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Sampling entropies and operational phase-space measurement. II. Detection of quantum coherences

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We use the operational phase-space distributions and sampling entropies developed in the preceding paper [V. Bužek, C.H. Keitel, and P.L Knight, Phys. Rev. A **51**, 2575 (1995)] to discuss the nature of quantum interference between components of superpositions of states. We show how the Wehrl entropy, a special case of the sampling entropy, is a useful discriminator between different kinds of superpositions and of statistical mixtures, and is determined essentially by the coherent-state content. Apart from interference terms, this content is given by the quantum uncertainty of a single coherent state and the classical contribution of the number of coherent states necessary to tile the dominant phase-space "patch" representing the quantum state of interest. We illustrate these ideas using nonclassical superpositions of coherent states, where interference modifies the phase-space distributions, and show how these features are sensitive to dissipation.

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I. INTRODUCTION

The phase-space description of a state of a classical system is based on the definition of a probability (or a classical probability density) that a classical system is found at a point (q,p) in phase space. In classical physics it is considered that a point in phase space can be located with arbitrary accuracy and that two distinct points in phase space are "independent" (see below). In quantum mechanics the situation is radically different. First of all, a "point" in the quantum-mechanical phase space cannot be localized precisely. There is always a fundamental limit with which the point of the quantum-mechanical phase space can be determined (see [1] to which we will refer as to paper I). It was shown by Schrödinger [2] that coherent states (see also [3]) represent the best approximation to points in the quantum-mechanical phase space. In other words, coherent states form a position-momentum phase-space patch of minimum uncertainty area and may be regarded as the quantum analog of the classical point in phase space. The second fundamental difference between the classical and quantum phase space is that points of the quantum-mechanical phase space can interfere whereas this is impossible classically. This quantum-mechanical interference among coherent states gives rise to *all* nonclassical properties of quantum systems (in particular light fields modeled as harmonic oscillators) [4].

In paper I, we employed the idea of an operational measurement, introduced by Wódkiewicz [5] (see also a review article by Stenholm [6] on phase-space measurements and references therein), to construct sampling entropies that are determined in part by the quantum state of interest, and in part by the "quantum ruler" used to make the measurement. When the ruler states are coherent states, the relevant sampling entropy is the Wehrl entropy [7].

The main purpose of this paper is to analyze whether an operational phase-space measurement as described in paper I reveals the character of quantum interference between coherent states, i.e., between the points of the quantum-mechanical phase space. In particular, we will discuss in detail whether the Wehrl entropy can convey useful information about the quantum interference between coherent states.

II. WEHRL ENTROPY OF SUPERPOSITIONS OF A FINITE NUMBER OF COHERENT STATES

Recently, many authors have studied various nonlinear processes in which optical superpositions of coherent states can be produced in principle (for a detailed discussion see [4] and references therein). In particular, it has been shown by Yurke and Stoler [8] that, in the presence of low dissipation, a nonlinear Kerr-like medium may convert an initial coherent state $|\alpha\rangle$ into a quantum-mechanical superposition of two coherent states that are 180° out of phase with respect to each other. This superposition state is described by the state vector

$$|\alpha\rangle_{\rm YS} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle + e^{i\pi/2}|-\alpha\rangle\right). \qquad (2.1a)$$

Another scheme for production of pure superpositions of coherent states has been proposed by Brune *et al.* [9]. These authors have shown that an atomic-phase detection quantum nondemolition scheme can serve to produce superposition states of a single-mode cavity field. These superposition states are described by the following state vectors:

$$|\alpha\rangle_{\rm even} = \frac{1}{\sqrt{2[1+\exp(-2|\alpha|^2)]}}\left(|\alpha\rangle + |-\alpha\rangle\right). \quad (2.1b)$$

and

$$|\alpha\rangle_{\rm odd} = \frac{1}{\sqrt{2[1-\exp(-2|\alpha|^2)]}}\left(|\alpha\rangle - |-\alpha\rangle\right). \quad (2.1c)$$

The state $|\alpha\rangle_{\rm even}$ is usually called the even coherent state, while the state $|\alpha\rangle_{\rm odd}$ is called the odd coherent state. The even (odd) coherent state in the Fock basis is represented only by even (odd) Fock states. In other words the photon number distribution corresponding to the even (odd) coherent state exhibits significant oscillations that have their origin in quantum interference among the components of the pure superposition state (2.1). Another consequence of this quantum interference is the reduction of quadrature fluctuations in the even coherent state and in the Yurke-Stoler coherent state (these states exhibit second-order squeezing as well as fourth-order squeezing [4]). On the other hand, the odd coherent state exhibits reduced fluctuations in the photon number, i.e., the odd coherent state is a sub-Poissonian state. Nonclassical effects that emerge as a direct consequence of quantum interference in phase space have been discussed in dètail previously (see [4] and references therein). The character of these nonclassical effects is intrinsically related to the character of quantum interference between coherent components of pure superposition states and depends on the relative phase φ defined as

$$|\alpha\rangle_{\varphi} = \frac{1}{\sqrt{2[1+\cos\varphi\exp(-2|\alpha|^2)]}}\left(|\alpha\rangle + e^{i\varphi}|-\alpha\rangle\right). \quad (2.2)$$

We see that for $\varphi = 0, \pi/2$, and $\varphi = \pi$ we obtain from Eq. (2.2) expressions for the even, the Yurke-Stoler, and the odd coherent states, respectively. Before discussing in detail whether the character of quantum interference between two coherent components of the superposition state can be reflected by the Wehrl entropy [i.e., whether the Wehrl entropy of the superposition state (2.2) depends on the relative phase φ] we present a general formalism for the phase-space description of a pure superposition of a finite number of coherent states.

