Propensities in discrete phase spaces: Q function of a state in a finite-dimensional Hilbert space

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(Received 26 April 1995)

We present a Q function of a state of a quantum-mechanical system in a finite-dimensional Hilbert space. This discrete Q function is defined with the help of the Wódkiewicz concept of propensities, i.e., we define the Q function as a discrete convolution of two Wigner functions based on Wootters's formalism, one of the state itself and one of the filter state. The discrete Q function takes nonnegative values in all "points" of the discrete phase space and is normalized and it is possible to reconstruct from it the density operator of the state under consideration. We analyze Q-function graphs for several states of interest.

PACS number(s): 42.50.Dv, 03.65.Bz, 05.30.Ch

I. INTRODUCTION

Phase-space methods have recently been widely used in quantum mechanics and in quantum optics. Within the framework of the phase-space formalism, a state of a quantum-mechanical system can be completely described with the help of quasiprobability density distributions. In classical mechanics dynamical variables can be measured to arbitrary accuracy. This, in principle, permits a precise measurement of conjugated variables such as the position and momentum. Consequently, a joint probability density distribution can be constructed that describes a classical system. On the other hand, in quantum mechanics two conjugated observables (operators) do not commute, which results in the fact that they cannot be measured *simultaneously* with infinite precision, i.e., there is always a fundamental limit with which a "point" of the quantum-mechanical phase space can be determined. Among other consequences of the fact that two conjugated observables do not commute is the lack of a unique rule by means of which quantum and classical variables can be associated. Therefore a number of (quasi)probability density distributions in quantum-mechanical phase space can be defined. To be more specific, depending on the operator ordering, various (quasi)probability density distributions can be defined, of which the best known are the Wigner function [1], the Husimi (Q) function [2], and the Glauber-Sudarshan (P) function [3], reflecting the symmetric (Weyl), antinormal, and normal ordering of operators in the corresponding characteristic functions [4]. The Wigner function plays an exceptional role among all quasiprobability density distributions. First, it generates proper marginal distributions for individual phase-space variables. Second, under the action of linear canonical transformations, the Wigner function behaves exactly in the same way as the classical probability density distributions [5]. The Wigner function contains *complete* information about the state of the system, i.e., it carries the same information as the density operator or the corresponding wave function. From the Wigner function one can evaluate all (symmetrically ordered) moments of the system operators. On the other hand, the inverse is also valid. It means that from the knowledge of the *complete* set of moments of system operators, the Wigner function (as well as the density operator) can be determined uniquely [6].

The Q function also plays a very important role in quantum mechanics. In particular, from an operational point of view it can be associated with a simultaneous measurement of two conjugated observables over the quantum-mechanical state described by a given Wigner function. To describe a process of a simultaneous measurement of two noncommuting observables Wódkiewicz [7] has proposed a formalism based on an operational probability density distribution that explicitly takes into account the action of the measurement device modeled as a "filter" (quantum ruler). A particular choice of the state of the ruler samples a specific type of accessible information concerning the system, i.e., information about the system is biased by the filtering process. The quantum-mechanical noise induced by filtering formally results in smoothing of the original Wigner function of the measured state [2,8], so that the operational probability density distribution can be expressed as a convolution of the original Wigner function and the Wigner function of the filter state [7]. In particular, if the filter is considered to be in a vacuum state, then the corresponding operational probability density distributions is equal to the Husimi (Q) function [2].

Recently, due attention has been paid to the investigation of quantum systems in finite-dimensional Hilbert spaces (FDHS's). These quantum systems can be associated with spin systems or with angular momentum, or recently they have been widely used for an introduction of the Hermitian phase operator into quantum mechanics [9]. States of quantum-mechanical systems in FDHS's can be described by N-dimensional vectors (pure states) or by corresponding density operators (which are equal to $N \times N$ matrices). At this point a natural question arises: Is it possible to apply a phase-space formalism for a description of quantum-mechanical states in FDHS's? The pioneering work in this direction has been done by Wootters [10] and by Galetti and de Toledo Piza [11]. These authors have introduced the Wigner function on discrete phase spaces that are associated with states in FDHS's. Vaccaro and Pegg [12] have applied Wootters's approach (see below) and they have introduced a Wigner function for number and phase within the framework of the Pegg-Barnett formalism (see also [13]).

The purpose of the present paper is to generalize Wootters's formalism and to introduce other discrete phase-space (quasi)probability distributions associated with quantum-mechanical states in FDHS's. In particular, we introduce the Q function corresponding to the Wigner function in the discrete phase space. We utilize recent results by Bužek, Keitel, and Knight [14], who have shown that the Wódkiewicz propensities can be defined also for states in the FDHS's.

The paper is organized as follows. In Sec. II we summarize the basic properties of FDHS quantum mechanics, including the phase-space treatment. Section III is devoted to general properties of propensities associated with states in the FDHS's. Finally, in Sec. IV we analyze properties of discrete Q functions. We also present graphs of Q functions for various states of interest.

II. QUANTUM MECHANICS IN THE FINITE-DIMENSIONAL HILBERT SPACE

Let the N-dimensional Hilbert space be spanned by N orthogonal normalized vectors $|u_k\rangle$ and equivalently by N vectors $|v_l\rangle$, $k,l = 0,\ldots,N-1$, where both bases are connected by the discrete Fourier transform

$$|u_k\rangle = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\exp\left(-i\frac{2\pi}{N}kl\right)|v_l\rangle,$$
$$|v_l\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\exp\left(i\frac{2\pi}{N}kl\right)|u_k\rangle. \tag{1}$$

It can be assumed that these bases are sets of eigenvectors of noncommuting operators $\hat{U}$ and $\hat{V}$:

$$\hat{U}|u_k\rangle = k|u_k\rangle, \quad \hat{V}|v_l\rangle = l|v_l\rangle. \tag{2}$$

