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Sampling entropies and operational phase-space measurement. I. General formalism

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We present an entropic description of quantum-mechanical states based on an operational approach to a phase-space measurement. We give a simple phase-space interpretation of sampling entropies in terms of which we derive very general entropic uncertainty relations reflecting the degree of the phase-space uncertainty of the quantum-mechanical state in the given measurement (i.e., for a given "quantum-ruler" state). We relate the sampling entropy to the von Neumann and Shannon entropy and show that the Wehrl entropy represents a particular example of a sampling entropy when the quantum ruler is represented by coherent states.

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I. INTRODUCTION

Classical dynamical variables can be measured to arbitrary accuracy in principle. This permits precise measurements of conjugate variables such as position and momentum, and allows joint probability distributions to be constructed for a phase-space description of dynamics. This situation is much more complicated in quantum mechanics, for a lack of commutability of conjugate variables prevents us from constructing joint probability distributions, and the lack of a unique rule by which quantum and classical variables are associated leads to a variety of ordering schemes for operators representing observables [1,2].

Nevertheless, phase-space methods can be constructed within quantum mechanics and have been widely used to describe quantum mechanics (see for instance [3,4]). Depending on operator ordering, a number of different quasiprobabilities can be defined of which the best known are the Wigner function [3], the Husimi (Q) function, and the Glauber-Sudarshan P function [4,5], reflecting symmetric, antinormal, and normal ordering of operators in the characteristic function that determines the appropriate quasiprobability. The P function can be singular or negative, the Wigner function can be negative but is regular, whereas the Q function is always non-negative. Cahill and Glauber [2] showed that all these can be contained in an s-ordered quasiprobability density distribution where the choice of the value of the parameter s determines the degree of "smoothing," from the P function to the Q function. The connection between quasiprobabilities and measurement schemes is delicate (Leonhardt and Paul [6]). The Wigner function, in particular, generates the proper marginal distributions for individual phase-space variables (position or momentum), and attention has recently focused on its measurability (this is closely related to the so-called Pauli problem of reconstructing the wave function [7]) by the introduction of the so-called "optical homodyne tomography" method for reconstructing quantum states (Vogel and Risken [8], Smithey *et al.* [9]). The tomographic scheme has reopened questions concerning the physical significance of the Wigner function given that it may not be positive everywhere [10]. The tomographic scheme is based on the measurement of a *single* observable, the rotated quadrature $\hat{x}_\theta$ in which case a detector may be prepared in an eigenstate of this observable so the measured probability distribution $P(x_\theta)$ of $\hat{x}_\theta$ is unbiased by the measurement process. From a set of the measured data for $P(x_\theta)$ for all values of θ $[-\pi \leq \theta \leq \pi]$ the Wigner function can be in principle reconstructed [8]. On the other hand, one can consider a simultaneous measurement of two *noncommuting* conjugate observables $\hat{q}$ (where $\hat{q} = \hat{x}_{\theta=0}$) and $\hat{p}$ (where $\hat{p} = \hat{x}_{\theta=\pi/2}$). It is not possible to construct an eigenstate of these two noncommuting observables, and, therefore, it is inevitable that the *simultaneous* measurement process introduces additional noise.

To describe the process of the simultaneous measurement of two noncommuting observables Wódkiewicz [11] has proposed a formalism based on an operational probability density distribution that explicitly takes into account the action of the measurement device, modeled as a "filter." A particular choice of the state basis for the quantum ruler samples a specific type of accessible information concerning the system. To be more specific, operational probability density distributions have been introduced in various forms by several authors. In particular, in a review article by Stenholm [12] the relation between the operational probability distributions and smoothed density distributions introduced by Husimi [13] and Arthurs and Kelly [14] has been analyzed (see also [15–18]).

In this paper, we employ Wódkiewicz's operational probability densities to characterize quantum states of the field in terms of sampling entropies, which we relate to the von Neumann, Shannon, and Wehrl entropies. In Sec. II we summarize the basic properties of Wigner func-

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tions, paying particular attention to their positivity (or lack of it). In Sec. III we define the relevant operational phase-space density distributions in terms of the states of the quantum ruler characterizing the measurement, and use them to derive "operational" uncertainty relations. In Sec. IV, we use these operational phase-space distributions to generate the sampling entropy that we relate to the von Neumann and Shannon entropy. In Sec. V we derive the appropriate entropic uncertainty relations, relate these to the mutual information and illustrate them with nonclassical states drawn from quantum optics. Finally in Sec. VI we relate the sampling entropies to the Wehrl entropy and address the question of the reconstruction of the quantum state of the measured system. Appendices contain relegated mathematical details. In the second paper [19] (in what follows we will refer to this paper as Ref. II) we use the ideas developed in this paper to investigate the nature of quantum interference in phase space in generating nonclassical states.

II. WIGNER FUNCTIONS AND QUANTUM OPTICS IN PHASE SPACE

Wigner [3] was the first to show that expectation values $\langle\hat{M}\rangle$ of certain physically important classes of operators $\hat{M}$ can be expressed as integrals similar to the phase-space integrals of classical probability theory [4]. This approach is based on the idea of the transfer of quantum-mechanical statistical information from the density operator $\hat{\rho}$ describing the quantum-mechanical state to the *Wigner* function $W(\xi)$ and a function $M(\xi)$ which refers to the operator $\hat{M}$,

$$\langle\hat{M}\rangle = \mathrm{Tr}\left\{\hat{\rho}\hat{M}\right\} = \frac{1}{\pi}\int d^2\xi\, W(\xi)M(\xi). \tag{2.1}$$

The integral in Eq. (2.1) is carried out over all possible states of the system, i.e., in the case of a harmonic oscillator over the entire complex ξ plane. The differential element $d^2\xi/\pi$ (the measure of the integration) is a real element of an area proportional to the phase-space element $dq\,dp$ in classical mechanics.

Let us consider a dynamical system that is described by a pair of canonically conjugated Hermitian observables $\hat{q}$ and $\hat{p}$,

$$[\hat{q},\hat{p}] = i\hbar, \tag{2.2}$$

and have eigenvalues that range continuously from $-\infty$ to $+\infty$. The annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ can be expressed as a complex linear combination of $\hat{q}$ and $\hat{p}$,

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\lambda\hat{q} + i\lambda^{-1}\hat{p}\right); \tag{2.3a}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\lambda\hat{q} - i\lambda^{-1}\hat{p}\right), \tag{2.3b}$$

where λ is an arbitrary real parameter. The operators $\hat{a}$ and $\hat{a}^\dagger$ obey the Weyl-Heisenberg commutation relation

$$[\hat{a},\hat{a}^\dagger] = 1, \tag{2.4}$$

and, therefore, possess the same algebraic properties as the operator associated with the complex amplitude of a harmonic oscillator (in this case $\lambda = \sqrt{m\omega}$, where m and ω are the mass and the frequency of the quantum-mechanical oscillator) or the photon annihilation and creation operators of a single mode of the quantum electromagnetic field (in this case $\lambda = \sqrt{\epsilon\omega}$, where ϵ is the dielectric constant and ω is the frequency of the field mode).

A particularly useful set of states is the overcomplete set of coherent states $|\alpha\rangle$ which are the eigenstates of the annihilation operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.5}$$

These coherent states can be generated from the vacuum state $|0\rangle$ (which is defined through $\hat{a}|0\rangle = 0$) by the unitary displacement operator $\hat{D}(\alpha)$ [5]

$$\hat{D}(\alpha) \equiv \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right], \qquad |\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{2.6}$$

The space of eigenvalues, i.e., the *phase space* for our dynamical system is the *infinite* plane of eigenvalues (q,p) of the Hermitian operators $\hat{q}$ and $\hat{p}$. An equivalent phase space is the complex plane of eigenvalues

$$\alpha = \frac{1}{\sqrt{2\hbar}}\left(\lambda q + i\lambda^{-1}p\right), \tag{2.7}$$

of the annihilation operator $\hat{a}$. We should note here that the coherent state $|\alpha\rangle$ is not an eigenstate of either $\hat{q}$ or $\hat{p}$. The quantities q and p in Eq. (2.7) can be interpreted as the expectation values of the operators $\hat{q}$ and $\hat{p}$ in the state $|\alpha\rangle$. Two invariant differential elements of the two phase spaces are related as

$$\frac{1}{\pi}d^2\alpha = \frac{1}{\pi}d[\mathrm{Re}(\alpha)]\,d[\mathrm{Im}(\alpha)] = \frac{1}{2\pi\hbar}dq\,dp. \tag{2.8}$$

The phase-space description of the quantum-mechanical oscillator which is in the state described by the density operator $\hat{\rho}$, is based on the definition of the Wigner function [3] $W(\xi)$. The Wigner function is related to the characteristic function $C^{(W)}(\eta)$ of the Weyl-ordered moments of the annihilation and creation operators of the harmonic oscillator as follows [2]:

$$W(\xi) = \frac{1}{\pi}\int C^{(W)}(\eta)\,\exp(\xi\eta^* - \xi^*\eta)\,d^2\eta. \tag{2.9}$$

The characteristic function $C^{(W)}(\eta)$ of the system described by the density operator $\hat{\rho}$ is defined as

$$C^{(W)}(\eta) \equiv \mathrm{Tr}[\hat{\rho}\hat{D}(\eta)], \tag{2.10}$$

where $\hat{D}(\eta)$ is the displacement operator given by Eq. (2.6). The characteristic function $C^{(W)}(\eta)$ can be used for the evaluation of the Weyl-ordered products of the annihilation and creation operators as follows [2]:

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle = \left.\frac{\partial^{(m+n)}}{\partial\eta^m\partial(-\eta^*)^n}C^{(W)}(\eta)\right|_{\eta=0}. \tag{2.11}$$

On the other hand, the mean value of the Weyl-ordered product $\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle$ can be obtained by using the Wigner function directly to generate moments

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle = \frac{1}{\pi}\int d^2\xi\,(\xi^*)^m\xi^n W(\xi). \tag{2.12}$$

For instance, the Weyl-ordered product $\langle\{\hat{a}^\dagger\hat{a}\}\rangle$ can be evaluated as

$$\langle\{\hat{a}^\dagger\hat{a}\}\rangle = \frac{1}{2}\langle\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\rangle \equiv \frac{1}{\pi}\int d^2\xi\,|\xi|^2 W(\xi). \tag{2.13}$$

Originally the Wigner function was introduced in a form different from (2.9). Namely, the Wigner function was defined as a particular Fourier transform of the density operator expressed in the basis of the eigenvectors $|q\rangle$ of the position operator $\hat{q}$,

$$W(q,p) \equiv \int_{-\infty}^{\infty} d\zeta\langle q-\zeta/2|\hat{\rho}|q+\zeta/2\rangle e^{ip\zeta/\hbar}, \tag{2.14a}$$

which for a pure state described by a state vector $|\Psi\rangle$ (i.e., $\hat{\rho}=|\Psi\rangle\langle\Psi|$) reads

$$W(q,p) \equiv \int_{-\infty}^{\infty} d\zeta\psi(q-\zeta/2)\psi^*(q+\zeta/2)e^{ip\zeta/\hbar}. \tag{2.14b}$$

It can be shown that both definitions (2.9) and (2.14) of the Wigner function are identical (see Hillery *et al.* [4]), providing the parameters ξ and ξ^* [see Eq. (2.9)] are related to the coordinates q and p of the phase space as

$$\xi = \frac{1}{\sqrt{2\hbar}}\left(\lambda q + i\lambda^{-1}p\right), \tag{2.15a}$$

$$\xi^* = \frac{1}{\sqrt{2\hbar}}\left(\lambda q - i\lambda^{-1}p\right). \tag{2.15b}$$

The Wigner function can be interpreted as the quasiprobability (see below) density distribution through which a probability can be expressed to find a quantum-mechanical system (harmonic oscillator) around the "point" (q,p) of the phase space.