A. *Q* function of superpositions of coherent states

Let us consider the superposition $|\Psi\rangle$ of coherent states $|\alpha_j\rangle$ given by the relation

$$|\Psi\rangle = A^{1/2}\left[\sum_{j=1}^{N} e^{i\varphi_j}|\alpha_j\rangle\right], \qquad \alpha_j = \alpha_j^r + i\alpha_j^i, \quad (2.3a)$$

where $A^{1/2}$ is the normalization constant. The phases φ_j are arbitrary and their values determine whether the quantum interference among the coherent states $|\alpha_j\rangle$ is constructive or destructive [4]. We note this determines observable effects such as squeezing or sub-Poissonian photon statistics.

The normalization constant $A^{1/2}$ can be written as

$$A = \left\{N + 2\sum_{\substack{j,k=1\\ k>j}} \exp\left[-\frac{1}{2}|\alpha_j - \alpha_k|^2\right]\cos\left[(\varphi_k - \varphi_j) + \alpha_j \otimes \alpha_k\right]\right\}^{-1}, \quad (2.3b)$$

where we have used the notation $\otimes$ for the antisymmetric product of two two-dimensional vectors (α_j^r, α_k^i) and (α_k^r, α_k^i),

$$\alpha_j \otimes \alpha_k \equiv \alpha_j^r\alpha_k^i - \alpha_j^i\alpha_k^r. \quad (2.3c)$$

Using the standard procedures discussed in paper I we find the Wigner function $W(\beta;\alpha_1,\ldots,\alpha_N)$ of the superposition state (2.3a) in the form

$$W(\xi;\alpha_1,\ldots,\alpha_N) = A\left\{\sum_{j=1}^{N} W_j(\xi;\alpha_j) + 2\sum_{\substack{j,k=1\\ j>k}}^{N} W_{jk}(\xi;\alpha_j,\alpha_k)\right\}, \quad (2.4)$$

where $W_j(\xi;\alpha)$ is the Wigner function corresponding to the coherent state $|\alpha_j\rangle$,

$$W_j(\xi;\alpha_j) = 2\exp(-2|\xi - \alpha_j|^2), \quad (2.5)$$

and $W_{jk}(\xi;\alpha_j,\alpha_k)$ is the quasiprobability density distribution emerging from the quantum interference between the coherent states $\exp(i\varphi_j)|\alpha_j\rangle$ and $\exp(i\varphi_k)|\alpha_k\rangle$,

$$W_{jk}(\xi;\alpha_j,\alpha_k) = 2\exp\left[-2\left|\xi - \frac{\alpha_j+\alpha_k}{2}\right|^2\right] \times \cos[\varphi_k - \varphi_j + 2\xi\otimes\alpha_k - 2\xi\otimes\alpha_j + \alpha_k\otimes\alpha_j]. \quad (2.6)$$

The oscillatory behavior of the interference part given by Eq. (2.6) of the Wigner function plays a crucial role in the appearance of nonclassical effects. For comparison purposes we determine the Wigner function of the statistical mixture described by the density operator

$$\hat{\rho} = \sum_{j=1}^{N} p_j|\alpha_j\rangle\langle\alpha_j|, \qquad \sum_{j=1}^{N} p_j = 1. \quad (2.7)$$

This Wigner function can be written as

$$W(\xi;\alpha_1,\ldots,\alpha_N) = \sum_{j=1}^{N} p_j W_j(\xi,\alpha_j), \quad (2.8)$$

where functions $W(\xi;\alpha_j)$ are given by Eq. (2.5). The Wigner function (2.8) does not contain a quantum-interference term and does not describe nonclassical effects.

Performing a convolution of the Wigner function (2.6) of the superposition state $|\psi\rangle$ with the Wigner function of the quantum ruler which is in a coherent state (i.e., $|\varphi\rangle = |\beta\rangle$) with the complex amplitude $\beta = \beta_r + i\beta_i$ we find the Q function of the superposition state under consideration,

$$Q(\beta;\alpha_1,...,\alpha_N) = A\left\{\sum_{j=1}^{N} Q_j(\beta;\alpha_j) + 2\sum_{\substack{j,k=1\\ j>k}}^{N} Q_{jk}(\beta;\alpha_j,\alpha_k)\right\}, \quad (2.9)$$

where $Q_j(\beta;\alpha)$ is the Glauber Q function corresponding to the coherent state $|\alpha_j\rangle$,

$$Q_j(\beta;\alpha_j) = \exp(-|\beta-\alpha_j|^2), \quad (2.10)$$

and $Q_{jk}(\beta;\alpha_j,\alpha_k)$ is the probability density distribution emerging from the quantum interference between the coherent states $\exp(i\varphi_j)|\alpha_j\rangle$ and $\exp(i\varphi_k)|\alpha_k\rangle$,

$$Q_{jk}(\beta;\alpha_j,\alpha_k) = \exp\left[-\frac{|\beta-\alpha_j|^2}{2} - \frac{|\beta-\alpha_k|^2}{2}\right] \times \cos[(\varphi_j-\varphi_k) + \beta\otimes(\alpha_j-\alpha_k)]. \quad (2.11)$$