For instance, we can assume that the operators $\hat{U}$ and $\hat{V}$ are related to a discrete position and momentum of a particle on a ring with a finite number of equidistant sites. Alternatively, $\hat{U}$ and $\hat{V}$ can be associated with a photon number and phase in the Pegg-Barnett formalism. The squared absolute values of the scalar product of eigenkets (2) do not depend on the indices k,l:

$$|\langle u_k|v_l\rangle|^2 = 1/N, \tag{3}$$

which means that pairs (k,l) form a discrete phase space [i.e., pairs (k,l) represent "points" of the discrete phase space] on which Wigner function can be defined [10]. Next we introduce operators that rotate (cyclic permute) the basis vectors [11]:

$$\hat{R}_u(n)|u_k\rangle = |u_{k+n}\rangle, \quad \hat{R}_v(m)|v_l\rangle = |v_{l+m}\rangle, \tag{4}$$

where the sums of indices are taken mod N (this summation rule is considered throughout the paper). It is seen that the operators $\hat{R}_u(n)$ and $\hat{R}_v(m)$ can be expressed as powers of the operators $\hat{R}_u(1)$ and $\hat{R}_v(1)$, respectively,

$$\hat{R}_u(n) = \hat{R}_u^n(1), \quad \hat{R}_v(m) = \hat{R}_v^m(1). \tag{5}$$

In the U basis these operators can be expressed as

$$\langle u_k|\hat{R}_u(n)|u_l\rangle = \delta_{k+n,l},$$
$$\langle u_k|\hat{R}_v(m)|u_l\rangle = \delta_{k,l}\exp\left(i\frac{2\pi}{N}ml\right). \tag{6}$$

Moreover, these operators fulfill Weyl's commutation relation [15–17]

$$\hat{R}_u(n)\hat{R}_v(m) = \exp\left(i\frac{2\pi}{N}mn\right)\hat{R}_v(m)\hat{R}_u(n); \tag{7}$$

although they do not commute, they form a representation of an Abelian group in a ray space. We can displace a state in arbitrary order using $\hat{R}_u(n)\hat{R}_v(m)$ or $\hat{R}_v(m)\hat{R}_u(n)$; the resulting state will be the same: the corresponding kets will differ only by an unessential multiplicative factor. We see that the product $\hat{R}_u(n)\hat{R}_v(m)$ acts as a displacement operator in the phase space (k,l) [14].

It is interesting to compare these displacements with the displacement operator $\hat{D}(q,p)$ in the continuous (q,p) phase space,

$$\hat{D}(q,p) \equiv \exp\left(\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right). \tag{8}$$

Because the commutator of $\hat{q}$ and $\hat{p}$ is a c number, we can use the Baker-Hausdorff identity and write the $\hat{D}$ operator in forms

$$\hat{D}(q,p) = \exp\left(\frac{i}{\hbar}p\hat{q}\right)\exp\left(-\frac{i}{\hbar}q\hat{p}\right)\exp\left(i\frac{pq}{2\hbar}\right)$$
$$= \exp\left(-\frac{i}{\hbar}q\hat{p}\right)\exp\left(\frac{i}{\hbar}p\hat{q}\right)\exp\left(-i\frac{pq}{2\hbar}\right). \tag{9}$$

The $\hat{R}$ operators in the discrete phase space can also be expressed in an exponential form

$$\hat{R}_u(n) = \exp\left(-i\frac{2\pi}{N}n\hat{V}\right),$$
$$\hat{R}_v(m) = \exp\left(i\frac{2\pi}{N}m\hat{U}\right). \tag{10}$$

A "displacement" in the discrete phase space can be described in a very close analogy with a continuous displacement in the (q,p) phase space as described by

Eq. (9), i.e.,

$$\exp\left(i\frac{2\pi}{N}m\hat{U}\right)\exp\left(-i\frac{2\pi}{N}n\hat{V}\right)\exp\left(i\frac{\pi}{N}mn\right)$$

$$=\exp\left(-i\frac{2\pi}{N}n\hat{V}\right)\exp\left(i\frac{2\pi}{N}m\hat{U}\right)\exp\left(-i\frac{\pi}{N}mn\right). \tag{11}$$

We note here that the dimension N of the Hilbert space plays a role similar to the Planck constant $2\pi\hbar$ [10]. Unfortunately, the commutator $[\hat{U},\hat{V}]$ is not a c number, so we cannot write the displacement in a symmetric form such as (8); the symmetrically constructed operators $\hat{D}^{(\mathrm{sym})}(m,n)$

$$\hat{D}^{(\mathrm{sym})}(m,n)\equiv\exp\left(i\frac{2\pi}{N}(m\hat{U}-n\hat{V})\right), \tag{12}$$

do not form the required group representation.

Now following Wootters [10], we introduce the Wigner-function formalism for quantum-mechanical systems in the FDHS. The W function in a phase-space point (k,l) is proportional to the mean value of a Hermitian phase-space point operator $\hat{A}(k,l)$, which can be written in the U representation

$$\langle u_r|\hat{A}(k,l)|u_s\rangle=\delta_{2k,r+s}\exp\left(i\frac{2\pi}{N}l(r-s)\right) \tag{13}$$

for a prime number $N\geq 3$. A different expression is valid for $N=2$ and also for composite numbers N, where we should work with direct products of the Hilbert subspaces (for details see [10,11]). The W function now reads

$$W(k,l)=(1/N)\langle\hat{A}(k,l)\rangle\equiv\mathrm{Tr}\,[\hat{\rho}\hat{A}(k,l)]. \tag{14}$$

This function is real, but it can take negative values. Nevertheless, sums of its values over "lines," i.e., sets of points (k,l) for which $(ak+bl+c)\bmod N=0$ holds, are always non-negative and can be understood as marginal probabilities that can be obtained as a result of a measurement. It is worth stressing the importance of calculations with mod N: the phase space is then topologically identical to a set of points on a torus; the lines are points of closed toroidal spirales. Inverting the expression (14), we obtain the density operator

$$\hat{\rho}=\sum_{k,l}W(k,l)\hat{A}(k,l), \tag{15}$$

which means that density operator and the W function represent equivalent descriptions of the quantum system state in the FDHS. An important property of W functions is that an overlap of two states (squared modulus of scalar product in the pure states case) can be expressed as the overlap of corresponding W functions

$$\mathrm{Tr}\,(\hat{\rho}\hat{\rho}')=N\sum_{k,l}W(k,l)W'(k,l). \tag{16}$$

From Eq. (16) we again see the analogy between N in the discrete case and $2\pi\hbar$ in the continuous case: roughly speaking an area $\mathcal{A}$ of continuous phase space contains $\mathcal{A}/(2\pi\hbar)$ orthogonal states; similarly if $\mathcal{A}$ means the number of phase-space points in the discrete case, then $\mathcal{A}$ contains $\mathcal{A}/N$ orthogonal states [10].

