are squeezed. Therefore, our classical analysis has again led us to a correct quantum mechanical conclusion: a beamsplitter input with equal mean amplitudes and correlated amplitude fluctuations will produce an output state with small difference-phase fluctuations.

7. Conclusion

By pursuing a classical analogy we have found that it is possible to use a beamsplitter to convert two-mode light with reduced amplitude fluctuations into light with reduced fluctuations in the phase difference between the modes. An interferometer detects this phase difference so that by reducing fluctuations in this quantity more accurate interferometric measurements become possible. Amplitude squeezing is generally easier to produce than phase squeezing so that the fact that a beamsplitter allows conversion from one kind to the other is useful. For example, constant-current-driven semiconductor lasers can produce amplitude squeezed light [13]. This suggests that two injection-locked semiconductor lasers, with equal intensities, sent into a beamsplitter would produce phase-difference squeezed light. The injection locking is necessary to control the phase noise so that we only pick up the phase-difference point at zero and not the one at π.

A Mach–Zehnder interferometer consists of two beamsplitters and two additional mirrors. The analysis we have presented here applies to the first of these beamsplitters. We have seen how to produce states with a well defined difference phase, which are suitable for measuring a phase shift, as well as states with two sharp peaks in their difference-phase distribution. The second beamsplitter in the interferometer converts the phase information back into photon-number information. One would like to understand this process more completely and to ask whether it is the optimal way to detect the phase information in the state between the beamsplitters. Phase-shift information can even be obtained when the state inside the interferometer has more than one sharp peak in its difference-phase distribution, as was shown by Holland and Burnett [7]. All of this suggests that a quantum phase analysis of a Mach–Zehnder interferometer would be useful both in gaining a better understanding of how it works and in finding ways to improve its accuracy.

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