As an example we evaluate the Wigner function of the coherent state $|\alpha\rangle$ given by Eq. (2.6) (with the complex amplitude $\alpha=\alpha_r+i\alpha_i$)

$$W(\xi;\alpha) = 2\exp\left(-2|\xi-\alpha|^2\right), \tag{2.16a}$$

or alternatively, in the (q,p) representation we have

$$W(q,p;\alpha) = \frac{1}{\sigma_q\sigma_p}\exp\left[-\frac{1}{2\hbar}\frac{(q-\bar{q})^2}{\sigma_q^2} - \frac{1}{2\hbar}\frac{(p-\bar{p})^2}{\sigma_p^2}\right], \tag{2.16b}$$

where $\bar{q}=\sqrt{2\hbar}\alpha_r/\lambda$; $\bar{p}=\sqrt{2\hbar}\alpha_i\lambda$, and

$$\sigma_q^2 = \frac{1}{2\lambda^2} \quad\text{and}\quad \sigma_p^2 = \frac{\lambda^2}{2}. \tag{2.16c}$$

Marginal probability density distributions obtained from the Wigner function are related to the momentum and position probability distributions $W(q)$ and $W(p)$ as follows:

$$W(q) = \frac{1}{\lambda\sqrt{2\pi\hbar}}\int dp\,W(q,p) \equiv \frac{\sqrt{2\pi\hbar}}{\lambda}\langle q|\hat{\rho}|q\rangle, \tag{2.17a}$$

where $|q\rangle$ is the eigenstate of the position operator $\hat{q}$. The probability density distribution $W(q)$ is normalized to unity,

$$\frac{\lambda}{\sqrt{2\pi\hbar}}\int dq\,W(q) = 1, \tag{2.17b}$$

where $\lambda dq/\sqrt{2\pi\hbar}$ is a dimensionless measure of integration. Analogously, the momentum probability density distribution $W(p)$ is defined as

$$W(p) = \frac{\lambda}{\sqrt{2\pi\hbar}}\int dq\,W(q,p) \equiv \lambda\sqrt{2\pi\hbar}\langle p|\hat{\rho}|p\rangle, \tag{2.18a}$$

where $|p\rangle$ is the eigenstate of the momentum operator $\hat{p}$. The momentum probability density distribution $W(p)$ is normalized as

$$\frac{1}{\lambda\sqrt{2\pi\hbar}}\int dp\,W(p) = 1. \tag{2.18b}$$

From the above definitions we can obtain expressions for the mean values of any power of the operator $\hat{q}$ ($\hat{p}$) as

$$\langle\hat{q}^n\rangle = \frac{\lambda}{\sqrt{2\pi\hbar}}\int dq q^n W(q),$$
$$\langle\hat{p}^n\rangle = \frac{1}{\lambda\sqrt{2\pi\hbar}}\int dp p^n W(p). \tag{2.19}$$

For example, using the above expressions we can easily find the variances $\langle(\Delta\hat{q})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ of the operators $\hat{q}$ and $\hat{p}$ in the coherent state $|\alpha\rangle$ given by Eq. (2.6). The variances $\langle(\Delta\hat{q})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ are defined in the usual way,

$$\langle(\Delta\hat{q})^2\rangle \equiv \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2, \qquad \langle(\Delta\hat{p})^2\rangle \equiv \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 \tag{2.20}$$

and we find that

$$\langle(\Delta\hat{q})^2\rangle = \hbar\sigma_q^2 \quad\text{and}\quad \langle(\Delta\hat{p})^2\rangle = \hbar\sigma_p^2. \tag{2.21}$$

From the definition of the parameters σ_q and σ_p [see Eq. (2.16c)] it follows that

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \frac{\hbar^2}{4}, \tag{2.22}$$

which demonstrates that coherent states are minimum uncertainty states [20].

By the definition the Wigner function $W(q,p)$ is normalized to unity in the whole phase space, i.e.,

$$\frac{1}{2\pi\hbar}\int dq\,dp\,W(q,p) = \frac{1}{\pi}\int d^2\xi\,W(\xi) = 1. \tag{2.23}$$

If $W_\Psi(q,p)$ and $W_\Phi(q,p)$ are two quasiprobability densities corresponding to the states $|\Psi\rangle$ and $|\Phi\rangle$, respectively, then it can be shown that the transition probability $|\langle\Psi|\Phi\rangle|^2$ between these two states can be expressed as

$$|\langle\Psi|\Phi\rangle|^2 = \left|\int dq\,\psi^*(q)\phi(q)\right|^2 = \frac{1}{2\pi\hbar}\int dq\,dp\,W_\Psi(q,p)W_\Phi(q,p). \qquad (2.24)$$

In particular, from Eq. (2.24) it follows that if the states $|\Psi\rangle$ and $|\Phi\rangle$ are orthogonal (i.e., $\langle\Psi|\Phi\rangle = 0$) then

$$\frac{1}{2\pi\hbar}\int dq\,dp\,W_\Psi(q,p)W_\Phi(q,p) = 0, \qquad (2.25)$$

which implies that the Wigner function cannot be everywhere positive and, therefore, it is not a probability density distribution, but rather a quasiprobability density distribution. Moreover, it can be shown that the Gaussian quasiprobability density distribution [21]

$$W_\Psi(q,p) = \frac{1}{\sigma_{q,\Psi}\sigma_{p,\Psi}\sqrt{1-r^2}}\exp\left\{-\frac{1}{2(1-r^2)\hbar}\times\left[\frac{(p-\bar{p})^2}{\sigma^2_{p,\Psi}} - \frac{2r(p-\bar{p})(q-\bar{q})}{\sigma_{p,\Psi}\sigma_{q,\Psi}} + \frac{(q-\bar{q})^2}{\sigma^2_{q,\Psi}}\right]\right\}, \qquad (2.26)$$

for which the parameters $\sigma_{q,\Psi}$ and $\sigma_{p,\Psi}$ are related as

$$\sigma^2_{q,\Psi}\sigma^2_{p,\Psi} = \frac{1}{4(1-r^2)}, \qquad (2.27)$$

is the *only* positive Wigner function for a pure state. From Eqs. (2.26) and (2.27) it follows that coherent states and squeezed states [i.e., the states described by the Wigner function (2.26) with the parameter r equal to zero] are the only minimum uncertainty states. Their Wigner functions are non-negative. All other Wigner functions that describe a pure state necessarily take negative values for some values of q and p.

Finally, we turn our attention to a very important property of Wigner functions which we will utilize later in our paper. Namely, the convolution of two Wigner functions of quantum-mechanical states is *always* a non-negative density distribution. This convolution is usually referred to as the smoothed Wigner function [11–14,22,23] and can be expressed as

$$P_{\Psi\Phi}(q,p) = \frac{1}{2\pi\hbar}\int dq'\,dp'\,W_\Psi(q+q',p+p')W_\Phi(q',p'), \qquad (2.28)$$

where $W_\Psi(q,p)$ is the Wigner function of the state described by the state vector $|\Psi\rangle$ and $W_\Phi(q,p)$ is a "smoothing" Wigner function corresponding to the state $|\Phi\rangle$.

III. OPERATIONAL APPROACH TO PHASE-SPACE MEASUREMENT IN QUANTUM OPTICS

From our previous discussion it is clear that the Wigner function cannot be straightforwardly interpreted as the phase-space probability density distribution because it may take negative values for some values of p and q. This negativity of the Wigner function is intrinsically related to the nonclassical nature of corresponding quantum-mechanical states. Simultaneously, it is an indication that the Wigner function is not a *directly* measurable probability density in the phase space. Nevertheless, it has to be stressed that the Wigner function can be (in principle) *indirectly* reconstructed from experimentally measured data [8–10], although due account must be taken of the problems noise introduces in this reconstruction.

There have been many attempts to overcome formally the problem connected with the fact that the Wigner function can take negative values. In particular, Husimi [13] defined a non-negative probability density distribution as a convolution between the smoothing Gaussian function and the Wigner function. Recently, Lalović, Davidović, and Bijedić [23] have proposed a formalism that allows the formulation of quantum mechanics in terms of generalized nonpositive smoothed Wigner functions in a systematic and unified way.

All these formal smoothing procedures result in definitions of probability density distributions which are non-negative as demanded, but the relation between the smoothing procedure and a realistic measurement procedure remained rather unclear. A more physical approach has been adopted by Arthurs and Kelly [14] (see also [12]) in which the smoothing procedure is explained in terms of the noise induced by a *simultaneous* measurement of two conjugated noncommuting observables. Following Wódkiewicz [11] we may ask: "Is it possible to define a realistic phase-space function that can be recorded in the laboratory?" Wódkiewicz has also proposed an answer to this question in which he derived a *positive* definite quantum probability density distribution $P(q,p)$ which is directly connected to a realistic measurement.

The fundamental feature of the operational probability density distribution proposed by Wódkiewicz [11] is that, in addition to the quantum-mechanical system to be measured and its detector, a device acting as a filter is introduced. This filter is needed in order to resolve for example the current position and momentum of the investigated system. Obviously, the particular realization (choice) of the filter strongly influences the outcome of the measurement.

If a quantum-mechanical state of the filtering device is described by the state vector $|\Phi\rangle$ and the system being measured is in the state $|\Psi\rangle$, then, as shown by Wódkiewicz [11] the operational probability density $P_{\Psi\Phi}(q,p)$ can be expressed as (see also [12,14])

$$P_{\Psi\Phi}(q,p) = \left|\int d\xi\,\exp\left(\frac{-ip\xi}{\hbar}\right)\Psi^*(\xi+q)\Phi(\xi)\right|^2, \qquad (3.1)$$

which is equivalent to the expression

$$P_{\Psi\Phi}(q,p) = \frac{1}{2\pi\hbar}\int dq'dp' W_\Psi(q+q',p+p')W_\Phi(q',p'). \tag{3.2a}$$

In other words, the operational probability density distribution $P_{\Psi\Phi}(q,p)$ is *equivalent* to the convolution (i.e., "overlap") of the detected and the displaced filtering Wigner functions. The operational probability density distribution (3.2a) can be also expressed as a squared modulus of a scalar product of the state vector $|\Psi\rangle$ of the measured system and the displaced state vector $\hat{D}(q,p)|\Phi\rangle$ of the quantum ruler, i.e.,

$$P_{\Psi\Phi}(q,p) = |\langle\Psi|\hat{D}(q,p)|\Phi\rangle|^2. \tag{3.2b}$$

Alternatively, if the quantum system and the quantum ruler are described by density operators $\hat{\rho}_\Psi$ and $\hat{\rho}_\Phi$, respectively, then the operational probability density distribution can be expressed as

$$P_{\Psi\Phi}(q,p) = \mathrm{Tr}\left\{\hat{\rho}_\Psi\hat{D}^{-1}(q,p)\hat{\rho}_\Phi\hat{D}(q,p)\right\}, \tag{3.2c}$$

where $\hat{D}(q,p) = \exp[i(p\hat{q}-q\hat{p})/\hbar]$ is the displacement operator in the (q,p) phase space. Equations (3.2) answer also the fundamental question of the proper relation between the Wigner function of the state of the measured quantum-mechanical system and a realistic phase-space measurement in quantum mechanics.

The operational phase-space probability density $P_{\Psi\Phi}(q,p)$ can be interpreted as a *propensity* [11,24] or a *tendency* of the measured state $|\Psi\rangle$ to take up certain states $|\Phi\rangle$ prescribed by a quantum measuring device [see Eq. (3.2b)]. The reference states $|\Phi\rangle$ of the measurement device (the filter) are often called the states of the "quantum ruler." We will demonstrate how different choices of the ruler states (for example coherent states) affect the measured information (data) about the nature of the quantum state.

A. Operational uncertainty relations for variances

Following Wódkiewicz [11] and Raymer [25] we derive an operational uncertainty relation for the variances of the position and momentum operators from which the role of quantum noise induced by the measurement process (filtering) will be transparent. Using the Wigner function of the quantum-mechanical state $|\Psi\rangle$ we can derive probability density distributions $W_\Psi(q)$ and $W_\Psi(p)$ of the position and momentum [see Eqs. (2.17–18)]. With the help of these probability distributions and Eq. (2.20) we can evaluate the variances of the position and momentum operators in the state $|\Psi\rangle$,

$$\langle(\Delta\hat{q})^2\rangle_\Psi = \langle\hat{q}^2\rangle_\Psi - \langle\hat{q}\rangle_\Psi^2, \qquad \langle(\Delta\hat{p})^2\rangle_\Psi = \langle\hat{p}^2\rangle_\Psi - \langle\hat{p}\rangle_\Psi^2, \tag{3.3}$$

where $\langle\hat{q}^n\rangle_\Psi$ and $\langle\hat{p}^n\rangle_\Psi$ are given by Eqs. (2.19). Using very general properties of the distributions $W_\Psi(p)$ and $W_\Psi(q)$ and the Schwartz inequality, it can be shown that for any quantum-mechanical system described by the density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$, the uncertainty relation

$$\langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Psi \geq \frac{\hbar^2}{4} \tag{3.4}$$

is valid. This uncertainty relation depends only on the state of the quantum-mechanical system and does not carry any information about the phase-space measurement of the position and momentum, i.e., the uncertainty relation (3.4) does not contain any information about the measuring device.

We briefly note here that there is a class of the quantum-mechanical states $|\Psi\rangle$, the minimum uncertainty states (MUS) for which the product of the variances $\langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Psi$ take the minimum value which is equal to $\hbar^2/4$. The MUS are described by the Gaussian Wigner function given by Eq. (2.27) with $r = 0$. Particular examples of the MUS are coherent states [see Eq. (2.6)] and squeezed states (for more details see Sec. V and Ref. II).

To take into account the intrinsic uncertainty of the quantum-mechanical system [i.e., the relation (3.4)] as well as the uncertainty related to the quantum measuring device we utilize the operational phase-space probability density $P_{\Psi\Phi}(q,p)$ given by Eq. (3.2). The corresponding marginal probability densities $P_{\Psi\Phi}(q)$ and $P_{\Psi\Phi}(p)$ defined as

$$P_{\Psi\Phi}(q) = \frac{1}{\lambda\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\, P_{\Psi\Phi}(q,p), \tag{3.5a}$$

$$P_{\Psi\Phi}(p) = \frac{\lambda}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dq\, P_{\Psi\Phi}(q,p) \tag{3.5b}$$

are related to the probabilities

$$\frac{\lambda}{\sqrt{2\pi\hbar}}P_{\Psi\Phi}(q)dq \qquad \text{and} \qquad \frac{1}{\lambda\sqrt{2\pi\hbar}}P_{\Psi\Phi}(p)dp \tag{3.5c}$$

that the value q of the position and the value p of the momenta of the quantum-mechanical system which is in the state $|\Psi\rangle$ are measured. The probability densities $P_{\Psi\Phi}(p)$ and $P_{\Psi\Phi}(q)$ carry information about the system to be measured (which is in the state $|\Psi\rangle$) and about the measuring apparatus which is in the state $|\Phi\rangle$.