It is interesting to note at this point that in the limit $|\alpha_j - \alpha_k| \gg 1$ for any $j \neq k$ (i.e., when all component states $|\alpha_j\rangle$ are "far away" from each other), the Q function (2.9) of the superposition state (2.3a) is equal to the Q function of the *mixture* described by the density operator (2.7) with $p_j = 1/N$. This means that for large values of $|\alpha_j - \alpha_k|$ the quantum interference terms in the Q function are completely suppressed and one cannot distinguish between the pure state (2.3a) and the corresponding statistical mixture. This deterioration of quantum interference terms is exclusively due to the quantum measurement process (here represented via the filtering with coherent states). If we turn our attention back to the Wigner function we find that the interference terms $W_{jk}(\xi;\alpha_j,\alpha_k)$ are not suppressed in the limit $|\alpha_j-\alpha_k| \gg 1$. We see that the phase-space measurement (which explicitly includes an account of the measurement through the "ruler states") leads to the deterioration of quantum interference terms and consequently suppression of nonclassical effects, and this can, in principle, be observed.

To illustrate the difference between the Wigner function and the Q function we present these two functions for the even coherent state (2.1b) with real amplitudes α of component states. The Wigner function of the even coherent state reads

$$W(\xi,\alpha,-\alpha) = \frac{1}{2[1+\exp(-2\alpha^2)]}\left\{W(\xi,\alpha) + W(\xi,-\alpha) + W_{\rm int}(\xi,\alpha,-\alpha)\right\}, \quad (2.12a)$$

where the Wigner functions $W(\xi,\pm\alpha)$ of coherent states $|\pm\alpha\rangle$ are given by Eq. (2.5) and the interference part $W_{\rm int}(\xi,\alpha,-\alpha)$ of the total Wigner function reads

$$W_{\rm int}(\xi,\alpha,-\alpha) = 4\exp(-2|\xi|^2)\cos(4\xi_i\alpha). \quad (2.12b)$$

The peak-to-peak ratio parameteh $R_W(\alpha)$ defined as

$$R_W(\alpha) = \frac{W_{\rm int}(\xi,\alpha,-\alpha)|_{\rm peak}}{2\left[W(\xi,\alpha)|_{\rm peak}W(\xi,-\alpha)|_{\rm peak}\right]^{1/2}}, \quad (2.13)$$

measuring the relative heights between the interference part and component parts of the Wigner function is for the even coherent state equal to unity, that is $R_W(\alpha) = 1$ irrespectively on the amplitude α of the component states.

The Q function of the even coherent state under consideration is given by the relation

$$Q(\beta,\alpha,-\alpha) = \frac{1}{2[1+\exp(-2\alpha^2)]}\left\{Q(\beta,\alpha) + Q(\beta,-\alpha) + Q_{\rm int}(\beta,\alpha,-\alpha)\right\}, \quad (2.14a)$$

where the Q functions $Q(\beta,\pm\alpha)$ of coherent states $|\pm\alpha\rangle$ are given by Eq. (2.10) and the interference part $Q_{\rm int}(\beta,\alpha,-\alpha)$ of the total Q function reads

$$Q_{\rm int}(\beta,\alpha,-\alpha) = 2\exp(-|\beta|^2-\alpha^2)\cos(2\beta_i\alpha). \quad (2.14b)$$

The peak-to-peak ratio parameter $R_Q(\alpha)$ for the Q function is defined in an analogous way as for the Wigner function, i.e.,

$$R_Q(\alpha) = \frac{Q_{\rm int}(\beta,\alpha,-\alpha)|_{\rm peak}}{2\left[Q(\beta,\alpha)|_{\rm peak}Q(\beta,-\alpha)|_{\rm peak}\right]^{1/2}}, \quad (2.15)$$

and its explicit expression for the considered even coherent state reads

$$R_Q(\alpha) = \exp(-\alpha^2). \quad (2.16)$$

From the above we can conclude that with the increase of the amplitude of the component states of the even coherent state, the interference part of the Q function (i.e., the operational probability density distribution measured with the help of a filter in a coherent state) becomes significantly reduced.

B. Wehrl entropy of mixture state

Here we turn our attention to the Wehrl entropy of a statistical mixture of N coherent states described by the density operator (2.7) with probabilities $p_j = 1/N$. This Wehrl entropy reads

$$S = k_B + k_B\ln N - \frac{k_B}{N}\sum_k\int d^2\beta\, e^{-|\beta|^2} \times \ln\left\{1 + \sum_{j\neq k}^{N}\exp[|\beta|^2 - |\beta-\alpha_j+\alpha_k|^2]\right\}, \quad (2.17)$$

and in the limit $|\alpha_j - \alpha_k| \gg 1$ it reaches its maximum value $S^{\max}$,

$$S^{\max} = S_{\rm QM} + S_B, \quad S_{QM} = k_B, \quad \text{and} \quad S_B = k_B \ln N. \tag{2.18}$$

We have said earlier that in the limit $|\alpha_j - \alpha_k| \gg 1$ the Q function (2.9) of the pure superposition state (2.3) is equal to the Q function of the corresponding mixture. Consequently, in this limit the Wehrl entropy of the pure state is equal to the Wehrl entropy of the corresponding mixture and is equal to (2.18). This equation has a rather appealing physical interpretation. The term $S_{\rm QM} = k_B$ arises as a consequence of the operational phase-space measurement and reflect the overcompleteness of the coherent-state basis (for details see paper I). The term $S_B = k_B \ln N$ is equal to the *Boltzmann* entropy and it reflects the fact that in the limit $|\alpha_j - \alpha_k| \gg 1$, the coherent states under consideration can be considered as orthogonal and, therefore, the density operator $\hat{\rho} = N^{-1}\sum_j |\alpha_j\rangle\langle\alpha_j|$ describes a state in an n-dimensional state space $\mathcal{U}_n$ composed of these N mutually "orthogonal" coherent states. Because each of these states is realized with the probability $1/N$, the corresponding Boltzmann entropy is equal to $k_B \ln N$.