III. PROPENSITIES AND THE Q FUNCTION IN THE FDHS: GENERAL PROPERTIES

According to Wódkiewicz [7], propensity means the tendency (or probability) of a measured object to take up certain states prescribed by a measuring device. Let the measuring device — the so-called quantum ruler—be in a pure state $|\Phi\rangle$. The quantum-ruler state can be "shifted" by an action of some generalized displacement operator $\hat{D}(g)$, where g is an element of a group G. If the measured system is in a pure state $|\Psi\rangle$, then its probability to be in the ruler state shifted by g (i.e., the propensity) is

$$P_{\Phi,\Psi}(g)=|\langle\Psi|\hat{D}(g)|\Phi\rangle|^2, \tag{17}$$

whereas if the system is in a mixed state described by the density operator $\hat{\rho}$, the propensity is

$$P_{\Phi,\rho}(g)=\mathrm{Tr}\left[\hat{\rho}\hat{D}(g)|\Phi\rangle\langle\Phi|\hat{D}^{+}(g)\right]. \tag{18}$$

In our case of finite-dimensional Hilbert space, the considered group G will be formed by discrete translations on a torus: if $g_1\equiv(n_1,m_1)$ and $g_2\equiv(n_2,m_2)$ are elements of G, then their group product is $g_1g_2\equiv[(n_1+n_2)\mathrm{mod}N,(m_1+m_2)\mathrm{mod}N]$. The corresponding displacement operator is then given by (7) or (11): we see that the displacement is not a representation of the group G in the studied Hilbert space; nevertheless, it is representation of this group in the ray space, which enables us to define the propensity uniquely. For a pure state $|\Psi\rangle$ we can write the propensity in a form (see [14])

$$P_{\Phi,\Psi}(n,m)=|\langle\Psi|\hat{R}_u(n)\hat{R}_v(m)|\Phi\rangle|^2. \tag{19}$$

In the case of a statistical mixture described by the density operator $\hat{\rho}$ the corresponding propensity reads

$$P_{\Phi,\rho}(n,m)=\mathrm{Tr}\left[\hat{\rho}\hat{R}_u(n)\hat{R}_v(m)|\Phi\rangle\langle\Phi|\hat{R}_v(-m)\hat{R}_u(-n)\right]. \tag{20}$$

It has been show in [14] that if $W_\rho(r,s)$ is the W function of the quantum-mechanical system and $W_\Phi(r,s)$ is the Wigner function of the quantum ruler, the corresponding propensity can be written as the discrete convolution

$$P_{\Phi,\rho}(n,m)=\sum_{r,s}W_\rho(r,s)W_\Phi(r-n,s-m). \tag{21}$$

A. Q function in discrete phase space

In an analogy with a continuous (q,p) phase space when the Q function is defined as propensity of the state to be in the vacuum state, we will define the discrete Q function as the propensity (21)

$$Q(n,m)\equiv P_{\Phi,\rho}(n,m), \tag{22}$$

with the quantum ruler being in a vacuum state. The problem is how to define a vacuum state corresponding to a FDHS and how to describe this state in the corresponding discrete phase space (i.e., determine the Wigner function of this vacuum state).

In Sec. IV we will address the problem how to find a proper (ruler) vacuum state. We will discuss properties of discrete Q functions related to various ruler states. Here we will analyze some general features of this function.

If we assume that the ruler state $|\Phi\rangle$ is chosen (i.e., the vacuum state is specified), then the Q function has the following properties: (i) it is uniquely defined, (ii) it is non-negative, (iii) it is normalized to N (see the Appendix)

$$\sum_{n,m} Q(n,m) = N, \tag{23}$$

and (iv) for *properly* chosen ruler states $|\Phi\rangle$ information about a system state can be completely reconstructed from the corresponding Q function. Here *properly chosen* means that the discrete Fourier transform of the ruler state Wigner function $\tilde{W}_\Phi(k,l)$ has no zero values. The proof is essentially the same as the proof that the Fourier transform of a convolution is a product of Fourier transforms. To be more specific, we can write the Fourier transform of the Q function in the form

$$\tilde{Q}(k,l) = \frac{1}{N}\sum_{n,m} Q(n,m)\exp\left(i\frac{2\pi}{N}(kn+lm)\right) \tag{24}$$

and using Eq. (21) we will write the Q function as the overlap of the W functions

$$\begin{aligned}\tilde{Q}(k,l) &= \frac{1}{N}\sum_{r,s} W_\rho(r,s)\sum_{n,m} W_\Phi(r-n,s-m)\\ &\quad\times\exp\left(i\frac{2\pi}{N}(kn+lm)\right)\\ &= \frac{1}{N}\sum_{r,s} W_\rho(r,s)\exp\left(i\frac{2\pi}{N}(rk+sl)\right)\\ &\quad\times\sum_{t,p} W_\Phi(t,p)\exp\left(-i\frac{2\pi}{N}(tk+pl)\right).\end{aligned} \tag{25}$$

Since the W function is always real, the last sum can be written as the complex conjugate of the Fourier transformed W_Φ, i.e., $N\tilde{W}_\Phi^*(k,l)$. Thus

$$\tilde{Q}(k,l) = N\tilde{W}_\rho(k,l)\tilde{W}_\Phi^*(k,l). \tag{26}$$

Inverting this equation we arrive at the reconstruction formula

$$W_\rho(n,m) = \frac{1}{N^2}\sum_{k,l}\frac{\tilde{Q}(k,l)}{\tilde{W}_\Phi^*(k,l)}\exp\left(-i\frac{2\pi}{N}(kn+lm)\right), \tag{27}$$

which can always be done if the denominators are nonzero. It is worth noting that there exist ruler states for which $\tilde{W}_\Phi$ has zero elements. An example is any state $|u_t\rangle$; its W function is

$$W_{u_t}(n,m) = \frac{1}{N}\delta_{n,t} \tag{28}$$

and the Fourier transform of this W function is

$$\tilde{W}_{u_t}(k,l) = \frac{1}{N}\delta_{l,0}\exp\left(i\frac{2\pi}{N}kt\right), \tag{29}$$

which has only N nonzero elements. It is clear that for such a ruler state the Q function reduces to the probability distribution of the observable $\hat{U}$, i.e., $Q(n,m) = P(U=n)$, which does not contain full information about the state. Nevertheless, for a properly chosen ruler state the formula (27) together with (15) enables us to reconstruct the density matrix from the measured propensities $Q(n,m)$.