From the probability densities $P_{\Psi\Phi}(q)$ and $P_{\Psi\Phi}(p)$ we can evaluate the mean values of the position $\hat{q}$ and momentum operators $\hat{p}$ using the standard relations

$$\langle\hat{q}^n\rangle_{\Psi\Phi} = \frac{\lambda}{\sqrt{2\pi\hbar}}\int dq\, q^n P_{\Psi\Phi}(q)$$

and

$$\langle\hat{p}^n\rangle_{\Psi\Phi} = \frac{1}{\lambda\sqrt{2\pi\hbar}}\int dp\, p^n P_{\Psi\Phi}(p). \tag{3.6}$$

Here we stress once more that these operational quantum-mechanical averages involve both the state $|\Psi\rangle$

of the system to be measured and the state $|\Phi\rangle$ of the filtering (reference) device. From (3.6) we can in particular find that

$$\langle\hat{q}\rangle_{\Psi\Phi} = \langle\Psi|\hat{q}|\Psi\rangle - \langle\Phi|\hat{q}|\Phi\rangle, \tag{3.7a}$$
$$\langle\hat{p}\rangle_{\Psi\Phi} = \langle\Psi|\hat{p}|\Psi\rangle - \langle\Phi|\hat{p}|\Phi\rangle, \tag{3.7b}$$

which reflects the fact that $\langle\hat{q}\rangle_{\Psi\Phi}$ can be interpreted as a relative position (distance) of the detected state $|\Psi\rangle$ with respect to a reference fixed by the filter (quantum-ruler) state $|\Phi\rangle$.

Analogously we can evaluate operational variances $\Sigma_{\Psi\Phi}(p)$ and $\Sigma_{\Psi\Phi}(q)$ defined as

$$\Sigma^2_{\Psi\Phi}(q) = \langle\hat{q}^2\rangle_{\Psi\Phi} - \langle\hat{q}\rangle^2_{\Psi\Phi},$$

and

$$\Sigma^2_{\Psi\Phi}(p) = \langle\hat{p}^2\rangle_{\Psi\Phi} - \langle\hat{p}\rangle^2_{\Psi\Phi} \tag{3.8}$$

for which we find

$$\Sigma^2_{\Psi\Phi}(q) = \langle(\Delta\hat{q})^2\rangle_\Psi + \langle(\Delta\hat{q})^2\rangle_\Phi, \tag{3.9a}$$

and

$$\Sigma^2_{\Psi\Phi}(p) = \langle(\Delta\hat{p})^2\rangle_\Psi + \langle(\Delta\hat{p})^2\rangle_\Phi, \tag{3.9b}$$

which means that there are two sources of measured quantum-mechanical fluctuations. The first part of the quantum-mechanical noise has its origin in the detected state $|\Psi\rangle$ itself and the second source of noise is from the measuring device (i.e., the state $|\Phi\rangle$). Both of these sources have a quantum-mechanical origin. Later in this paper we will discuss how to describe other sources of noise that can appear during the operational phase-space measurement.

Following Wódkiewicz we can derive for the operational variances $\Sigma_{\Psi\Phi}(q)$ and $\Sigma_{\Psi\Phi}(p)$ the uncertainty relation which reads

$$\Sigma^2_{\Psi\Phi}(q)\Sigma^2_{\Psi\Phi}(p) = \langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Psi + \langle(\Delta\hat{q})^2\rangle_\Phi\langle(\Delta\hat{p})^2\rangle_\Phi + \langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Phi + \langle(\Delta\hat{q})^2\rangle_\Phi\langle(\Delta\hat{p})^2\rangle_\Psi \geq \hbar^2. \tag{3.10}$$

This uncertainty relation depends on the quantum-mechanical properties of the detecting device as well as the measured system. It differs from the intrinsic uncertainty relation (3.4) by a factor of 2 in the quantum noise, or a factor of 4 for the product of variances.

B. Example A

In what follows we will consider a harmonic oscillator with a unit mass and a unit frequency, i.e., the parameter λ in Eq. (2.15) is equal to unity. If this harmonic oscillator is prepared in a coherent state $|\Psi\rangle = |\alpha\rangle$ [see Eq. (2.6)] with the corresponding Wigner function given by Eq. (2.16) then the variances in $\hat{q}$ and $\hat{p}$ read

$$\langle(\Delta\hat{q})^2\rangle_\Psi = \langle(\Delta\hat{p})^2\rangle_\Psi = \frac{\hbar}{2}. \tag{3.11}$$

From Eq. (3.11) we see that the coherent state is a MUS. The squeezed vacuum state (for more details see Sec. V) is also a MUS but its variances are not equal. For a particular phase of squeezing these variances can take the following values:

$$\langle(\Delta\hat{q})^2\rangle_\Psi = \frac{\hbar}{2}\frac{1+\eta}{1-\eta} \equiv \hbar\sigma^2_{q,\Psi} > \frac{\hbar}{2},$$

and

$$\langle(\Delta\hat{p})^2\rangle_\Psi = \frac{\hbar}{2}\frac{1-\eta}{1+\eta} \equiv \hbar\sigma^2_{p,\Psi} < \frac{\hbar}{2}, \tag{3.12}$$

where the squeezing parameter η takes the values from the interval $(-1,1)$. From Eq. (3.12) it follows that for $\eta > 0$ ($\eta < 0$) the variance in the momentum (position) is squeezed below the coherent-state (vacuum) level at the expense of the increase of the variance in the momentum. Nevertheless, the product of these variances is still equal to $\hbar^2/4$, i.e., the squeezed vacuum is the MUS.

Now we can assume that the quantum filter (the quantum ruler) is in the coherent state $|\beta\rangle \equiv |\Phi\rangle$ (for more discussion on this choice of the state of the quantum filter see Sec. VI). The variances of the position and momentum operators in this filter state are

$$\langle(\Delta\hat{q})^2\rangle_\Phi = \langle(\Delta\hat{p})^2\rangle_\Phi = \frac{\hbar}{2} \tag{3.13}$$

and, therefore, we obtain for operational variances of the coherent state the following expressions:

$$\Sigma^2_{\Psi\Phi}(q) = \hbar = \Sigma^2_{\Psi\Phi}(p), \tag{3.14}$$

while for the operational variances of the squeezed vacuum state we find

$$\Sigma^2_{\Psi\Phi}(q) = \frac{\hbar}{1-\eta} \equiv \hbar\Sigma^2_{q,\Psi\Phi}$$

and

$$\Sigma^2_{\Psi\Phi}(p) = \frac{\hbar}{1+\eta} \equiv \hbar\Sigma^2_{p,\Psi\Phi}. \tag{3.15}$$

From the above, it follows that coherent states are the minimum uncertainty states with respect to the operational uncertainty relations providing the filter state (i.e., the quantum ruler) is taken to be in a coherent state. On the other hand, the squeezed vacuum state, which is intrinsically a minimum uncertainty state, is *not* a minimum uncertainty state with respect to the operational uncertainty relation, i.e.,

$$\Sigma^2_{\Psi\Phi}(q)\Sigma^2_{\Psi\Phi}(p) = \frac{\hbar^2}{1-\eta^2} > \hbar^2. \tag{3.16}$$

We can say that a phase-space measurement with the quantum detection apparatus modeled as a filtering device which is prepared in a coherent state is an *optimized* quantum measurement for coherent states in a sense that the product of the corresponding operational variances minimizes the operational uncertainty relation.

Here we note that if the measured state $|\Psi\rangle$ is a squeezed vacuum state and if we choose the filter (the quantum ruler) to be in a squeezed state characterized by the variances (3.12) such that the condition

$$\langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Phi = \langle(\Delta\hat{q})^2\rangle_\Phi\langle(\Delta\hat{p})^2\rangle_\Psi = \frac{\hbar^2}{4}, \tag{3.17}$$

is fulfilled, then we find for the operational variances of the squeezed vacuum in this particular kind of measurement the expression

$$\Sigma^2_{\Psi\Phi}(q) = \hbar = \Sigma^2_{\Psi\Phi}(p). \tag{3.18}$$

This equation reflects the fact that squeezed states can be minimum uncertainty states with respect to a particular phase-space measurement with properly chosen filter states. From here it follows that we can optimize the measurement in such a way that the operational uncertainty relation can be minimized [26].

IV. ENTROPIC UNCERTAINTY MEASURES

Gaussian quantum-mechanical states $|\Psi\rangle$ of the harmonic oscillator are *fully* characterized by the mean amplitude [i.e., the mean value of the annihilation operator, $\langle\hat{a}\rangle$, and the second-order variances $\langle(\Delta\hat{q})^2\rangle_\Psi$ and $\langle(\Delta\hat{p})^2\rangle_\Psi$]. Therefore, the uncertainty relation $\langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Psi \geq \hbar^2/4$ completely reflects the intrinsic uncertainty of the quantum-mechanical state.

For non-Gaussian pure states of the harmonic oscillator the second-order variances $\langle(\Delta\hat{q})^2\rangle_\Psi$ and $\langle(\Delta\hat{p})^2\rangle_\Psi$ do not contain enough information to characterize completely the intrinsic uncertainty of the state. In particular, let us consider a pure quantum-mechanical superposition of two coherent states [27]

$$|\Psi\rangle = \frac{1}{\{2[1+\exp(-2\alpha^2)]\}^{1/2}}\,[|\alpha\rangle + |-\alpha\rangle] \qquad (\alpha \text{ is real}), \tag{4.1}$$

which is described by a non-Gaussian Wigner function. The variances $\langle(\Delta\hat{q})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ of the operators $\hat{q}$ and $\hat{p}$ in the superposition state (4.1) read

$$\langle(\Delta\hat{q})^2\rangle_\Psi = \frac{\hbar}{2}\left[1 + \frac{2\alpha^2}{1+\exp(-2\alpha^2)}\right], \tag{4.2a}$$

$$\langle(\Delta\hat{p})^2\rangle_\Psi = \frac{\hbar}{2}\left[1 - \frac{2\alpha^2\exp(-2\alpha^2)}{1+\exp(-2\alpha^2)}\right], \tag{4.2b}$$

from which it directly follows that the state under consideration is not a MUS. Moreover, in the limit of a large amplitude α the product of two variances $\langle(\Delta\hat{q})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ goes to infinity as α^2

$$\lim_{\alpha^2\to\infty}\langle(\Delta\hat{q})^2\rangle_\Psi\langle(\Delta\hat{p})^2\rangle_\Psi \to \frac{\hbar^2}{2}\alpha^2 \to \infty. \tag{4.3}$$

If we consider the filter (the quantum ruler) to be in a coherent state then the product of two operational variances $\Sigma^2_{\Psi\Phi}(q)$ and $\Sigma^2_{\Psi\Phi}(p)$ also goes to infinity for large values of α. On the other hand, we understand that the superposition of two coherent states can be "classically" interpreted as a superposition of two points in the phase space. Therefore, if we perform a phase-space measurement (with the coherent-state quantum ruler) the uncertainty corresponding to a superposition of two coherent states has to be *finite* (i.e., we should expect some uncertainty related to a two-state system [for more details see Ref. II]). This discrepancy between the uncertainty measures based on the second-order variances and the intuitive picture corresponding to the phase-space measurement has its origin in the fact that using $\langle(\Delta\hat{q})^2\rangle_\Psi$ and $\langle(\Delta\hat{p})^2\rangle_\Psi$ as the uncertainty measures for non-Gaussian states we *deliberately* neglect higher-order variances that carry essential information about non-Gaussian states. Therefore, in order to quantify more carefully the intrinsic uncertainty of non-Gaussian states we have to consider a global (complete) information about the state. One possibility is to take into account higher-order variances of the operators $\hat{q}$ and $\hat{p}$ and to use them to quantify the degree of uncertainty of the state. Nevertheless, a more consistent approach would be to utilize the total information available about the quantum-mechanical system and define an uncertainty measure based on this information.

The most natural measure of the uncertainty of the quantum-mechanical state is the entropy [28–36]. If we are talking about the quantum-mechanical state *per se*, i.e., without any reference to a measurement process, then the corresponding quantum-mechanical entropy due to von Neumann [35] reads

$$S_{\hat{\rho}} = -k_B \mathrm{Tr}\,(\hat{\rho}_\Psi \ln\hat{\rho}_\Psi), \tag{4.4}$$

where $\hat{\rho}_\Psi$ is the density operator of the quantum-mechanical system under consideration and k_B is Boltzmann's constant. This definition of the quantum-mechanical entropy generalizes the classical expression of the entropy due to Boltzmann and Gibbs.

If we relate the von Neumann entropy to the number of pure states $|\Psi\rangle$ that contribute to the mixture described by the density operator $\hat{\rho}_\Psi$ then the von Neumann entropy of a pure state (i.e., the state that is microscopically uniquely prescribed) is equal to zero. For quantum-mechanical mixtures the von Neumann entropy is larger than zero, because there exist several pure states that realize the same quantum-mechanical mixture described by the density operator $\hat{\rho}_\Psi$.