C. Wehrl entropy and character of quantum interference

Now we turn our attention back to superpositions of just two coherent states given by the state vector (2.2). We will assume the amplitude α of component states to be real and small enough so that quantum-interference effects play a significant role. We will study the dependence of the Wehrl entropy on the relative phase φ, i.e., we will analyze the relation between the character of quantum interference and the value of the Wehrl entropy.

The Q function of the superposition state (2.2) is given by the relation

$$Q(\beta,\alpha,-\alpha) = \frac{1}{2[1+\cos\varphi\exp(-2\alpha^2)]}\Big\{Q(\beta,\alpha) + Q(\beta,-\alpha) + Q_{\rm int}(\beta,\alpha,-\alpha)\Big\}, \tag{2.19a}$$

where the Q functions $Q(\beta,\pm\alpha)$ of coherent states $|\pm\alpha\rangle$ are given by Eq. (2.10) and the interference part $Q_{\rm int}(\beta,\alpha,-\alpha)$ of the total Q function reads

$$Q_{\rm int}(\beta,\alpha,-\alpha) = 2\exp(-|\beta|^2 - \alpha^2)\cos(\varphi + 2\beta_i\alpha). \tag{2.19b}$$

In Fig. 1 we plot the Wehrl entropy evaluated from the Q function (2.19) as a function of the relative phase φ between the component state $|\alpha\rangle$ and $|-\alpha\rangle$. For comparison purposes we also present the value (the dashed line in Fig. 1) of the Wehrl entropy of the corresponding mixture state. From Fig. 1 it is clearly seen that the Wehrl entropy is sensitive to the character of the quantum interference between coherent states. Depending on the character of the quantum interference the value of the Wehrl entropy can be either higher or lower than its value for the corresponding mixture state. If the relative phase φ is equal to zero then the Wehrl entropy takes its minimum value (for a given α), while for $\varphi = \pi$ the entropy takes its maximum value. It is interesting to notice that the Wehrl entropy of a pure state can be larger than the entropy of a corresponding mixture state. In other words quantum interference can act in a "destructive" way and to increase the Wehrl entropy of a pure state over the entropy value of the corresponding mixture state. The mean photon number of the superposition state (2.2) is also sensitive with respect to the character of quantum interference and it depends on the relative phase φ as

$$\bar{n}(\varphi) = {}_\varphi\langle\alpha|\hat{n}|\alpha\rangle_\varphi = \alpha^2\frac{1-\cos\varphi\exp(-2\alpha^2)}{1+\cos\varphi\exp(-2\alpha^2)}. \tag{2.20}$$

From Eq. (2.20) it follows that when the Wehrl entropy increases due to the quantum interference, so does the mean photon number of a corresponding superposition state. In particular, the mean photon number of the odd coherent state ($\varphi = \pi$) is larger than the mean photon number of the mixture state equal to α^2.

We have to stress that the sensitivity of the Wehrl entropy with respect to the character of the quantum interference can only be seen for small amplitudes of coherent components of superposition states. In Fig. 2 we plot the Wehrl entropy as a function of the parameter α^2 for the even coherent state (short-dashed line); for the odd coherent state (long-dashed line) and the corresponding mixture (solid line). We note that for $0 < \alpha^2 \approx 3$ the Wehrl entropy of the odd (even) coherent state has values larger (smaller) than the value of the Wehrl entropy of the corresponding statistical mixture. On the other hand, for the values of the parameter α such that $\alpha^2 > 4$ all three lines basically coincide. From Fig. 2 we can conclude that quantum effects that emerge as a consequence of quantum interference in phase space can be detected

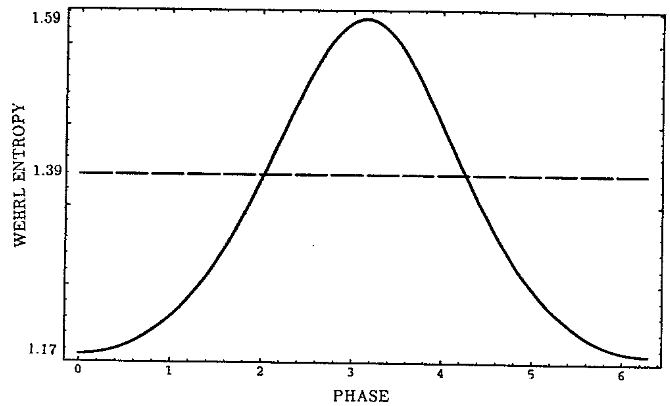

FIG. 1. The Wehrl entropy of the superposition state (2.2) as a function of the relative phase φ. The real amplitude α of the coherent component states $|\alpha\rangle$ and $|-\alpha\rangle$ is equal to 0.8. The dashed line corresponds to the value of the Wehrl entropy of the mixture state with the same amplitude α (we use units such that $k_B = 1$).