IV. ANALYSIS OF PARTICULAR Q FUNCTIONS

Now we are in a position to choose a particular ruler state and to construct from it the discrete Q functions for various states. We think that the following requirements on the ruler state are reasonable: (i) it should be in some sense centered at the phase space point $(0,0)$; (ii) it should be "symmetrical" with regard to the quantities U and V, i.e., its wave function should have a similar form in both representations (perhaps up to scalings); and (iii) it should be in some sense a minimum uncertainty state, which means that in the phase space it should be represented by a peak that is as narrow as possible. These requirements follow the properties of the continuous Q function and its quantum ruler, the vacuum state. The narrowness of the ruler is useful also for the state reconstruction: a narrow Wigner function has a broad Fourier transform; therefore it has smaller values present in the denominator of (27), which would increase the measurement errors of $Q(n,m)$.

Now we should be more precise in specifying the uncertainty that has to be minimized; we will devote a few words to this problem. Usually one considers uncertainty as the square root of *variance*, i.e., for an observable X its uncertainty is $\Delta X \equiv \langle(\hat{X}-\langle\hat{X}\rangle)^2\rangle^{1/2}$. Nevertheless, in the case of angular or phase variables (e.g., our quantities U and V with the mod N summing rule) it has been observed that there are problems with the unique definition of mean value and variance. To be specific, if we shift the phase window over which the mean is calculated, these quantities change. One could of course define them uniquely by postulating that such a phase window is chosen for which the variance takes a minimal value, but this step is rather artificial. Instead, another measure of phase uncertainty was suggested by Bandilla and Paul [18] (see also [19]), the so-called *dispersion*. This uncertainty measure is useful for any quantity with circular symmetry; here we will present it in the case of $\hat{U}$. It is always possible to find a mean value $\langle\exp\left(i\frac{2\pi}{N}\hat{U}\right)\rangle$ and write it in the goniometric form

$$\left\langle \exp\left(i\frac{2\pi}{N}\hat{U}\right)\right\rangle = \mathrm{R}e^{i\bar{\varphi}}, \tag{30}$$

where $\bar{\varphi}$ is the uniquely defined (for $R > 0$) mean (or central) phase [as a mean value of $\hat{U}$ we could treat $N\bar{\varphi}/(2\pi)$]. The dispersion of the phase is then defined as

$$\sigma^2_{\varphi,U} \equiv 1 - R^2. \tag{31}$$

This quantity takes values between zero (sharp value of the phase) and unity (uniform phase spread). It was shown in Ref. [20] that a very intuitive measure of a phase uncertainty $\Delta\varphi_U$ is simply connected to the dispersion

$$\Delta\varphi_U \equiv \arcsin\sigma_{\varphi,U}, \tag{32}$$

which takes values between zero and $\pi/2$. These uncertainty measures have the advantages that there is a simple rule for the uncertainty of a sum of independent quantities and a clear connection between the uncertainty and probability expressed by a version of the Chebyshev inequality. Therefore, in this paper we will work mostly with these quantities and in searching for the ruler state we will require a minimization of dispersions, although other uncertainty measures are possible as well.

A state that fulfills all the requirements (i)–(iii) can be found as the ground state of the Hamiltonian $\hat{H}_0$ (see Ref. [20])

$$\hat{H}_0 \equiv -\cos\left(\frac{2\pi}{N}\hat{U}\right) - \cos\left(\frac{2\pi}{N}\hat{V}\right). \tag{33}$$

It is clear that the ground state $|\Phi_0\rangle$ of the Hamiltonian (33) fulfills properties (i) and (ii). Now let us pay attention to property (iii): If we assume that the mean phases $\bar{\varphi}_U$ and $\bar{\varphi}_V$ in the state $|\Phi_0\rangle$ are equal to zero, then the dispersions can be written as

$$\sigma^2_{\varphi,U} = 1 - \left\langle \cos\left(\frac{2\pi}{N}\hat{U}\right)\right\rangle^2,$$
$$\sigma^2_{\varphi,V} = 1 - \left\langle \cos\left(\frac{2\pi}{N}\hat{V}\right)\right\rangle^2. \tag{34}$$

From our definition it follows that in the ground state of $\hat{H}_0$ the sum of the dispersions is minimized. Consequently, there cannot exist a state that would have the same dispersions for $\hat{U}$ and $\hat{V}$ and these dispersions would be less than those of $|\Phi_0\rangle$; hence $|\Phi_0\rangle$ is a minimum uncertainty state with respect to dispersions. A similar situation is in the continuous case, when the ruler state fulfilling requirements (i)–(iii) could be defined as the ground state of the harmonic-oscillator Hamiltonian

$$\hat{H}_{\rm osc} = \tfrac{1}{2}\hat{x}^2 + \tfrac{1}{2}\hat{p}^2 \tag{35}$$

($m = \hbar = \omega = 1$). Here condition (iii) is considered with respect to variances, of course. The Hamiltonian $\hat{H}_0$ has in the U representation a simple form

$$\langle u_k|\hat{H}_0|u_l\rangle = -\delta_{k,l}\cos\left(\frac{2\pi}{N}k\right) - \frac{1}{2}\left(\delta_{k+1,l} + \delta_{k,l+1}\right). \tag{36}$$