Formally we can express the von Neumann entropy in the framework of the phase-space formalism, i.e., in terms of the Wigner function $W_{\hat{\rho}}(q,p)$,

$$S_{\hat\rho} = -\frac{k_B}{2\pi\hbar}\int dp\, dq\, W_{\hat\rho}(q,p) W_{\ln\hat\rho}(q,p), \quad (4.5a)$$

where $W_{\ln\hat\rho}(q,p)$ is the Wigner function of the operator $\ln\hat\rho$ which can be defined as the Fourier transform of the Weyl-ordered characteristic function $C^{(W)}(\eta)$ of the operator $\ln\hat\rho$, i.e.,

$$W_{\ln\hat\rho}(\xi) = \frac{1}{\pi}\int d^2\eta \,\mathrm{Tr}\left[\ln\hat\rho \exp(\eta\hat a^\dagger - \eta^*\hat a)\right] \times \exp(\xi\eta^* - \xi^*\eta). \quad (4.5b)$$

We have to stress that this "Wigner logarithm function" is not equal to the logarithm of the Wigner function $W_{\hat\rho}(q,p)$ [i.e., $W_{\ln\hat\rho}(q,p) \neq \ln W_\Psi(q,p)$], which means we cannot express the von Neumann entropy as

$$S_\Psi = -\frac{k_B}{2\pi\hbar}\int dp\, dq\, W_\Psi(q,p)\ln W_\Psi(q,p), \quad (4.6)$$

because the function S_Ψ has no physical meaning for negative values of $W_\Psi(q,p)$. Besides this purely formal reason there exists a conceptual problem with the entropy defined only via the Wigner function of the quantum-mechanical state. The point is that in the definition of the Wigner function there is no reference to a measurement process by means of which the information about the quantum-mechanical state $|\Psi\rangle$ is obtained. We can think about the quantum-mechanical entropy in terms of a "number of possible realizations" of the state of the measured system by quantum-filter states. To this end we can adopt a schematic phase-space picture describing overlaps of the error contour corresponding to the measured state $|\Psi\rangle$ and the filter states $|\Phi\rangle$ (see Fig. 1). If this overlap is unique, then the corresponding entropy has to take its minimal value. If it is not unique, we have to expect that this entropy is larger than zero.

To formalize the above idea we utilize the operational probability density distribution $P_{\Psi\Phi}(p,q)$ which can be interpreted as the overlap of the measured state $|\Psi\rangle$ and

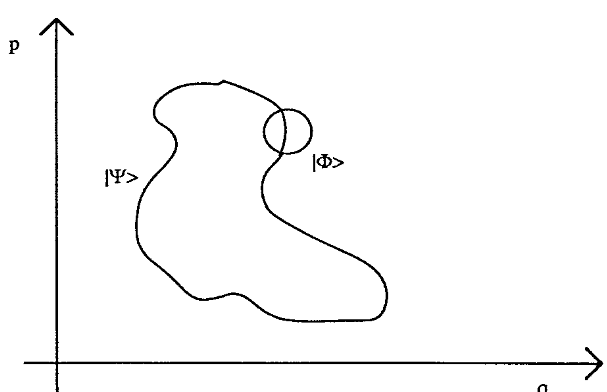

FIG. 1. Phase-space portrait of a quantum state and its associated quantum ruler: what is shown is the "error" contour of the quasiprobability density distribution of the system in the state $|\Psi\rangle$ and the quantum ruler in the state $|\Phi\rangle$ (imagined here to be in a coherent state).

the quantum-ruler state $|\Phi\rangle$. Then the corresponding entropy can be defined as [26]

$$S_{\Psi\Phi}^{(q,p)} = -\frac{k_B}{2\pi\hbar}\int dp\, dq\, P_{\Psi\Phi}(p,q)\ln P_{\Psi\Phi}(p,q). \quad (4.7)$$

This entropy has a very attractive and simple interpretation analogous to that which is contained in Boltzmann's definition of entropy ($S = k_B \ln W$, where W is the number of microstates that realize a given macrostate). Namely we can interpret the entropy $S_{\Psi\Phi}$ as a function of the number of quantum-ruler states $|\Phi\rangle$ that *sample* the quantum-mechanical state $|\Psi\rangle$. This interpretation motivates us to call the entropy $S_{\Psi\Phi}$ the phase-space *sampling entropy.*

It is worth to note here that Eq. (4.7) for the entropy $S_{\Psi\Phi}^{(q,p)}$ can be formally reduced to the expression (4.6) describing an entropy in terms of only the Wigner function $W_\Psi(q,p)$. This reduction is possible providing the Wigner function W_Φ of the quantum filter is defined as

$$W_\Phi(q,p) \sim \delta(p-p')\delta(q-q'), \quad (4.8)$$

which means that nonphysical states of the quantum filter are used to perform a quantum-mechanical measurement of the state $|\Psi\rangle$. The states of the quantum filter "described" by the probability density distribution (4.8) are classical states because they allow simultaneous determination of $\hat q$ and $\hat p$ with an infinite accuracy, which violates the physical laws of the quantum mechanics. This is the physical reason why the expression (4.6) has no physical meaning.

A. Sampling entropy and Shannon entropy

In addition to the entropy $S_{\Psi\Phi}^{(q,p)}$ that measures the uncertainty related to a *simultaneous* measurement of two canonically conjugated operators $\hat q$ and $\hat p$, we can introduce two other entropies $S_{\Psi\Phi}^{(q)}$ and $S_{\Psi\Phi}^{(p)}$. These two entropies are defined through the marginal probability density distributions $P(q)_{\Psi\Phi}$ and $P(p)_{\Psi\Phi}$ of the position and momentum, respectively,

$$S_{\Psi\Phi}^{(q)} = -\frac{\lambda k_B}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dq\, P_{\Psi\Phi}(q)\ln P_{\Psi\Phi}(q), \quad (4.9a)$$

and

$$S_{\Psi\Phi}^{(p)} = -\frac{k_B}{\lambda\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp P_{\Psi\Phi}(p)\ln P_{\Psi\Phi}(p). \quad (4.9b)$$

The entropies $S_{\Psi\Phi}^{(q)}$ and $S_{\Psi\Phi}^{(p)}$ describe the uncertainty in the *measured* values of the position or momentum providing the quantum ruler is in the state $|\Phi\rangle$. If the quantum ruler is supposed to be in the eigenstate of the measured observable (let us say $\hat q$), i.e., $|\Phi\rangle = |q\rangle$, so that

$$P(q)_{\Psi\Phi} = |\langle q|\Psi\rangle|^2, \quad (4.10)$$

then the entropy $S_{\Psi\Phi}^{(q)}$ is equal to the Shannon entropy [31] corresponding to the measurement of the single observable $\hat q$.

If we assume that the filter states are equal to eigenstates (here it is implicitly assumed that these eigenstates form a complete orthonormal basis) of one of the two noncommuting observables (we consider the position operator $\hat{q}$) then the fluctuations associated with the quantum ruler are equal to zero, i.e., $\langle(\Delta\hat{q})^2\rangle_{\Phi=q} = 0$, so that $\Sigma_{\Psi,\Phi=q}(q) = \langle(\Delta\hat{q})^2\rangle_\Psi$ [see Eq. (3.9a)], which means that fluctuations in the distribution $P(q)_{\Psi,\Phi=q}$ are related to the intrinsic uncertainty of the operator $\hat{q}$ in the state $|\Psi\rangle$. This reduction of operational fluctuations in the $\hat{q}$ observable is done at the expense of infinite filter fluctuations in the conjugated $\hat{p}$ observable. To be more specific, adjusting the filter state to the eigenstate of the position operator we cannot obtain any information about the distribution in the momentum of the state under consideration and consequently we are not able to perform an operational phase-space measurement.

The Shannon entropy is a very useful measure of the uncertainty related to a measurement of a *single* observable. In this case the filter state can be taken as the eigenstate of this observable and filter fluctuations are consequently reduced to zero. On the other hand, in this kind of measurement the information about the conjugated variable is completely lost. In the case when a *simultaneous* measurement of two noncommuting observables is performed one cannot introduce straightforwardly the concept of the Shannon entropy because two noncommuting observables do not share the same eigenstates. Therefore, the filter has to be prepared in a state that is not an eigenstate of the two observables under consideration and thus the filtering process inevitably introduces fluctuations into measured data. From this point of view the sampling entropy $S_{\Psi\Phi}^{(q,p)}$ as introduced by Eq. (4.7) can be interpreted as the Shannon entropy for a *simultaneous* measurement of two noncommuting observables.

B. Sampling entropy for $\hat{n}$ and $\hat{\phi}$

In quantum optics the Shannon entropy is usually associated with the photon number distribution $P_n = |\langle n|\Psi\rangle|^2$ of a given state of a single-mode light field. The Shannon entropy in this case reads

$$S_\Psi^{(n)} = -k_B \sum_n P_n \ln P_n. \tag{4.11}$$

We will briefly describe this entropy in the framework of the sampling-entropies formalism. To do so, we firstly introduce two conjugated observables $\hat{n}$ and $\hat{\phi}$, that is the number operator and the conjugated Hermitian phase operator. Following Pegg and Barnett [37] we consider a finite-dimensional Hilbert space of the dimension $(s+1)$ in which these two operators do act. The corresponding phase space [38,39] is represented by $(s+1)^2$ discrete "points" associated with states of the harmonic oscillator in the finite-dimensional Hilbert space. We can introduce Wigner-function-like probabilities $W_\Psi(n_i,\phi_j)^{(s)}$ (here the index s indicates the dimension of the Hilbert space under consideration) that the harmonic oscillator which is in the state $|\Psi\rangle$ will be found in a particular point (n_i,ϕ_j) of the phase space. Following Wootters [39] and Vaccaro and Pegg [38] we can express the Wigner function $W_\Psi(n_i,\phi_j)^{(s)}$ as the mean value

$$W_\Psi(n_i,\phi_j)^{(s)} \equiv \frac{1}{s+1}\langle\Psi|\hat{A}(n_i,\phi_j)^{(s)}|\Psi\rangle \tag{4.12}$$

of the Hermitian operator $\hat{A}(n_i,\phi_j)^{(s)}$ which is defined as

$$\hat{A}(n_i,\phi_j)^{(s)} = \sum_{k=0}^{s} \exp(-2ikn_i\Delta)|\phi_{j+k}\rangle\langle\phi_{j-k}|. \tag{4.13}$$

In our notation $n_i = i$ $(i = 0,1,...,s)$; $\Delta = 2\pi/(s+1)$, and the phase states $|\phi_j\rangle$ in the finite-dimensional Fock space read [37]

$$|\phi_j\rangle = \frac{1}{\sqrt{1+s}}\sum_{k=0}^{s}\exp(ik\phi_j)|k\rangle, \tag{4.14}$$

where $\phi_j = \phi_0 + j\Delta$ with $j = 0,1,...,s$ and $|k\rangle$ is the Fock state. The Hermitian phase operator is defined as the projector $\hat{\Phi} = \sum_{j=0}^{s}\phi_j|\phi_j\rangle\langle\phi_j|$ and has eigenvalues within the interval $[\phi_0, \phi_0 + 2\pi]$.

In an analogy with a case of the (q,p) phase space we can introduce operational probability distributions $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$ [see Appendix A]

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \sum_{k,l=0}^{s} W_\Psi(n_{i+k},\phi_{j+l})^{(s)} \times W_\Phi(n_k,\phi_l)^{(s)}, \tag{4.15}$$

[here the periodic properties of the operator $\hat{A}(n_i,\phi_j)^{(s)}$ have to be taken into account when the summation over k and l is performed]. The corresponding sampling entropy is given by the expression

$$S_{\Psi\Phi}^{(n,\phi)}(s) = -k_B \sum_{i,j=0}^{s} P_{\Psi\Phi}(n_i,\phi_j)^{(s)} \times \ln P_{\Psi\Phi}(n_i,\phi_j)^{(s)}. \tag{4.16}$$

Using the distribution $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$ we can also introduce probability distributions in the measured values of n and ϕ, respectively,

$$P_{\Psi\Phi}(n_i)^{(s)} = \sum_{j=0}^{s} P_{\Psi\Phi}(n_i,\phi_j)^{(s)}, \tag{4.17a}$$

and

$$P_{\Psi\Phi}(\phi_j)^{(s)} = \sum_{i=0}^{s} P_{\Psi\Phi}(n_i,\phi_j)^{(s)}. \tag{4.17b}$$

Entropies measuring uncertainties in the distributions $P_{\Psi\Phi}(n_i)^{(s)}$ and $P_{\Psi\Phi}(\phi_j)^{(s)}$ read

$$S_{\Psi\Phi}^{(n)}(s) = -k_B \sum_{i=0}^{s} P_{\Psi\Phi}(n_i)^{(s)} \ln P_{\Psi\Phi}(n_i)^{(s)} \tag{4.18a}$$

and

$$S_{\Psi\Phi}^{(\phi)}(s) = -k_B \sum_{j=0}^{s} P_{\Psi\Phi}(\phi_j)^{(s)} \ln P_{\Psi\Phi}(\phi_j)^{(s)}. \quad (4.18b)$$

If the filter state $|\Phi\rangle$ is assumed to be equal to the Fock state $|n\rangle$ $(0 \leq n \leq s)$, then $P_{\Psi\Phi=n}(n_i)^{(s)}$ represents a "photon" number distribution in the finite-dimensional Hilbert space and the corresponding entropy $S_{\Psi,\Phi=n}^{(n)}(s)$ in the limit $s \to \infty$ is equal to the Shannon entropy (4.11).