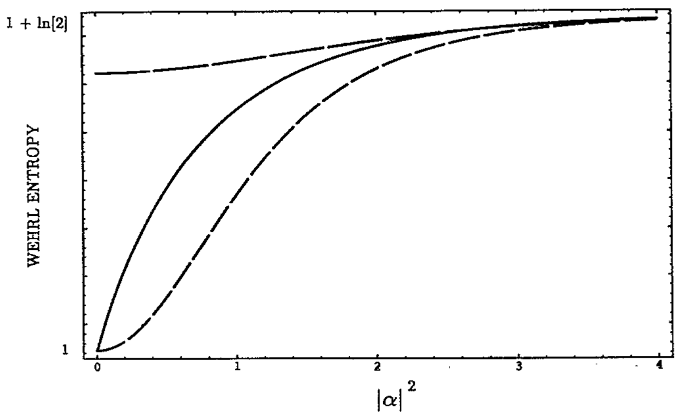

FIG. 2. The Wehrl entropy as a function of the parameter $|\alpha|^2$ for the even coherent state (short-dashed line); for the odd coherent state (long-dashed line) and the corresponding mixture (solid line); we use units such that $k_B = 1$.

via the operational phase-space measurement only providing the amplitudes of coherent components are relatively small. In the limit $\alpha^2 \to 0$ the Wehrl entropy of the even coherent state is equal to unity (i.e., the value of the Wehrl entropy of the vacuum state). Simultaneously, in the same limit $\alpha^2 \to 0$, we note of course that the Wehrl entropy of the odd coherent state is greater than that of the mixture and approaches $1 + C$ (here C is the Euler constant) which is the value of the Wehrl entropy of the Fock state $|1\rangle$ [10].

D. Wehrl entropy and phase-space uncertainty

The Q function of pure coherent states is given by the relation

$$Q(q,p) = \frac{1}{\Sigma_p \Sigma_q} \exp\left[-\frac{1}{2\hbar}\frac{(q-\bar{q})^2}{\Sigma_q^2} - \frac{1}{2\hbar}\frac{(p-\bar{p})^2}{\Sigma_p^2}\right], \tag{2.21a}$$

where $\Sigma_q^2 = 1/2 + \sigma_q^2$ and $\Sigma_p^2 = 1/2 + \sigma_p^2$. The corresponding Wehrl entropy reads

$$S_{\Psi\Phi}^{(p,q)} = k_B + k_B \ln(\Sigma_p \Sigma_q) \ . \tag{2.21b}$$

The parameter $A = \Sigma_q \Sigma_p$ can be interpreted as an area (in units of $2\pi\hbar$) of the phase space "covered" by the Q function at $1/e$ of its maximum value. From this point of view we can relate the Wehrl entropy of Gaussian states to the specifically defined phase-space uncertainty area A of the phase space. Because the area A in units of $2\pi\hbar$ for the Q function of a coherent state is equal to unity, therefore, $\ln A = 0$ and so the corresponding Wehrl entropy takes the minimum value equal to k_B. For a superposition (or a mixture) of N coherent states "well-separated" in the phase space, the total uncertainty area A in units of $2\pi\hbar$ is equal to N, so the Wehrl entropy is equal to $k_B + k_B \ln N$. This relation between the uncertainty area and the Wehrl entropy is not exact for non-Gaussian states. Nevertheless qualitatively this relation holds. That is, the larger the phase-space uncertainty area the larger the value of the Wehrl entropy (and vice versa). To illustrate this relation we plot in Fig. 3 the Q function of the even coherent state, the odd coherent state, and the corresponding statistical