Physically we can interpret this Hamiltonian as follows. A particle moves along a ring with a finite number of sites; the ring is placed in a uniform force field parallel to the plane of the ring. Therefore the potential energy is the same as that for the mathematical pendulum—proportional to $-\cos(2\pi U/N)$—and the kinetic energy is given by the jumps between neighboring sites—in the Hamiltonian represented by the $\delta_{k\pm1,l}$ terms. The eigenstates of such a Hamiltonian can always be calculated (at least numerically); from them we choose that corresponding to the minimum eigenvalue, our ruler state. In Fig. 1 we can see Wigner functions of such ground states for several dimensions N. It is seen that the W function consists of three peaks and one anti-peak, placed against the main (central) peak. [We have chosen the axes to be numbered from $-(N-1)/2$ to $(N-1)/2$ so that the main peak is centered; recall the modulo N summation.] From the position of the peaks we see that the marginal distribution of n (m) is effectively nonzero only around $n \approx 0$ ($m \approx 0$); for n,m near $N/2$ the contributions of the auxiliary peaks and the antipeak cancel each other. Increasing the dimension N to infinity, the antipeak and the two auxiliary peaks are taken away to infinity and we find the well known Gaussian shape of the vacuum state. The width of the main peak increases with $\sqrt{N}$, again we see the correspondence between N and $2\pi\hbar$ as the vacuum state in the continuous case has width $\sqrt{\hbar/2}$. Note that we get very similar W functions (for $N > 10$ almost identical) for ruler states defined as ground states of the Hamiltonians

$$\hat{H}_1 \equiv (1/2)\hat{U}^2 + (1/2)\hat{V}^2. \tag{37}$$

Such states minimize variances of $\hat{U}$ and $\hat{V}$ and may also have other useful properties for a definition of the Q function. We also turn our attention to the fact that the shape of the Wigner function under consideration is invariant with respect to rotations in the discrete phase space; see Fig. 1(d), where the W function of a shifted ruler is presented.

In Fig. 2 we plot Q functions for several important states. In particular, in Figs. 2(a) and 2(b) the Q function of the "position" or "number" states $|u_n\rangle$ are presented: Fig. 2(a) shows the Q function of the Fock vacuum $|u_0\rangle$: notice the peak in the n distribution located at $n = 0$, which continues at the "opposite end" of the phase space at $n \approx N-1$; in Fig. 2(b) we can see the number state $|u_5\rangle$. In Fig. 2(c) the Q function of "momentum" or "phase" states $|v_m\rangle$ is plotted. In Figs. 2(a)–2(c), as well as in Fig. 3, we have chosen the axes numbering to be from 0 to $N-1$ because the discussed states are relevant to the Pegg-Barnett model and n really corresponds to a photon number (on the other hand, we can consider this "shift" of axis parameters as another choice of the phase-number window). The Q function of the quantum-ruler state $|\Phi_0\rangle$, i.e., the vacuum state of the Hamiltonian (33), is shown in Fig. 2(d). This is a single-

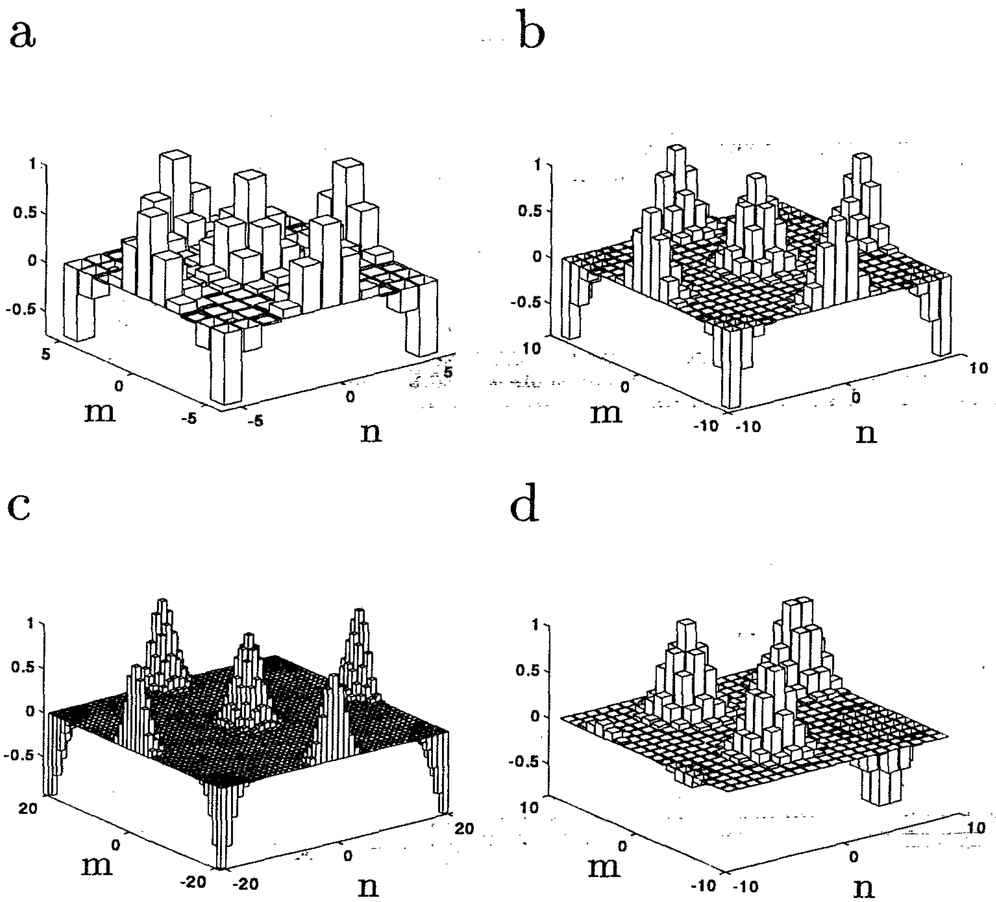

FIG. 1. Wigner functions (14) of the quantum ruler state $|\Phi_0\rangle$. (a)–(c) show the ruler state as the ground state of the Hamiltonian (33): (a) $N = 11$, (b) $N = 19$, and (c) $N = 41$. In (d) the ruler state for $N = 19$ is shifted by the displacement operator $\hat{R}_u(-2)\hat{R}_v(5)$.