V. ENTROPIC UNCERTAINTY RELATIONS

Using the definition of the phase-space sampling entropy we can define in a very natural way the entropic uncertainty relations (EUR) which are based on the total information about the measured system $|\Psi\rangle$ rather than just on second-order variances of the operators $\hat{q}$ and $\hat{p}$ (the so-called Heisenberg uncertainty relations). Obviously, we should expect that for Gaussian states, the fluctuations of which are characterized by the second-order variances, the entropic and the Heisenberg uncertainty relations yield the same results.

A. Mutual information and EUR

To introduce entropic uncertainty relations we utilize the concept of a mutual information between two random variables as defined by Kolmogoroff [40] and Shannon [31]: Let A (B) be a random variable defined on a stochastic object that takes different values x_i (y_j) $[i = 1,...,n$ $(j = 1,...,m)]$ with probabilities $P^{(A)}(x_i)$ $[P^{(B)}(y_j)]$. The joint probability distribution of the random variables A and B is defined as $P^{(A,B)}(x_i,y_j)$ and the marginal probabilities $P^{(A)}(x_i)$ and $P^{(B)}(y_j)$ are defined in the usual way,

$$P^{(A)}(x_i) = \sum_j P^{(A,B)}(x_i,y_j),$$

and

$$P^{(B)}(y_j) = \sum_i P^{(A,B)}(x_i,y_j). \quad (5.1)$$

The mutual information $I^{(A,B)}$ between the random variables A and B is defined as (see for instance [31,40]):

$$I^{(A,B)} = k_B \sum_{i,j} P^{(A,B)}(x_i,y_j) \ln \left[\frac{P^{(A,B)}(x_i,y_j)}{P^{(A)}(x_i)P^{(B)}(y_j)} \right]. \quad (5.2)$$

The parameter $I^{(A,B)}$ represents a measure of information contained in the random variable A about the random variable B (and vice versa). For statistically independent random variables A and B, i.e., if $P^{(A,B)}(x_i,y_j) = P^{(A)}(x_i)P^{(B)}(y_j)$, the parameter $I^{(A,B)}$ is equal to zero. Otherwise it is a positive number [40],

$$I^{(A,B)} \geq 0. \quad (5.3)$$

With the use of the probability density $P_{\Psi\Phi}(p,q)$ and the corresponding marginal probability densities $P_{\Psi\Phi}(q)$ and $P_{\Psi\Phi}(p)$ we can define the mutual information $I_{\Psi\Phi}^{(q,p)}$ as

$$I_{\Psi\Phi}^{(q,p)} = \frac{k_B}{2\pi\hbar} \int dq\, dp\, P_{\Psi\Phi}(q,p) \ln \left[\frac{P_{\Psi\Phi}(q,p)}{P_{\Psi\Phi}(q)P_{\Psi\Phi}(p)} \right]. \quad (5.4)$$

The parameter $I_{\Psi\Phi}^{(q,p)}$ can be interpreted as the amount of information contained in the measured value of the position (momentum) of the quantum-mechanical state $|\Psi\rangle$ about the momentum (position) of this state, providing the quantum-mechanical measurement with the quantum ruler in the state $|\Phi\rangle$ is performed.

Using the definitions of the entropies $S_{\Psi\Phi}^{(q,p)}$, $S_{\Psi\Phi}^{(q)}$, and $S_{\Psi\Phi}^{(p)}$ [see Eqs. (4.7) and (4.9), respectively] we can rewrite the mutual information $I_{\Psi\Phi}^{(q,p)}$ in terms of these entropies as

$$I_{\Psi\Phi}^{(q,p)} = S_{\Psi\Phi}^{(q)} + S_{\Psi\Phi}^{(p)} - S_{\Psi\Phi}^{(q,p)}. \quad (5.5)$$

Taking into account non-negativity of the mutual information [i.e., $I_{\Psi\Phi}^{(q,p)} \geq 0$, see Eq. (5.3)] then from Eq. (5.5) we directly obtain the entropic uncertainty relation which reads

$$S_{\Psi\Phi}^{(q)} + S_{\Psi\Phi}^{(p)} \geq S_{\Psi\Phi}^{(q,p)}. \quad (5.6)$$

We have to stress here that these entropic uncertainty relations do depend not only on the quantum-mechanical state $|\Psi\rangle$ but also on the quantum filter state $|\Phi\rangle$. If we fix some quantum filter state $|\Phi\rangle$, i.e., we choose a particular quantum-mechanical measurement procedure, then we call the quantum-mechanical state $|\Psi\rangle$, an *intelligent state* [41] with respect to the given measurement, providing the left and right side of the uncertainty relation (5.6) become equal, i.e.,

$$S_{\Psi\Phi}^{(q)} + S_{\Psi\Phi}^{(p)} = S_{\Psi\Phi}^{(q,p)}. \quad (5.7)$$

If in addition both the left and right side of the relation (5.7) reach their minimum values then we can call the state $|\Psi\rangle$ the *minimum uncertainty state* with respect to the given measurement.

From the above definitions it follows that the mutual information between the momentum and position variables of intelligent states is equal to zero. These states are described by Gaussian probability density distributions $P_{\Psi\Phi}(q,p)$ for which p and q variables are completely independent.

We note that the uncertainty relation (5.6) has a counter part for von Neumann entropies which is one of the two well-known Araki-Lieb inequalities [28]. Here the total system was separated into two subsystems where the corresponding density matrix is evaluated by tracing over the variables of the other subsystem. Both subsystems then on their own include less information together than the whole system due to the neglect of the influence of any of both systems on the other. In our case the total "system" is described by the quasiprobability function $P_{\Psi\Phi}(p,q)$ rather than the density matrix and one "subsystem" describes the spatial component $P_{\Psi\Phi}(q)$ only whereas the other describes the momentum component $P_{\Psi\Phi}(p)$. Again we expect the position of a system to have influence on its own dynamics and vice versa and thus information is lost due to the restriction on the marginal entropies in analogy to the well-known Araki-Lieb inequality for the von Neumann entropy. Thus we anticipate some connection between both uncertainty relations for these principally different entropies for quantum-mechanical states. The entropies in relation (5.6) are also defined for classical distributions with narrow probability distributions where one Araki-Lieb inequality was shown to be violated [28].

B. Example B

To illustrate the above definitions let us consider that the quantum-ruler (i.e., the filter) states are the coherent states with the Wigner functions given by Eq. (2.16) which means that $\sigma^2_{q,\Phi} = \sigma^2_{p,\Phi} = 1/2$. In this case the corresponding sampling entropy is equal to the Wehrl entropy [28] (for more details see Ref. II). Furthermore, let us assume that the state that is going to be measured is the pure Gaussian state described by the Wigner function (2.27) with $r = 0$, i.e., the parameters $\sigma_{q,\Psi}$ and $\sigma^2_{p,\Psi}$ are related as $\sigma^2_{q,\Psi}\sigma^2_{p,\Psi} = 1/4$, i.e.,

$$W_\Psi(p,q) = \frac{1}{\sigma_{p,\Psi}\sigma_{q,\Psi}} \exp\left[-\frac{1}{2\hbar}\frac{(q-\bar{q})^2}{\sigma^2_{q,\Psi}} - \frac{1}{2\hbar}\frac{(p-\bar{p})^2}{\sigma^2_{p,\Psi}}\right]. \tag{5.8}$$

The state described by the Wigner function (5.8) can be obtained by the action of the squeezing operator $\hat{S}(\xi)$ and the displacement operator $\hat{D}(\bar{q},\bar{p})$ on the vacuum state $|0\rangle$,

$$|\Psi\rangle = \hat{D}(\bar{q},\bar{p})\hat{S}(\xi)|0\rangle, \tag{5.9}$$

where the squeezing operator $\hat{S}(\xi)$ is defined as [42]

$$\hat{S}(\xi) = \exp\left[\frac{i\xi}{2\hbar}(\hat{q}\hat{p} + \hat{p}\hat{q})\right]. \tag{5.10}$$

The corresponding parameters $\sigma^2_{A,\Psi}$ can be expressed as

$$\sigma^2_{q,\Psi} = \tfrac{1}{2}e^{-2\xi}, \qquad \sigma^2_{p,\Psi} = \tfrac{1}{2}e^{2\xi}, \tag{5.11}$$

which represents another parametrization of squeezing parameters given by Eq. (3.12). From Eq. (5.11) it is seen that if $\xi > 0$ then the fluctuations in the position operator are reduced below the vacuum level [i.e., $\langle(\Delta\hat{q})\rangle_\Psi = \hbar\sigma^2_{q,\Psi} < \hbar/2$] at the expense of the increase of fluctuations in the conjugated momentum operator.

After some algebra we can find the operational phase-space probability distribution $P_{\Psi\Phi}(p,q)$ related to this particular phase-space measurement of the state $|\Psi\rangle$ in the form

$$P_{\Psi\Phi}(p,q) = \frac{1}{\Sigma_{p,\Psi\Phi}\Sigma_{q,\Psi\Phi}} \exp\left[-\frac{1}{2\hbar}\frac{(q-\bar{q})^2}{\Sigma^2_{q,\Psi\Phi}} - \frac{1}{2\hbar}\frac{(p-\bar{p})^2}{\Sigma^2_{p,\Psi\Phi}}\right], \tag{5.12}$$

where $\Sigma^2_{q,\Psi\Phi} = \sigma^2_{q,\Psi} + \sigma^2_{q,\Phi} = 1/2 + \sigma^2_{q,\Psi}$ and $\Sigma^2_{p,\Psi\Phi} = \sigma^2_{p,\Psi} + \sigma^2_{p,\Phi} = 1/2 + \sigma^2_{p,\Psi}$ and $\sigma^2_{q,\Psi}\sigma^2_{p,\Psi} = 1/4$. The marginal probability distributions $P_{\Psi\Phi}(q)$ and $P_{\Psi\Phi}(p)$ in this case read

$$P_{\Psi\Phi}(q) = \frac{1}{\Sigma_{q,\Psi\Phi}} \exp\left[-\frac{1}{2\hbar}\frac{(q-\bar{q})^2}{\Sigma^2_{q,\Psi\Phi}}\right], \tag{5.13a}$$

and

$$P_{\Psi\Phi}(p) = \frac{1}{\Sigma_{p,\Psi\Phi}} \exp\left[-\frac{1}{2\hbar}\frac{(p-\bar{p})^2}{\Sigma^2_{p,\Psi\Phi}}\right]. \tag{5.13b}$$

Now we can easily derive the corresponding entropies for which we find

$$S^{(q,p)}_{\Psi\Phi} = k_B + k_B \ln \Sigma_{p,\Psi\Phi}\Sigma_{q,\Psi\Phi} = k_B[1 + \ln(\cosh\xi)], \tag{5.14}$$

and

$$S^{(q)}_{\Psi\Phi} = \frac{k_B}{2} + k_B \ln \Sigma_{q,\Psi\Phi} = \frac{k_B}{2}\left[\ln\left(\frac{1+e^{-2\xi}}{2}\right)\right], \tag{5.15a}$$

$$S^{(p)}_{\Psi\Phi} = \frac{k_B}{2} + k_B \ln \Sigma_{p,\Psi\Phi} = \frac{k_B}{2}\left[\ln\left(\frac{1+e^{2\xi}}{2}\right)\right]. \tag{5.15b}$$

From Eqs. (5.14–15) we learn the following:

(1) The Wehrl entropy (5.14) of Gaussian states is proportional to the uncertainty area in units $2\pi\hbar$, i.e., the area this quasiprobability distribution covers in phase space at height $1/e$ from its maximum value. This issue will be addressed in more detail in Ref. II.

(2) The squeezed state (5.9) is an intelligent state with respect to the given operational phase-space measurement but the lower bound on the EUR, i.e., the Wehrl entropy $S^{(q,p)}_{\Psi\Phi}$ given by Eq. (5.14), depends on the degree of squeezing. Namely, the larger the degree of squeezing the larger is the sampling entropy. In the limit $\xi \to \infty$ the sampling entropy $S^{(q,p)}_{\Psi\Phi}$ is diverging but simultaneously $S^{(q)}_{\Psi\Phi}$ reaches its minimum value equal to $k_B(1 - \ln 2)/2$. This value is smaller than $k_B/2$ which reflects a reduction of quantum fluctuations in the position of the harmonic oscillator. Nevertheless, because the filter is in a coherent state characterized by finite fluctuations in the position, the outcome of the measurement of $\hat{q}$ cannot be infinitely precise even in the limit of infinite squeezing of the mea-

sured state (for further discussion see Example C).

(3) Solving the simple variational problem we find that $S_{\Psi\Phi}^{(q,p)}$ given by Eq. (5.14) reaches its minimum value when $\sigma_{p,\Psi}^2 = \sigma_{q,\Psi}^2 = 1/2$, i.e., when the state $|\Psi\rangle$ is a coherent state ($\xi = 0$). In this case $\Sigma_{p,\Psi\Phi} = \Sigma_{q,\Psi\Phi} = 1$ and

$$S_{\Psi\Phi}^{(q,p)} = k_B. \tag{5.16}$$

We can conclude that coherent states are the minimum uncertainty states with respect to the chosen phase-space measurement (filter is in a coherent state). Squeezed states are only intelligent states with respect to this phase-space measurement.