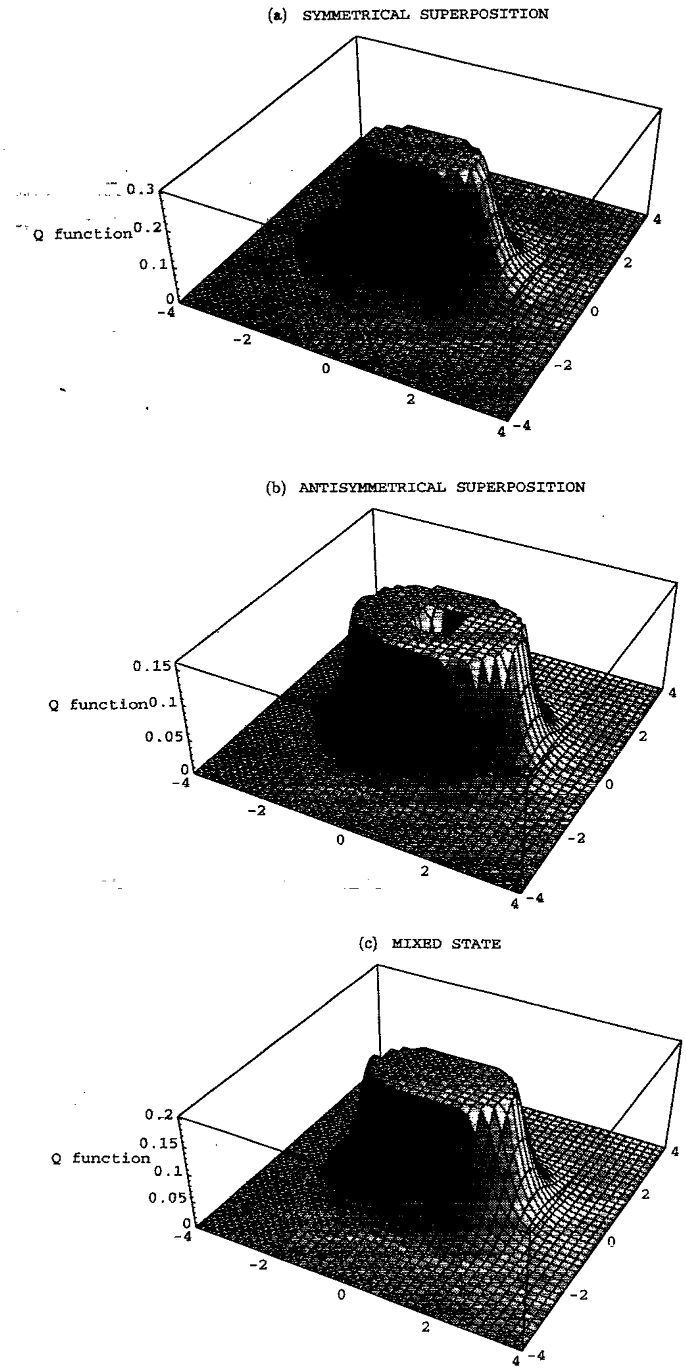

FIG. 3. The Q-function quasiprobabilities in phase space for the cases considered in Fig. 2 where the real amplitude α of the coherent component states $|\alpha\rangle$ and $|-\alpha\rangle$ is equal to 0.8: (a) symmetrical superposition, (b) anti-symmetrical superposition, and (c) mixed coherent state. The tops of the Q functions are removed at heights $1/e$ of the maximal values to show the corresponding uncertainties in phase space. The absolute value squared of the coherent state parameter α is assumed to be unity.

mixture. These three Q functions are sliced at $1/e$ of their maximal height. From these figures it is easy to estimate the area A. To do so, we note that with the grid used in the figures the unit area $2\pi\hbar$ is represented by approximately 57 phase-space "cells", so the parameters A related to the even coherent state, the odd coherent state, and the statistical mixture read 1.32, 2.84, and 2.09, respectively. Using these values one can evaluate the parameter $k_B + k_B \ln A$ for these three different cases and find reasonably good agreement with corresponding values of the Wehrl entropy.

E. Wehrl entropy of decaying superpositions

All quantum systems interact in some way with an environment. The environment can act either as an attenuator or an amplifier. In what follows we will discuss how the Wehrl entropy of the superposition state (2.2) changes when the quantum system under consideration is decaying into a zero-temperature heat bath. The dynamics of this simplest version of the system-environment interaction is governed by the Fokker-Planck equation for the Q function [11]

$$\frac{\partial Q(\beta,t)}{\partial t} = \gamma\left[\frac{\partial^2}{\partial\beta^*\partial\beta} + \frac{1}{2}\left(\beta^*\frac{\partial}{\partial\beta^*} + \beta\frac{\partial}{\partial\beta}\right)\right] Q(\beta,t), \tag{2.22}$$

where γ is the decay rate. If the initial Q function is given by Eq. (2.19) then using the coarse-graining procedure as described in paper I, we find an explicit expression for the Q function at time t in the form

$$Q(\beta,t) = \frac{1}{2[1+\cos\varphi\exp(-2\alpha^2)]} \times\Big\{Q(\beta,\alpha;t) + Q(\beta,-\alpha;t) + Q_{\rm int}(\beta,\alpha,-\alpha;t)\Big\}, \tag{2.23a}$$

where the Q functions $Q(\beta,\pm\alpha)$ of the decaying coherent states $|\pm\alpha\rangle$ are given by expression

$$Q(\beta,\pm\alpha) = \exp\left[-|\beta \mp \mu^{1/2}\alpha|^2\right]. \tag{2.23b}$$

The parameter μ is defined as usual, i.e., $\mu = \exp(-\gamma t)$. Equation (2.23b) reflects the fact that a coherent state decaying into the zero-temperature heat bath preserves its quantum-statistical properties except its amplitude which decays exponentially. The interference part $Q_{\rm int}(\beta,\alpha,-\alpha;t)$ of the total Q function reads

$$Q_{\rm int}(\beta,\alpha,-\alpha;) = 2\exp[-|\beta|^2 - (2-\mu)\alpha^2] \times\cos(\varphi + 2\beta_i\mu^{1/2}\alpha). \tag{2.23c}$$

Recently many authors have analyzed the influence of dissipative reservoirs on quantum-mechanical superposition states of light. In particular, the influence of damping at zero temperature on quantum coherences has been discussed in detail (see [4,12] and references therein). It has been shown that the decay rate of quantum coherences is proportional to the "distance" in phase space between coherent components of a superposition state. To see this we evaluate the peak-to-peak ratio $R_Q(\alpha;t)$ given by Eq. (2.15) for the Q function of the decaying even coherent state [see Eq. (2.23) with $\varphi = 0$],

$$R_Q(\alpha;t) = R_Q(\alpha;t=0)\exp[-(1-\mu)\alpha^2]. \tag{2.24a}$$

For times such that $\gamma t \ll 1$ we can approximate $R_Q(\alpha;t)$ as follows:

$$R_Q(\alpha;t) \simeq R_Q(\alpha;t=0)\exp[-\gamma\alpha^2 t], \tag{2.24b}$$

from which we directly see that the quantum interference term in the Q function is suppressed at the rate proportional to $\gamma\alpha^2$. Because of this rapid suppression of quantum coherences in the decay process, the Wehrl entropy of the initial superposition state (2.2) approaches rapidly the expression for the Wehrl entropy of the decaying statistical mixture. Moreover, the larger the amplitudes of component states, the faster the initial pure superposition state is transformed into the corresponding statistical mixture. In Fig. 4 we plot the time evolution of the Wehrl entropy of the even coherent state (short-dashed line), the odd coherent state (long-dashed line), and the corresponding statistical mixture (solid line) decaying into the zero-temperature heat bath. We choose the amplitude $\alpha = 0.8$ (because for this value of α the even coherent state exhibits the largest value of the quadrature squeezing). For this value of α the difference between the values of the Wehrl entropy of the three states under consideration is still "visible" even for relatively long times such as $\gamma t = 1$ when a substantial portion of the initial energy of the light mode has been dissipated. From here it follows that for very small initial amplitudes of superposition states nonclassical effects can be observed when the operational phase-space measurement process is used. On the other hand, as soon as the amplitudes become large enough quantum coherences are suppressed almost instantly.

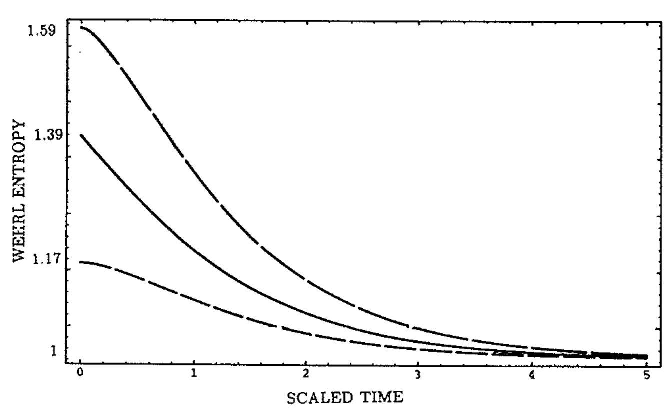

FIG. 4. The time evolution of the Wehrl entropy for the even coherent state (short-dashed line), for the odd coherent state (long-dashed line), and the corresponding mixture (solid line) which decay into the zero-temperature heat bath. The scaled time is measured in units of the decay constant γ (we use units such that $k_B = 1$).

III. WEHRL ENTROPY OF CONTINUOUS SUPERPOSITIONS OF COHERENT STATES

In the previous section we have studied in detail how the Wehrl entropy is sensitive to the character of quantum interference established between a finite number of coherent states $|\alpha_j\rangle$. We have shown that as soon as the "distance" between coherent components is large enough (i.e., if $|\alpha_i - \alpha_j| > 1$) then the interference terms cannot be "detected" by the operational phase-space measurement with the filter being in a coherent state. In other words, the value of the Wehrl entropy of a pure state is equal to the Wehrl entropy of a corresponding mixture. On the other hand, if the coherent states are "close" enough then the Wehrl entropy reflects the character of quantum interference between coherent components. Therefore, we should expect that for continuous superpositions of coherent states nonclassical effects that have their origin in quantum interference can be observed. Consequently, we should expect that the Wehrl entropy for a continuous superposition of coherent states is different from the Wehrl entropy of the corresponding statistical mixture.

To be specific we will consider a squeezed vacuum state $|\eta\rangle$ which in the Fock basis is represented as

$$|\eta\rangle = (1-\eta^2)^{1/4}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{2^n n!}\eta^n|2n\rangle, \tag{3.1}$$

where the squeezing parameter η is assumed to be real and $0 \le \eta \le 1$. The mean photon number of the squeezed vacuum state (3.1) is given by the relation

$$\langle\eta|\hat{n}|\eta\rangle = \frac{\eta^2}{1-\eta^2}, \tag{3.2}$$

and the variances of the position ($\hat{q}$) and the momentum ($\hat{p}$) operators are

$$\langle\eta|(\Delta\hat{q})^2|\eta\rangle = \frac{\hbar}{2}\,\frac{1+\eta}{1-\eta} = \frac{\hbar}{2} + \frac{\hbar\eta}{(1-\eta)} > \frac{\hbar}{2}, \tag{3.3a}$$

$$\langle\eta|(\Delta\hat{p})^2|\eta\rangle = \frac{\hbar}{2}\,\frac{1-\eta}{1+\eta} = \frac{\hbar}{2} - \frac{\hbar\eta}{(1+\eta)} < \frac{\hbar}{2}. \tag{3.3b}$$

From Eq. (3.3) we see that for the particular choice of the phase of the squeezing parameter η fluctuations in the momentum operator $\hat{p}$ are reduced below the vacuum limit $\hbar/2$ at the expense of increased fluctuations in the position operator.