peaked function very similar to the Q function of the vacuum state of the ordinary harmonic oscillator (except the mean photon number in this case is not equal to zero, but is equal to $N/2$). Under the action of the rotation operators $\hat{R}_u(n)$ and $\hat{R}_v(m)$, we obtain from the vacuum state $|\Phi_0\rangle$ the state

$$|\Phi_{(m,n)}\rangle = \hat{R}_u(n)\hat{R}_v(m)|\Phi_0\rangle. \tag{38}$$

The Q function of this shifted (rotated) state has the *same* shape as the Q function of the vacuum state $|\Phi_0\rangle$ except it is shifted in the discrete phase space. In a sense we can consider the state $|\Phi(m,n)\rangle$ (38) as a "coherent state" of a harmonic oscillator in a finite-dimensional Hilbert space. In Figs. 2(e) and 2(f) we plot Q functions of shifted vacuum states $|\Phi_0\rangle$ for different values of m and n. In particular, the Q function in Fig. 2(e) corresponds to the Wigner function shown in Fig. 1(d).

Recently, two different definitions of coherent states of harmonic oscillator in finite-dimensional Hilbert spaces have been discussed in a framework of the Pegg-Barnett formalism. In this case the coordinate n should be treated as the photon number and m as the index of phase φ_m. One definition of coherent states generalized to the case of the finite-dimensional Hilbert spaces was proposed by Bužek *et al.* [21] and studied further by Miranowicz *et al.* [22]: this definition is very similar to the definition of coherent states in the semi-infinite-dimensional Hilbert space

$$|\alpha\rangle \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|u_0\rangle. \tag{39}$$

Here the creation operator is defined as

$$\hat{a}^\dagger|u_n\rangle = \sqrt{n+1}|u_{n+1}\rangle, \quad n < N-1 \tag{40}$$

and

$$\hat{a}^\dagger|u_{N-1}\rangle = 0 \tag{41}$$

and the annihilation operator $\hat{a}$ is its conjugate. The Q functions of these coherent states are depicted in Figs. 3(a) and 3(b) for two values of $\alpha = -0.5$ and $\alpha = -4.4$, respectively.

Another definition of the FDHS coherent states was proposed by Kuang *et al.* [23], essentially based on truncating the Fock expansion of the usual coherent states. We will call such states truncated coherent states; they can be defined as

$$|\bar{\alpha}\rangle \equiv \mathcal{N}\exp(\bar{\alpha}\hat{a}^\dagger)|u_0\rangle, \tag{42}$$

where $\mathcal{N}$ is a normalization constant and the creation operator is the same as (40) and (41). Examples of Q functions of the truncated coherent states are in Figs. 3(c) and 3(d) for two values of $\bar{\alpha} = -0.5$ and $\bar{\alpha} = -4.4$, respectively. Note that for small $|\alpha|$ (more precisely for

$|\alpha|^2 \ll N/2$) both the coherent states (39) and the truncated coherent states (42) are very similar [in Figs. 3(a), 3(c) $\alpha = \bar{\alpha} = -0.5$]; we can recognize the deformed shape of the Fock vacuum from Fig. 2(a). For higher excitations the unusual property (41) of the finite-dimensional creation operator becomes apparent and both definitions give different results. The coherent states (39) behave quasiperiodically with increasing $|\alpha|$, several times returning very close to the starting vacuum state, whereas the truncated coherent states (42) monotonically increase their mean photon number and for $|\bar{\alpha}| \gg N$ tend to the number state $|u_{N-1}\rangle$. In Figs. 3(b) and 3(d) we can see the situation for $\alpha = \bar{\alpha} = -4.4$; the coherent state is just in the middle of its returning quasiperiod and is very close to an even state: we observe a typical two-peak structure of even states. More details about both approaches to the coherent states can be found in [13], where their number-phase Wigner functions are also presented.

Within the Pegg-Barnett formalism all physical quantities are usually analyzed in the limit $N \to \infty$; therefore it is instructive to discuss the behavior of the number-phase Q function in this limit. The widths of our quantum-ruler state increase with $\sqrt{N}$, which means that the noise in photon number measurements increases in this way, whereas the phase uncertainty behaves as $\sqrt{N}2\pi/N$, i.e., it decreases with $\sqrt{N}$. This behavior is reasonable if we measure sufficiently strong signals, with mean photon numbers comparable to N; the relative er-