C. Minimum value of the sampling entropy

It has been conjectured by Wehrl, and subsequently proven by Lieb [28], that the absolute minimum value of the Wehrl entropy (i.e., the sampling entropy with a coherent filter) is equal to k_B. As we have seen earlier this minimum is obtained when a pure coherent state $|\Psi\rangle$ is filtered by a coherent quantum ruler [see Eq. (5.16)]. It may seem strange that when the coherent state $|\alpha\rangle$ is "filtered" by a coherent-state quantum ruler then the sampling entropy (even though it takes its minimum value) is not equal to zero. The reason is that the coherent-state basis is overcomplete, i.e., coherent states are not mutually orthogonal and, therefore, the corresponding sampling entropy cannot be equal to zero. The physical meaning of this statement is simply that a point of the quantum-mechanical phase space cannot be in principle located (measured) with an infinite accuracy (a "point" in quantum-mechanical phase space is always associated with an area $2\pi\hbar$). The accuracy expressed by the relation (5.16) is the best one can achieve when performing a simultaneous (phase-space) measurement of two noncommuting observables.

Using the same arguments as Lieb [28] it can be shown [26] that for *any* choice of the filter state the minimum value of the sampling entropy $S_{\Psi\Phi}^{(q,p)}$ is equal to k_B. Therefore, we can write the entropic uncertainty relation for the pair of two observables $\hat{q}$ and $\hat{p}$ as

$$S_{\Psi\Phi}^{(q)} + S_{\Psi\Phi}^{(p)} \geq S_{\Psi\Phi}^{(q,p)} \geq k_B. \tag{5.17}$$

The proof of the entropic uncertainty relation (5.17) is based on an assumption that to perform an optimal phase-space measurement that minimizes the sampling entropy $S_{\Psi\Phi}^{(q,p)}$ we have to use a filter that is in a state characterized by the minimum intrinsic noise. In other words, the filter state is considered to be in a pure minimum uncertainty state. *All* pure minimum uncertainty states in the (q,p)-phase space are characterized by Gaussian Wigner functions of the form (5.8). Moreover, these states are related by simple unitary transformations (see for instance the review article by Schumaker [21]) which preserve the measure in (q,p)-phase space. With the help of these transformations we can transform any pure minimum uncertainty filter state into a coherent state $[W_\Phi(q,p) \to W_{\rm coh}(q,p)]$. Obviously under these transformations the Wigner function $W_\Psi(q,p)$ of the measured state is transformed as well, i.e., $W_\Psi(q,p) \to W_{\tilde{\Psi}}(q,p)$. This means the propensity $P_{\Psi\Phi}$ can be transformed into the Q function of the state described by the Wigner function $W_{\tilde{\Psi}}(q,p)$ (see Appendix B). Moreover, this unitary transformation has the property that the phase-space "overlap" between the measured state $|\Psi\rangle$ and the filter state $|\Phi\rangle$ is invariant, from which it follows that the sampling entropy is invariant under the action of the unitary transformations under consideration, i.e., the sampling entropy of the state $|\Psi\rangle$ which is filtered by the state $|\Phi\rangle$ is *equal* to the Wehrl entropy described by the Wigner function $W_{\tilde{\Psi}}(q,p)$. The inverse is true as well. Therefore, using the result by Lieb [28] we can conclude that the entropy $S_{\Psi\Phi}^{(q,p)}$ is bounded from below by the value k_B.

D. Example C

Now we evaluate the lower bound on the EUR when the harmonic oscillator under consideration is in an eigenstate $|Q\rangle$ of the position operator $\hat{q}$. To analyze this rather nontrivial example we first of all have to introduce a phase-space description of the state vector $|Q\rangle$, that is we have to find the Wigner function of the position state. In order to find this Wigner function we represent the position state $|Q\rangle$ as the displaced squeezed state $|Q,\xi\rangle \equiv \hat{D}(Q,0)\hat{S}(\xi)|0\rangle$ given by Eq. (5.9) in the limit of infinite squeezing,

$$|Q\rangle = \lim_{\xi\to\infty} \hat{D}(Q,0)\hat{S}(\xi)|0\rangle. \tag{5.18}$$

The action of the position operator $\hat{q}$ on the state $|Q,\xi\rangle$ is

$$\hat{q}|Q,\xi\rangle = Q|Q,\xi\rangle + e^{-\xi}\sqrt{\frac{2}{\hbar}}\hat{D}(Q,0)\hat{S}(\xi)|1\rangle, \tag{5.19}$$

where $|1\rangle$ is a Fock state with one excitation quantum. In the limit $\xi \to \infty$ we formally obtain from (5.19) the eigenvalue equation for the position operator. Alternatively, we can prove Eq. (5.18) in a weak sense. To do so, we turn our attention to the fact that the Wigner function corresponding to the state $|Q,\xi\rangle$ is given by expression (5.8), from which we find the marginal probability density distribution $W_\Psi(q)$ for the position in the form:

$$\begin{aligned} W_\Psi(q) &= \frac{1}{\sqrt{2\pi\hbar}}\int dp W_\Psi(q,p) \\ &= \frac{1}{\sigma_{q,\Psi}}\exp\left[-\frac{1}{2\hbar}\frac{(q-Q)^2}{\sigma_{q,\Psi}^2}\right], \end{aligned} \tag{5.20}$$

where $\sigma_{q,\Psi}^2 = \exp(-2\xi)/2$. With the help of the marginal distribution $W_\Psi(q)$ we evaluate the mean value of all powers of the operator $\hat{q}$

$$\begin{aligned} \langle \hat{q}^n\rangle_\Psi &= \frac{1}{\sqrt{2\pi\hbar}}\int dq\, q^n W_\Psi(q) \\ &= \sum_{m=0}^{[n/2]} \binom{n}{2m} Q^{n-2m}(2m-1)!!(\hbar\sigma_{q,\Psi}^2)^m, \end{aligned} \tag{5.21}$$

where $[x]$ denotes the largest integer smaller than x. From Eq. (5.21) it is obvious that in the limit of infinite squeezing we have $\langle Q,\xi|\hat{q}^n|Q,\xi\rangle \to Q^n$, which proves in a weak sense that $|Q,\xi\rangle$ in the limit $\xi \to \infty$ is equal to the position state $|Q\rangle$.

Now we assume that the state $|Q,\xi\rangle$ is filtered by the state $|\Phi\rangle = \hat{D}(q,p)\hat{S}(\xi)|0\rangle$, that is, the filter state is essentially the same as the measured state except it is displaced in the phase space (but the direction of squeezing is the same for the measured state $|\Psi\rangle$ as well as for the filter state $|\Phi\rangle$). This means that we perform a measurement of the position state with the help of the filter which is also in the position state (in the limit $\xi \to \infty$). The operational probability density distribution $P_{\Psi\Phi}(q,p)$ is in this case given by Eq. (5.12) with $\Sigma^2_{q,\Psi\Phi} = \exp(-2\xi)$ and $\Sigma^2_{p,\Psi\Phi} = \exp(2\xi)$. The corresponding marginal distribution in the limit of infinite squeezing has the form

$$P_{\Psi\Phi}(q-Q) \equiv \lim_{r\to\infty} \frac{1}{\sqrt{2\pi\hbar}} \int dp\, P_{\Psi\Phi}(q-Q,p;r) = \sqrt{2\pi\hbar}\,\delta(q-Q), \tag{5.22}$$

which one expects, because the eigenstates of the position operator (which are characterized by continuous eigenvalues) are mutually orthogonal. We also find explicit expressions for the sampling entropy and the corresponding marginal entropies of the state $|Q,\xi\rangle$ filtered by the state $|q,\xi\rangle$ which read

$$S^{(q,p)}_{\Psi\Phi} = k_B, \quad S^{(q)}_{\Psi\Phi} = \frac{k_B}{2} - 2k_B\xi, \quad S^{(p)}_{\Psi\Phi} = \frac{k_B}{2} + 2k_B\xi. \tag{5.23}$$

From Eq. (5.23) it directly follows that the position state is an intelligent state if it is filtered by another position state. Moreover, the lower bound on the EUR is equal to k_B, so the position state is the minimum uncertainty state with respect to the given operational phase-space measurement. From Eq. (5.23) we also find that in the limit $\xi \to \infty$ the marginal entropy $S^{(p)}_{\Psi\Phi}$ is equal to $+\infty$ which reflects a complete uncertainty in the distribution of the momentum of the position state. On the other hand $S^{(q)}_{\Psi\Phi}$ is equal to $-\infty$ which is related to the fact that the marginal distribution (5.22) has the form of the δ function analogous to a density distribution of a classical continuous variable. It is well known [28] that entropies related to classical density distributions of the form (5.22) can be equal to $-\infty$.

VI. WEHRL ENTROPY AND SAMPLING ENTROPIES

In this paper we have introduced the idea of sampling entropies through the operational phase-space probability density distributions $P_{\Psi\Phi}(q,p)$. These density distributions contain information not only about the measured quantum-mechanical state $|\Psi\rangle$ but also about the state $|\Phi\rangle$ of the quantum ruler. In this paper, we have not specified the most natural quantum-ruler (filter) states for the quantum-mechanical phase-space measurement. To do so we remind ourselves that coherent states represent the best approximation to "classical" phase-space points in the framework of quantum theory [20]. Moreover these states have very exceptional properties when the influence of the larger external environment on an individual quantum-mechanical system is taken into account. In particular, coherent states are the most "robust" quantum-mechanical states in a sense that they preserve their quantum-statistical properties for times much longer than any other quantum-mechanical states [27,43]. Because of their exceptional properties, coherent states are the best candidates for the quantum-ruler states in the operational phase-space measurement described in our paper. In particular, if we choose the quantum ruler to be in a coherent state with a zero amplitude, i.e., when $|\Phi\rangle \equiv |\beta = 0\rangle$, then the operational probability density distribution $P_{\Psi\Phi}(q,p)$ given by Eq. (3.2) can be expressed in the form

$$P_{\Psi\Phi}(q,p) = \langle 0|\hat{D}^\dagger(q,p)\hat{\rho}_\Psi\hat{D}(q,p)|0\rangle \equiv Q_\Psi(q,p), \tag{6.1}$$

which means that with this particular choice of the ruler state the operational probability density distribution $P_{\Psi\Phi}(q,p)$ is equal to the Husimi (Q) function [13]. It is interesting to note here that with the help of this phase-space probability density distribution function one can evaluate the antinormally ordered products $\langle\hat{a}^n(\hat{a}^\dagger)^m\rangle_\Psi$ of the system operators in the state $|\Psi\rangle$ via the phase-space integration similar to Eq. (2.12),

$$\langle\hat{a}^n(\hat{a}^\dagger)^m\rangle_\Psi = \frac{1}{\pi}\int d^2\beta\,\beta^n(\beta^*)^m Q_\Psi(\beta). \tag{6.2a}$$

The Q function itself can be expressed as the Fourier transform of the antinormally ordered characteristic function $C^{(a)}(\eta)$ of the density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$, which is defined as

$$C^{(a)}_\Psi(\eta) = \mathrm{Tr}[\hat{\rho}\exp(-\eta^*\hat{a})\exp(\eta\hat{a}^\dagger)], \tag{6.2b}$$

so that

$$Q_\Psi(\beta) = \frac{1}{\pi}\int d^2\eta\, C^{(a)}_\Psi(\eta)\exp(\beta\eta^* - \beta^*\eta). \tag{6.2c}$$

We see that the Husimi function has a twofold meaning. First, it can be interpreted as the operational probability density distribution of the state $|\Psi\rangle$ given that the quantum ruler (filter) is in a coherent state. On the other hand, this function can be interpreted as the Fourier transform of the characteristic function $C^{(a)}(\eta)$ of antinormally ordered moments of the operators $\hat{a}$ and $\hat{a}^\dagger$.

Because of the fact that the Husimi function does not take negative values for any β it has been used by Wehrl [28] for a definition of what he has called the "classical entropy":

$$S_\Psi = -\frac{k_B}{\pi}\int d^2\beta\, Q_\Psi(\beta)\ln Q_\Psi(\beta). \tag{6.3}$$

According to Wehrl this entropy represents a classical analogue of the von Neumann entropy which in terms of the Q function can be written as

$$S_\Psi = -\frac{k_B}{\pi} \int d^2\beta \, Q_\Psi(\beta) P_{\ln\hat\rho}(\beta), \qquad (6.4)$$

where $P_{\ln\hat\rho}(\beta)$ is the phase-space probability density distribution (the so-called P function or Glauber-Sudarshan diagonal coherent state quasiprobability) corresponding to the Fourier transform of the normally ordered characteristic function $C^{(n)}_{\ln\hat\rho}(\eta)$ of the operator $\ln\hat\rho$ which is defined as

$$C_{\ln\hat\rho}(\eta) = \mathrm{Tr}\left[\exp(\eta\hat a^\dagger)\exp(-\eta^*\hat a)\ln\hat\rho\right], \qquad (6.5a)$$

and

$$P_{\ln\hat\rho}(\beta) = \frac{1}{\pi}\int d^2\eta \, C_{\ln\hat\rho}(\eta)\exp[\beta\eta^* - \beta^*\eta]. \qquad (6.5b)$$

We have to stress here that for nonclassical states, the P functions in general are tempered distributions. It can be shown that the "classical" approximation due to Wehrl [28] consists of the substitution of the function $P_{\ln\hat\rho}(\beta)$ by the logarithm of the Q function of the state $\hat\rho = |\Psi\rangle\langle\Psi|$, i.e.,

$$P_{\ln\hat\rho}(\beta) \to \ln Q_\Psi(\beta). \qquad (6.6)$$

From our previous discussion it clearly follows that the entropy introduced by Wehrl may have a rather attractive physical interpretation in terms of the operational phase-space probability density distributions. This interpretation is seen to have nothing to do with classical approximations and is based purely on the postulates of quantum theory. For this reason we prefer to call the sampling entropy (6.3) expressed through the Q functions as the *Wehrl* entropy.