The squeezed vacuum state (3.1) can be represented as a pure one-dimensional continuous superposition of coherent states on a q axis [13], that is

$$|\eta\rangle = (2\pi\eta)^{-1/2}(1-\eta^2)^{1/4}\int_{-\infty}^{\infty}d\alpha \times \exp\left[-\frac{1-\eta}{2\eta}\alpha^2\right]|\alpha\rangle. \tag{3.4a}$$

The density operator of the squeezed vacuum state (3.4) can be expressed as

$$\hat{\rho}_{sq} = \frac{(1-\eta^2)^{1/2}}{2\pi\eta}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta \times \exp\left[-\frac{1-\eta}{2\eta}(\alpha^2+\beta^2)\right]|\alpha\rangle\langle\beta|. \tag{3.4b}$$

The off-diagonal terms $|\alpha\rangle\langle\beta|$ ($\alpha \ne \beta$) in the coherent-state basis correspond to quantum interferences between coherent states in the phase space. The density operator of a statistical mixture corresponding to the squeezed vacuum state (3.4) can be written in the form of the following one-dimensional integral

$$\hat{\rho}_{\rm mix} = \left(\frac{1-\eta}{\pi\eta}\right)^{1/2}\int_{-\infty}^{\infty}d\alpha\,\exp\left[-\frac{1-\eta}{2\eta}\alpha^2\right]|\alpha\rangle\langle\alpha|. \tag{3.5}$$

The variances of the operators $\hat{q}$ and $\hat{p}$ in the statistical mixture (3.5) are

$$\mathrm{Tr}[\hat{\rho}_{\rm mix}(\Delta\hat{q}_1)^2] = \frac{\hbar}{2} + \frac{\hbar\eta}{(1-\eta)}, \tag{3.6a}$$

$$\mathrm{Tr}[\hat{\rho}_{\rm mix}(\Delta\hat{p}_1)^2] = \frac{\hbar}{2}, \tag{3.6b}$$

i.e., the fluctuations in the $\hat{q}$ quadrature are equal to those for the squeezed vacuum state [compare with Eq. (3.3a)], but because there is no interference between coherent components the fluctuations in the $\hat{p}$ quadrature are not squeezed and they are equal to the vacuum-level fluctuations.

The Q function corresponding to the squeezed vacuum state reads

$$Q(\beta) = \frac{1}{\Sigma_r\Sigma_i}\exp\left[-\frac{\beta_r^2}{\Sigma_r^2} - \frac{\beta_i^2}{\Sigma_i^2}\right], \tag{3.7}$$

where

$$\Sigma_r^{-2} = 1-\eta, \qquad \Sigma_i^{-2} = 1+\eta. \tag{3.8}$$

The Q function of the mixture state (3.5) has the same form as for the squeezed vacuum state, except the parameters Σ_r and Σ_i are given by the relations

$$\Sigma_r^{-2} = 1-\eta, \qquad \Sigma_i^{-2} = 1. \tag{3.9}$$

The Wehrl entropy of the squeezed vacuum state (3.4) and the corresponding statistical mixture (3.5) can be found in the form (see paper I)

$$S_{\rm sq} = k_B + k_B\ln\Sigma_r + k_B\ln\Sigma_i \tag{3.10a}$$

and

$$S_{\rm mix} = k_B + k_B\ln\Sigma_r, \tag{3.10b}$$

respectively. Comparing the last two expressions we conclude that the Wehrl entropy of the squeezed vacuum state is *always* (i.e., for any $\eta > 0$) smaller than the entropy of the corresponding statistical mixture,

$$S_{sq} - S_{\rm mix} = k_B \ln \Sigma_i = -\frac{k_B}{2} \ln(1+\eta) < 0. \quad (3.11)$$

In the limit of infinite squeezing (when $\eta \to 1$) the difference between the two entropies takes its maximum value which is equal to $(-k_B \ln \sqrt{2})$.

IV. CONCLUSIONS

We can conclude that by performing an operational phase-space measurement over a nonclassical state that can be represented as a pure continuous superposition of coherent states, we always can distinguish between this superposition state and the corresponding mixture state. This property of continuous superpositions of coherent states is a direct consequence of quantum interference between an *infinite* (continuous) number of coherent components. These component states mutually interfere in such a way that the total quantum interference is "robust" with respect to the operational phase-space measurement and can be detected.

The difference between the pure continuous superpositions of coherent states and the corresponding statistical mixtures is preserved also in the case when the environmental influence on the quantum-mechanical system under consideration is taken into account. Let us consider for instance that an initial squeezed vacuum state decays into a zero-temperature heat bath. The Q function of the decaying squeezing vacuum can be evaluated as shown in paper I and has the Gaussian form given by Eq. (3.7) with the time-dependent parameters $\Sigma_{r,i}(t)$ given by the following expressions:

$$\Sigma_r^{-2} = \frac{1-\eta}{1-\eta(1-\mu)}, \qquad \Sigma_i^{-2} = \frac{1+\eta}{1+\eta(1-\mu)}, \quad (4.1)$$

where $\mu = \exp(-\gamma t)$. The Q function of the statistical mixture decaying into the zero-temperature heat bath is also described by the Gaussian function (3.7) but the time-dependent parameters $\Sigma_{r,i}(t)$ in this case are

$$\Sigma_r^{-2} = \frac{1-\eta}{1-\eta(1-\mu)}, \qquad \Sigma_i^{-2} = 1. \quad (4.2)$$

The Wehrl entropy of the decaying squeezed vacuum state is given by Eq. (3.10a) with the parameters $\Sigma_{r,i}$ given by Eq. (4.1) (see also [14]). Analogously, the Wehrl entropy of the decaying statistical mixture is given by Eq. (3.10b) with the parameters $\Sigma_{r,i}$ given by Eq. (4.2). Comparing these entropies we can conclude that the total quantum coherence present in pure continuous superpositions of coherent states is "robust" not only with respect to the operational phase-space measurement but also with respect to the decay into a zero-temperature heat bath.

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