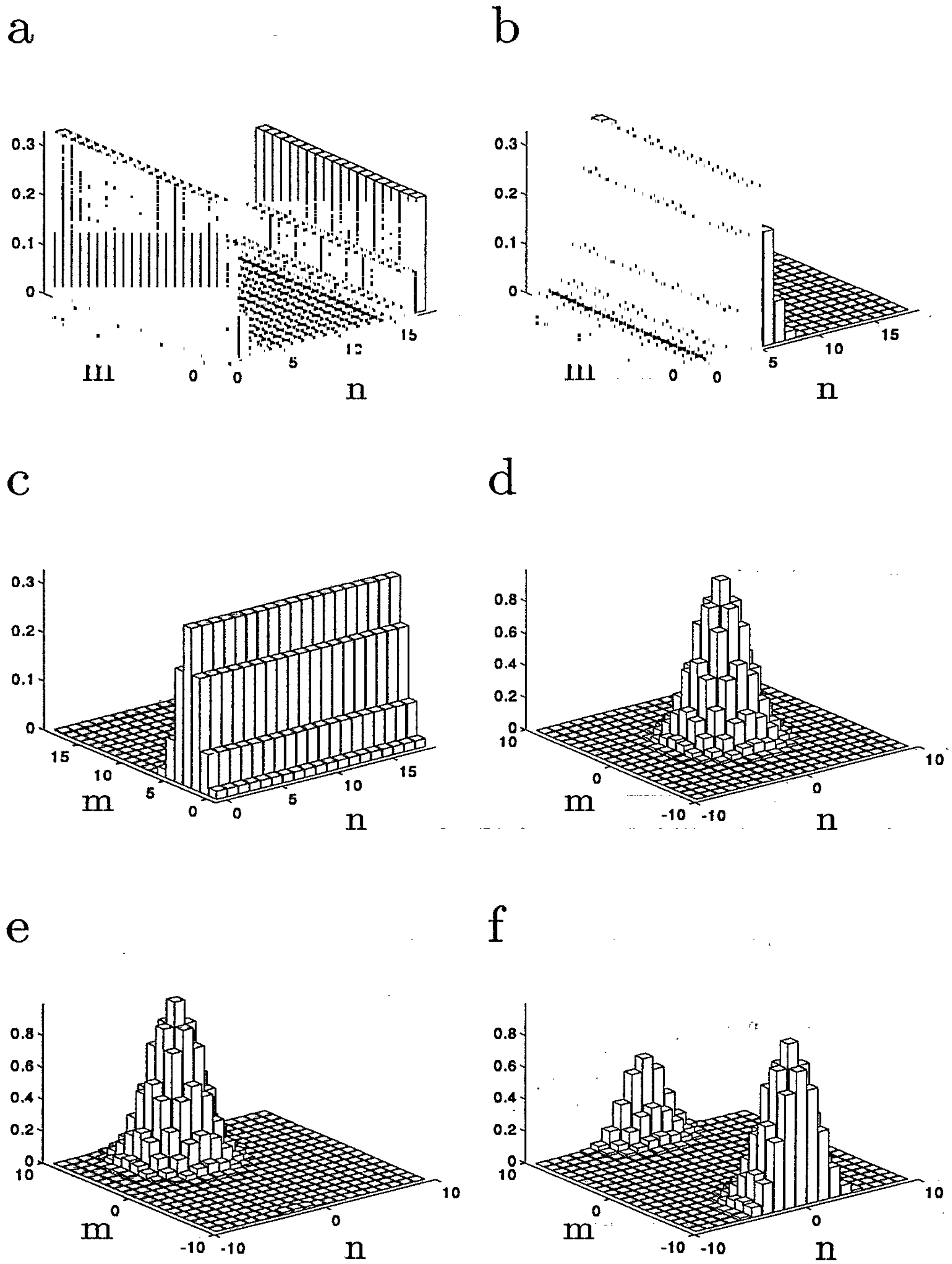

FIG. 2. Q-functions of several states for $N = 19$: (a) $|u_0\rangle$ (the Fock vacuum), (b) $|u_5\rangle$ (a Fock number state), and (c) $|v_3\rangle$ (a phase state), and (d) ruler state $|\Phi\rangle$. In (e) and (f), respectively, we plot the Q functions of the rotated (displaced) vacuum states $\hat{R}_u(n)\hat{R}_v(m)|\Phi_0\rangle$, which can be considered as coherent states of a harmonic state in the finite-dimensional Hilbert space. In particular, in (e) we plot the Q function of the state $\hat{R}_u(-2)\hat{R}_v(5)|\Phi_0\rangle$, while in (f) we plot the Q function of the state $\hat{R}_u(0)\hat{R}_v(8)|\Phi_0\rangle$.

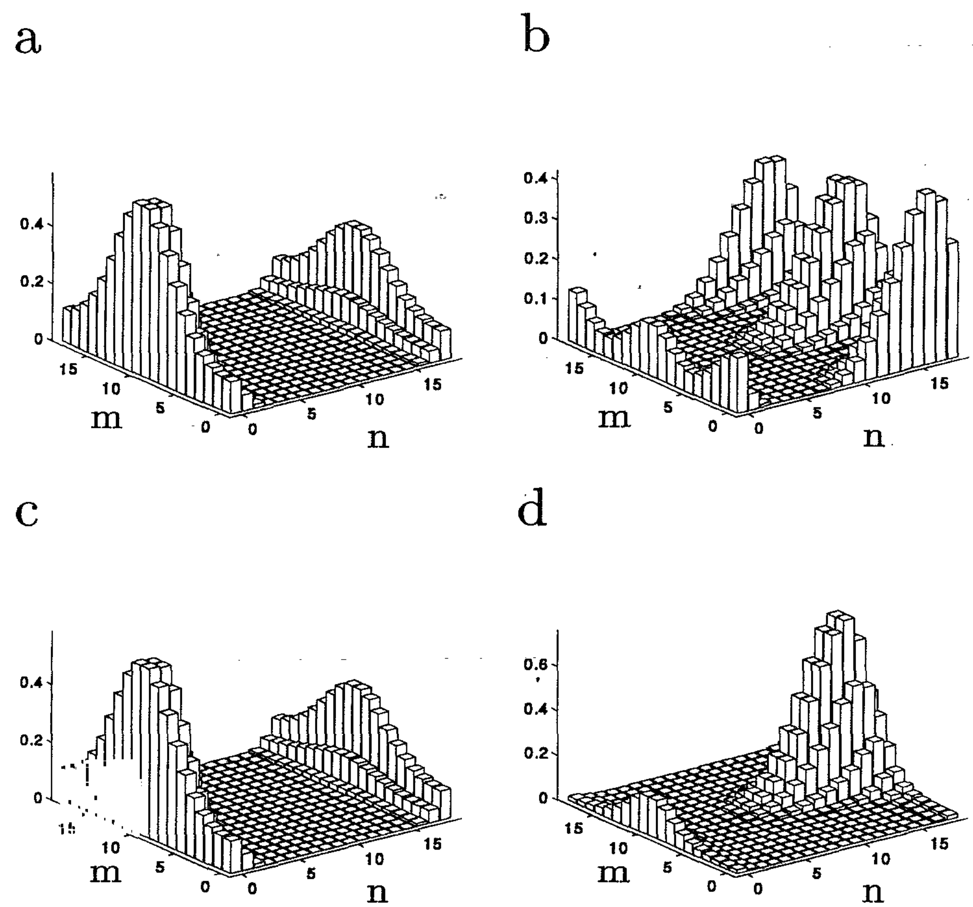

FIG. 3. Q functions of (a) and (b) the coherent states (39) and (c) and (d) the truncated coherent states (42), for $N = 19$. Coherent state $|\alpha\rangle$: (a) $\alpha = -0.5$ and (b) $\alpha = -4.4$. Truncated coherent state $|\bar{\alpha}\rangle$: (c) $\bar{\alpha} = -0.5$ and (d) $\bar{\alpha} = -4.4$.