There are at least two experimental schemes with the help of which the Q functions of single-mode light fields have been measured [15,16] (see also [17,18]). More generally, the direct experimental reconstruction of the operational phase-space probability density distribution $P_{\Psi\Phi}(q,p)$ is feasible in the experimental setup used by Noh, Fougéres, and Mandel [16]. In this eight-port device consisting of four beam splitters the signal state which is going to be measured is launched into one port of the first beam splitter while the filter state is launched into the unused port of this beam splitter. The essence of this setup is that a *simultaneous* (phase-space) measurement of two noncommuting observables can be performed and that the corresponding operational phase-space probability distribution can be reconstructed (i.e., the EUR can be experimentally verified) from the measurement of two *independent* quadrature distributions at the output of two different beam splitters.

Because the Q function can be experimentally measured it is important to understand whether, in principle, it is possible to reconstruct the Wigner function (i.e., the complete information about the measured state *per se*) from the Q function. As follows from our previous discussion, the Wigner function contains complete information about the quantum-mechanical system, in the sense that (at least in principle) the density operator $\hat\rho$ of the system can be found through the knowledge of the Wigner function,

$$\hat\rho = \frac{1}{\pi^2}\int d^2\eta \, \hat D^{-1}(\eta)\int d^2\xi \exp(\xi^*\eta - \xi\eta^*) W_\Psi(\xi). \qquad (6.7)$$

As we already know, the Q function can be expressed through the Wigner function as

$$Q_\Psi(\beta) = \frac{2}{\pi}\int d^2\xi \exp(-2|\beta-\xi|^2) W_\Psi(\xi). \qquad (6.8)$$

On the other hand, there does not exist a direct inverse transformation (deconvolution) of the form

$$W_\Psi(\xi) = \frac{1}{\pi}\int d^2\beta F(\beta,\xi) Q_\Psi(\beta), \qquad (6.9)$$

from which the Wigner function of the quantum-mechanical system can be reconstructed from the Q function [i.e., there does not exist a function $F(\beta,\xi)$ such that Eq. (6.9) is valid for an arbitrary quantum-mechanical state]. Nevertheless it is possible to reconstruct uniquely the Wigner function from the knowledge of the Q function. To do so we first have to perform a Fourier transform inverse to that described by Eq. (6.2c) from which we obtain the characteristic function $C^{(a)}_\Psi(\eta)$ of the antinormally ordered moments of bosonic operators. From here we obtain the characteristic function of the symmetrically ordered moments $C^{(W)}_\Psi(\eta)$ using the relation

$$C^{(W)}_\Psi(\eta) = e^{|\eta|^2/2} C^{(a)}_\Psi(\eta). \qquad (6.10)$$

The Wigner function then can be obtained as the Fourier transform of the characteristic function $C^{(W)}_\Psi(\eta)$ [see Eq. (2.9)]. With the help of the above prescription one can derive the formula that relates the Wigner function and the Q function

$$W_\Psi(\xi) = \exp\left[-\frac{1}{8}\frac{\partial^2}{\partial\xi_r^2} - \frac{1}{8}\frac{\partial^2}{\partial\xi_i^2}\right] Q_\Psi(\xi), \qquad (6.11)$$

where $\xi = \xi_r + i\xi_i$. This one-to-one correspondence between the Wigner and the Q functions reflects the fact that when an *ideal* operational phase-space measurement with a quantum ruler in a coherent state is performed then the information about the measured state $|\Psi\rangle$ can be completely reconstructed. In other words, using the transformation (6.11) one can extract unbiased information about the state $|\Psi\rangle$ from the data that contain noise due to the quantum-mechanical measurement.

In this paper, we have discussed only situations when the source of the uncertainty in $P_{\Psi\Phi}(q,p)$ has exclusively a quantum-mechanical origin. Remaining in the framework of the phase-space formalism we can quite naturally include in our discussion additional sources (even classical) of noise. To do so we utilize the procedure of a phase-space coarse graining. The main idea of the coarse-graining procedure is based on an assumption that under the influence of stochastic noise a point in the phase space cannot be localized precisely; a *probabilistic* description has to be introduced with the help of which a probability to find the point in a region Ω of the phase space is given. Using this description, a new phase-space probability density distribution $\bar{P}_{\Psi\Phi}(q,p)$ is defined as

$$\bar{P}_{\Psi\Phi}(q,p) \equiv \frac{1}{2\pi\hbar}\int_{\Omega} P_{\Psi\Phi}(q+q',p+p')G(q',p')dq'\,dp', \tag{6.12a}$$

where the weight (coarsening) function $G(q',p')$, which is normalized to unity as

$$\frac{1}{2\pi\hbar}\int_{\Omega} dq'\,dp'\,G(q',p') = 1, \tag{6.12b}$$

contains information about additional sources of stochastic fluctuations. In particular, we show in Appendix C that using the coarse-graining procedure it is possible to describe the decay of a quantum-mechanical harmonic oscillator into a phase-sensitive environment (a squeezed reservoir). In this way we are able to evaluate the sampling entropy of the decaying quantum-mechanical state $|\Psi\rangle$ which is measured by the quantum ruler in the state $|\Phi\rangle$ and thus obtain a realistic estimation of the uncertainty for our particular measurement.

Finally, we note that Leonhardt and Paul [44] have recently analyzed the problem of a reconstruction of the Wigner function from experimental data when detectors with efficiency smaller than unity are used. It has been shown that in this case a complete reconstruction of the Wigner function of the measured state is impossible because noise due to the imperfect measurement irreversibly deteriorates the information about the measured state.

VII. CONCLUSION

We have shown how operational phase-space distribution functions can be used to describe the effect of specific measurement schemes, and have shown that different choices of the states of the quantum ruler give rise to different sampling entropies. These lead to the idea of operational variances and operational uncertainty relations, as well as to entropic uncertainty relations. In this process, we reveal the true operational significance of the Wehrl entropy as a specific kind of sampling entropy that employs coherent states as quantum-ruler states. The Wehrl entropy generates an information-theoretic measure of the size of the intrinsic state fluctuations [45]. As coherent states are the most robust in dissipative environments [43], this suggests the utility of the Wehrl entropy in characterizing the decoherence process and the nature of the decoherent histories approach to quantum mechanics (e.g., [46]). We will address this problem elsewhere.

In the following paper, we will utilize the concept of sampling entropies (in particular, the Wehrl entropy) to analyze the decay of quantum coherences between coherent components of pure quantum-mechanical superposition states.

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APPENDIX A: SOME PROPERTIES OF $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$

Let us consider states $|\Psi\rangle$ and $|\Phi\rangle$ which in the phase-state basis read

$$|\Psi\rangle = \sum_{k=0}^{s} C_{\Psi}(k)|\phi_k\rangle \tag{A1}$$

and

$$|\Phi\rangle = \sum_{k=0}^{s} C_{\Phi}(k)|\phi_k\rangle. \tag{A2}$$

Using Wootter's definition [see Eq. (4.12)] of the Wigner function of a harmonic oscillator in a finite-dimensional Hilbert space we can write the Wigner functions of the state $|\Psi\rangle$ and $|\Phi\rangle$ as

$$W_{\Xi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\sum_{p=0}^{s} C_{\Xi}^{*}(j+p)C_{\Xi}(j-p) \times \exp[-2ipn_i\Delta], \qquad \Xi = \Psi,\Phi. \tag{A3}$$

In the derivation of Eq. (A3) we have used the definition of a periodic Kronecker δ

$$\bar{\delta}_{k,l} = \frac{1}{s+1}\sum_{p=0}^{s}\exp[ip(k-l)\Delta], \tag{A4}$$

which has the following properties:

$$\bar{\delta}_{k,l} = \bar{\delta}_{k,l+(s+1)} = \bar{\delta}_{k+(s+1),l}. \tag{A5}$$

Wootter's Wigner function (A3) is normalized to unity, i.e.,

$$\sum_{i,j}^{s} W_{\Xi}(n_i,\phi_j)^{(s)} = 1, \tag{A6}$$

and has the property of the usual Wigner function that after performing a summation over number (phase) variables we obtain from $W_{\Psi}(n_i,\phi_j)^{(s)}$ the phase (number) distribution of the state $|\Psi\rangle$,

$$\sum_{i=0}^{s} W_{\Psi}(n_i,\phi_j)^{(s)} = |\langle\Psi|\phi_j\rangle|^2 \equiv P(\phi_j); \tag{A7a}$$

$$\sum_{j=0}^{s} W_\Psi(n_i,\phi_j)^{(s)} = |\langle\Psi|n_i\rangle|^2 \equiv P(n_i). \quad \text{(A7b)}$$

Now we evaluate an explicit expression for the propensity $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$. If we insert Wigner functions given by Eq. (A3) into the definition (4.15) we obtain

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{(s+1)^2}\sum_{k,l=0}^{s}\sum_{p=0}^{s} C_\Psi^*(j+l+p) \times C_\Psi(j+l-p)\exp[-2ipn_{i+k}\Delta] \times \sum_{r=0}^{s} C_\Phi^*(l+r)C_\Phi(l-r) \times \exp[-2irn_k\Delta]. \quad \text{(A8)}$$

Using the definition of the periodic Kronecker δ we can rewrite Eq. (A8) as

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\sum_{l,p=0}^{s} C_\Psi^*(j+l+p)C_\Psi(j+l-p) \times \exp[-2ipn_i\Delta]C_\Phi^*(l-p)C_\Phi(l+p). \quad \text{(A9)}$$

In Eq. (A9) we can rearrange the summation over l and p using a linear substitution $l+p=a$ and $l-p=b$, so that we obtain

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\sum_{a=0}^{s} C_\Psi^*(j+a)C_\Phi(a)\exp[-ian_i\Delta] \times \sum_{b=0}^{s} C_\Psi(j+b)C_\Phi^*(b)\exp[ibn_i\Delta], \quad \text{(A10)}$$

or, alternatively,

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\left|\sum_{l=0}^{s} C_\Psi^*(j+l)C_\Phi(l)\exp[-iln_i\Delta]\right|^2 \geq 0. \quad \text{(A11)}$$

From Eq. (A11) it directly follows that the propensity $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$ is a non-negative function. Equation (A11) can still be rewritten in a simpler form. To do so, we use expressions for the amplitudes $C_\Psi^*(j+l)$ and $C_\Phi(l)$ [see Eq. (A1)]

$$C_\Psi^*(j+l) = \langle\Psi|\phi_{j+l}\rangle, \qquad C_\Phi(l) = \langle\phi_l|\Phi\rangle. \quad \text{(A12)}$$

In the finite-dimensional Hilbert space we can introduce a "rotation" (translation) operator $\hat{R}_{\hat{n}}(\phi_l)$ such that

$$|\phi_{j+l}\rangle = \hat{R}_{\hat{n}}(\phi_l)|\phi_j\rangle = \hat{R}_{\hat{n}}(\phi_j)|\phi_l\rangle, \quad \text{(A13)}$$

i.e., the operator $\hat{R}_{\hat{n}}(\phi_l)$ rotates the phase state $|\phi_j\rangle$ by an "angle" $\phi_l = l\Delta$ (in what follows we will assume that ϕ_0 in the definition of the phase ϕ_j is equal to zero). Taking into account periodic properties of the rotation operator we can find it in an explicit form (see [47]),

$$\hat{R}_{\hat{n}}(\phi_j) = \exp[i\phi_j\hat{n}], \quad \text{(A14)}$$

so that the propensity (A11) can be rewritten as

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\left|\sum_{l=0}^{s}\langle\Psi|\phi_{j+l}\rangle\langle\phi_l|\Phi\rangle\exp[-iln_i\Delta]\right|^2, \quad \text{(A15a)}$$

or

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\left|\langle\Psi|\hat{R}_{\hat{n}}(\phi_j)\left\{\sum_{l=0}^{s}|\phi_l\rangle\langle\phi_l|\exp[-iln_i\Delta]\right\}|\Phi\rangle\right|^2. \quad \text{(A15b)}$$

If we take into account the expression for the phase state $|\phi_l\rangle$ in the Fock basis [see Eq. (4.14)] then we can find the following relation:

$$\sum_{l=0}^{s}|\phi_l\rangle\langle\phi_l|\exp[-il\Delta] = \sum_{k=0}^{s-1}|k+1\rangle\langle k| + |0\rangle\langle s| = \exp[-i\hat{\phi}], \quad \text{(A16)}$$

where $\hat{\phi}$ is the Hermitian phase operator as defined by Pegg and Barnett [37],

$$\hat{\phi} = \sum_{j-0}^{s}\phi_j|\phi_j\rangle\langle\phi_j|. \quad \text{(A17)}$$

Taking into account the definition (A16) we can rewrite the propensity $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$ as [48]

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\left|\langle\Psi|\hat{R}_{\hat{n}}(\phi_j)\hat{R}^\dagger_{\hat{\phi}}(n_i)|\Phi\rangle\right|^2, \quad \text{(A18)}$$

where the operator $\hat{R}^\dagger_{\hat{\phi}}(n_i)$ describes "rotations" (translations) in the periodic finite-dimensional Fock basis and has the following form:

$$\hat{R}^\dagger_{\hat{\phi}}(n_i) = \exp[-in_i\hat{\phi}]. \quad \text{(A19)}$$

The action of the operator $\hat{R}^\dagger_{\hat{\phi}}(n_i)$ on Fock states is as follows:

$$|n_{l+i}\rangle = \hat{R}^\dagger_{\hat{\phi}}(n_i)|n_l\rangle = \hat{R}^\dagger_{\hat{\phi}}(n_l)|n_i\rangle. \quad \text{(A20)}$$

The propensity $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$ in the discrete (n,ϕ)-phase space given by Eq. (A18) has a very similar form to the propensity $P_{\Psi\Phi}(q,p)$ in the continuous (q,p)-phase space [see Eq. (3.2b)]. To see this we rewrite the displacement operator $\hat{D}(q,p) = \exp[i(p\hat{q} - q\hat{p})]$ (in this appendix we use units such that $\hbar = 1$) as a product of two translation ("rotation") operators $\hat{R}_{\hat{q}}(p)$ and $\hat{R}_{\hat{p}}(q)$,

$$\hat{D}(q,p) = \hat{R}_{\hat{q}}(p)\hat{R}^{\dagger}_{\hat{p}}(q)e^{ipq/2}, \quad \text{(A21a)}$$

where

$$\hat{R}_{\hat{q}}(p) = e^{ip\hat{q}}, \qquad \hat{R}^{\dagger}_{\hat{p}}(q) = e^{-iq\hat{p}}. \quad \text{(A21b)}$$

With the help of the definition (A21) we can rewrite the propensity $P_{\Psi\Phi}(q,p)$ given by Eq. (3.2b) in the form very similar to Eq. (A18) for the propensity $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$,

$$P_{\Psi\Phi}(q,p) = \left|\langle\Psi|\hat{R}_{\hat{q}}(p)\hat{R}^{\dagger}_{\hat{p}}(q)|\Phi\rangle\right|^2. \quad \text{(A22)}$$

The similarity between Eqs. (A18) and (A22) is transparent. Simultaneously we have to stress the difference between the two expressions. The right-hand side of Eq. (A22) is invariant under a transposition of the two translation operators $\hat{R}_{\hat{q}}(p)$ and $\hat{R}^{\dagger}_{\hat{p}}(q)$. This property of the propensity $P_{\Psi\Phi}(q,p)$ is intrinsically related to the fact that the commutator of the two generators $\hat{q}$ and $\hat{p}$ is equal to a c number. Therefore, we have

$$\begin{aligned}\left|\langle\Psi|\hat{R}_{\hat{q}}(p)\hat{R}^{\dagger}_{\hat{p}}(q)|\Phi\rangle\right|^2 &= \left|\langle\Psi|\hat{R}^{\dagger}_{\hat{p}}(q)\hat{R}_{\hat{q}}(p)|\Phi\rangle\right|^2 \\ &= \left|\langle\Psi|\hat{D}(q,p)|\Phi\rangle\right|^2. \quad \text{(A23)}\end{aligned}$$

On the other hand, the commutator of two operators $\hat{n}$ and $\hat{\phi}$ is not a c number and, therefore,

$$\begin{aligned}\left|\langle\Psi|\hat{R}_{\hat{n}}(\phi_j)\hat{R}^{\dagger}_{\hat{\phi}}(n_i)|\Phi\rangle\right|^2 &\neq \left|\langle\Psi|\hat{R}^{\dagger}_{\hat{\phi}}(n_i)\hat{R}_{\hat{n}}(\phi_j)|\Phi\rangle\right|^2 \\ &\neq \left|\langle\Psi|\hat{D}(n_i,\phi_j)|\Phi\rangle\right|^2. \quad \text{(A24)}\end{aligned}$$

The "displacement" operator $\hat{D}(n_i,\phi_j)$ in Eq. (A24) can be defined in an analogy with the operator $\hat{D}(q,p)$,

$$\hat{D}(n_i,\phi_j) = \exp[i(\phi_j\hat{n} - n_i\hat{\phi})]. \quad \text{(A25)}$$

The ambiguity in the definition of the propensity $P_{\Psi\Phi}(n_i,\phi_j)^{(s)}$ in the (n,ϕ)-phase space has its origin in Wootter's nonunique definition of the Wigner function in the discrete phase space [48].

Let us consider that the state $|\Psi\rangle$ is a phase state $|\phi_M\rangle$ and the filter state $|\Phi\rangle$ is a phase state $|\phi_0\rangle$. The Wigner functions of these states read

$$\begin{aligned}W_{\Psi}(n_i,\phi_j)^{(s)} &= \frac{1}{s+1}\delta_{j,M}, \\ W_{\Phi}(n_i,\phi_j)^{(s)} &= \frac{1}{s+1}\delta_{j,0}, \quad \text{(A26)}\end{aligned}$$

and the corresponding propensity has the form

$$P_{\Psi\Phi}(n_i,\phi_j)^{(s)} = \frac{1}{s+1}\delta_{j,M}. \quad \text{(A27a)}$$

The corresponding marginal propensities are

$$P_{\Psi\Phi}(n_i)^{(s)} = \frac{1}{s+1}, \qquad P_{\Psi\Phi}(\phi_j)^{(s)} = \delta_{j,M}. \quad \text{(A27b)}$$

Using expression (A27) we can evaluate the sampling entropy $S^{(n,\phi)}_{\Psi\Phi}(s)$ and the corresponding marginal entropies $S^{(n)}_{\Psi\Phi}(s)$ and $S^{(\phi)}_{\Psi\Phi}(s)$ [given by Eqs. (4.16) and (4.18), respectively],

$$S^{(n,\phi)}_{\Psi\Phi}(s) = S^{(n)}_{\Psi\Phi}(s) = k_B\ln(s+1), \qquad S^{(\phi)}_{\Psi\Phi}(s) = 0. \quad \text{(A28)}$$

From Eq. (A28) it follows that the phase space $|\phi_M\rangle$ is an intelligent state with respect to the given operational phase-space measurement. But unlike the eigenstate of the position operator (see Sec. V D) the phase space is not a minimum uncertainty state. Moreover, the lower bound on the sampling entropy $S^{(n,\phi)}_{\Psi\Phi}(s)$ is logarithmically diverging with the increase of the dimension s of the Hilbert space.

APPENDIX B: TRANSFORMATION PROPERTIES OF THE SAMPLING ENTROPY

Let us assume that the filter state is in a pure Gaussian state $|\Phi\rangle$ characterized by the Wigner function (5.8). Then there exists a unitary transformation $\hat{U}$ such that

$$|\beta = 0\rangle = \hat{U}|\Phi\rangle, \quad \text{(B1)}$$

where $|\beta\rangle$ is a coherent state. Consequently we can rewrite the propensity expressed by Eq. (3.2a) as

$$P_{\Psi\Phi}(q,p) = |\langle\Psi|\hat{U}^{\dagger}\hat{U}\hat{D}(q,p)\hat{U}^{\dagger}|\beta = 0\rangle|^2. \quad \text{(B2)}$$

Alternatively, we can introduce a propensity $P_{\bar{\Psi}\Phi}(q',p')$

$$P_{\bar{\Psi}\Phi}(q',p') = |\langle\bar{\Psi}|\hat{D}(q',p')|\beta = 0\rangle|^2, \quad \text{(B3)}$$

where q' and p' are the eigenvalues of the transformed operators $\hat{q}' = \hat{U}\hat{q}\hat{U}^{\dagger}$ and $\hat{p}' = \hat{U}\hat{p}\hat{U}^{\dagger}$, respectively, and $Q_{\bar{\Psi}}(q',p')$ is the Q function of the unitary transformed state $|\bar{\Psi}\rangle \equiv \hat{U}|\Psi\rangle$ characterized by the transformed Wigner function $W_{\bar{\Psi}}(q',p')$. The integration measure under the unitary transformation $\hat{U}$ is unchanged, i.e.,

$$\frac{dq\,dp}{2\pi\hbar} = \frac{dq'\,dp'}{2\pi\hbar}, \quad \text{(B4)}$$

and, therefore,

$$\begin{aligned}S^{(q,p)}_{\Psi\Phi} &= -\frac{k_B}{2\pi\hbar}\int dp\,dq\,P_{\Psi\Phi}(p,q)\ln P_{\Psi\Phi}(p,q) \\ &= -\frac{k_B}{2\pi\hbar}\int dp'\,dq'\,Q_{\bar{\Psi}}(q',p')\ln Q_{\bar{\Psi}}(q',p') = S_{\bar{\Psi}}, \quad \text{(B5)}\end{aligned}$$

where $S_{\bar{\Psi}}$ is the Wehrl entropy of the state $|\bar{\Psi}\rangle$.

APPENDIX C: DESCRIPTION OF THE DECAY OF A QUANTUM-MECHANICAL HARMONIC OSCILLATOR VIA COARSE GRAINING IN PHASE SPACE

In recent years, correlated (phase-sensitive) multimode reservoirs, sometimes called "rigged reservoirs" based on the establishment of squeezed light have been studied extensively. These reservoirs are characterized by the mean photon number N of a field mode of the reservoir at the particular frequency and by the correlation between modes which are symmetrically displaced around some carrier frequency. Correlation between modes is described by a correlation parameter M. An ideally squeezed reservoir is characterized by the equality $M^2 = N(N+1)$, while for a nonideally correlated reservoir we have $M^2 < N(N+1)$. For an uncorrelated (phase-insensitive) reservoir we put $M = 0$. For an uncorrelated reservoir at zero temperature we have $N = M = 0$. The dynamics of the field mode (the harmonic oscillator) coupled to a squeezed reservoir is in the Born and Markov approximation described by the Fokker-Planck equation, which in the interaction picture can be written as

$$\frac{\partial Q(\beta,t)}{\partial t} = \gamma\left[V\frac{\partial^2}{\partial\beta^*\partial\beta} + \frac{W}{2}\left(\beta^*\frac{\partial}{\partial\beta^*} + \beta\frac{\partial}{\partial\beta}\right) + \frac{M}{2}\frac{\partial^2}{\partial\beta^2} + \frac{M^*}{2}\frac{\partial^2}{\partial\beta^{*2}}\right]Q(\beta,t), \qquad (C1)$$

where γ is the coupling constant between the field and the phase-sensitive reservoir. The parameters V and W are defined as $V = N + 1$ and $W = 1$, respectively. (If instead the environment plays the role of an amplifier, $V = N$, $W = -1$.)

The solution of the Fokker-Planck equation (C1) can be obtained via "coarsening" of the initial Q function of the harmonic oscillator. Phase-space coarsening is generally associated with a measurement process with a nonunitary effective Hamiltonian. Analogously, the attenuation process can be considered as a model for a quantum-mechanical measurement.

The main idea of the coarse-graining procedure is based on the assumption that a point in phase space cannot be localized precisely but a probabilistic description is introduced, with the help of which the probability to find the point in a region Ω of the phase space is computed. Instead of considering a region with sharp boundaries, we will analyze the coarsened quasiprobability function defined as a convolution of the quasiprobability function with a properly chosen weight function. In particular, we will consider a Gaussian weight function $G(u,v)$,

$$G(u,v) = \frac{1}{\Delta_r\Delta_i}\exp\left(-\frac{u^2}{2\Delta_r^2} - \frac{v^2}{2\Delta_i^2}\right), \qquad (C2)$$

normalized as

$$\frac{1}{\pi}\int\int_{-\infty}^{\infty} G(u,v)\,du\,dv = 1, \qquad (C3)$$

so that the coarsened quasiprobability function $\bar{Q}(\beta)$ can be defined as

$$\bar{Q}(\beta_r,\beta_i) = \frac{1}{\pi}\int\int_{-\infty}^{\infty} G(u,v)Q(\beta_r+u,\beta_i+v)\,du\,dv. \qquad (C4)$$

The solution $Q(\beta',t)$ of the Fokker-Planck equation (C1) can be expressed through the coarsened Q function of the initial state as follows [49]:

$$\bar{Q}(\beta_r',\beta_i') = \frac{1}{\pi^2}\int_{-\infty}^{\infty} d\beta_r\,\delta(\beta_r' - \mu^{1/2}\beta_r) \times \int_{-\infty}^{\infty} d\beta_i\,\delta(\beta_i' - \mu^{1/2}\beta_i) \times \int\int_{-\infty}^{\infty} G(u,v)Q(\beta_r+u,\beta_i+v)\,du\,dv, \qquad (C5)$$

where the Gaussian weight function $G(u,v)$ (C2) is characterized by the time-dependent parameters $\Delta_r(t)$ and $\Delta_i(t)$

$$\Delta_r^2(t) = \frac{1-\mu}{2\mu}\left[1+N+M\right], \qquad (C6a)$$

$$\Delta_i^2(t) = \frac{1-\mu}{2\mu}\left[1+N-M\right], \qquad (C6b)$$

with $\mu = \exp(-\gamma t)$. We note here that the two integrals containing δ functions in the expression (C5) correspond to the classical dynamics of a point in the phase space (i.e., the decay of the classical harmonic oscillator), while the coarsening corresponds to added fluctuation noise. The expression (C5) is valid for *any* initial state of the harmonic oscillator.

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