ror in photon number then decreases also with $\sqrt{N}$. Except for measurement of less excited states, where the mean photon number is much less than N, this ruler is not too convenient: it measures the phase precisely but the photon number imprecisely. In this case we can change the ruler by changing the Hamiltonian: instead of (33) we can define the ruler as the ground state of the "tuned" Hamiltonian $\hat{H}_0^{(p)}$

$$\hat{H}_0^p \equiv -2p\cos\left(\frac{2\pi}{N}\hat{U}\right) - 2(1-p)\cos\left(\frac{2\pi}{N}\hat{V}\right), \quad (43)$$

where p is between zero and unity (for $p = 1/2$ we get $\hat{H}_0$). Changing the parameter p we can "squeeze" the ruler into the desired shape. This operation could appear too arbitrary, but exactly the same situation is in the case of continuous Q function: the ruler state (vacuum) depends on the oscillator frequency ω, as the Hamiltonian (35) changes:

$$\hat{H}_{\text{osc},\ \omega} = \tfrac{1}{2}\omega^2\hat{x}^2 + \tfrac{1}{2}\hat{p}^2. \quad (44)$$

Again we can choose which oscillator states should be used for our measurements: if we need precise x (position) measurements, we take higher ω, while for precise p we choose smaller ω.

V. DISCUSSION AND CONCLUSION

In the present paper we have defined the Q function of a state of a quantum-mechanical system in a finite-dimensional Hilbert space. This discrete Q function has been defined with the help of the Wódkiewicz concept of propensities, i.e., we have defined the Q function as a discrete convolution of two Wigner functions based on Wootters's formalism: one of the state itself and one of the filter state. The discrete Q function takes nonnegative values in all points of the discrete phase space, is normalized, and it is possible to reconstruct from it the density operator of the state under consideration. Surely, there is a problem how to chose the vacuum state (i.e., the quantum-ruler state) with the help of which the Q function is defined. One can, for instance, try to optimize quantum-ruler states for special measurements.

Some other open problems are as follows. It would be interesting to study the influence of noisy (nonunit efficiency) measurements of the Q function on a reconstruction of the density operator. It is also worth studying propensities defined based on other group representations, e.g., we can define the ruler state to be one of the "U states" $|u_k\rangle$ and the "displacements" would transform one line of the discrete phase space into another. Reconstruction of states from such propensities would be a discrete analog of the tomographical method [24,25].

ACKNOWLEDGMENTS

We are grateful to J. Peřina, V. Majerník, and Z. Hradil for discussions and other help. This work was supported by an internal grant of Palacký University and by the East-West Program of the Austrian Academy of Sciences under Contract No. 45.367/1-IV/6a/94 of the Österreichisches Bundesministerium für Wissenschaft und Forschung.

APPENDIX A

In this appendix we show that the Q function is normalized as shown by Eq. (23). To do so, we assume that the system is in a pure state $|\Psi\rangle$ (the extension to mixed states is straightforward); the scalar product $\langle\Psi|\hat{R}_u(n)\hat{R}_v(m)|\Phi\rangle$ can be expanded in the U representation as

$$\begin{aligned}\langle\Psi|\hat{R}_u(n)\hat{R}_v(m)|\Phi\rangle \\ &= \sum_{r,s}\langle\Psi|u_r\rangle\langle u_r|\hat{R}_u(n)\hat{R}_v(m)|u_s\rangle\langle u_s|\Phi\rangle \\ &= \sum_{r,s}\langle\Psi|u_r\rangle\langle u_{r-n}|\hat{R}_v(m)|u_s\rangle\langle u_s|\Phi\rangle \\ &= \sum_{r}\langle\Psi|u_r\rangle\langle u_{r-n}|\Phi\rangle \exp\left(i\frac{2\pi}{N}(r-n)m\right), \qquad \text{(A1)}\end{aligned}$$

where we have used (4) and (6). The Q function is then the last expression times its complex conjugate

$$Q(n,m) = \sum_{r,t}\langle\Psi|u_r\rangle\langle u_{r-n}|\Phi\rangle\langle\Phi|u_{t-n}\rangle\langle u_t|\Psi\rangle \times \exp\left(i\frac{2\pi}{N}(r-t)m\right) \qquad \text{(A2)}$$

and the sum over the arguments n, m is

$$\sum_{n,m} Q(n,m) = \sum_{r,t,n}\langle\Psi|u_r\rangle\langle u_{r-n}|\Phi\rangle\langle\Phi|u_{t-n}\rangle\langle u_t|\Psi\rangle \times \sum_{m}\exp\left(i\frac{2pi}{N}(r-t)m\right). \qquad \text{(A3)}$$

Since the last sum is equal to $N\delta_{r,t}$, we can write

$$\sum_{n,m} Q(n,m) = N\sum_{r}\langle\Psi|u_r\rangle\langle u_r|\Psi\rangle \times \sum_{n}\langle\Phi|u_{r-n}\rangle\langle u_{r-n}|\Phi\rangle = N, \qquad \text{(A4)}$$

which proves Eq. (23).

Finally, we point out that we can redefine the propensity $P_{\Phi,\rho}(n,m)$ given by Eq. (21) in such a way that the discrete Q function will be normalized to unity. To be specific, if we define the convolution in Eq. (21) as

$$\bar{P}_{\Phi,\rho}(n,m) = \frac{1}{N}\sum_{r,s} W_\rho(r,s)W_\Phi(r-n,s-m), \qquad \text{(A5)}$$

then we find the normalization condition for $\bar{P}_{\Phi,\rho}(n,m)$ to be equal to unity. In particular, if the ruler state is chosen to be the vacuum state as discussed above, then the $\bar{Q}$ function obtained from Eq. (A5) is normalized as

$$\sum_{n,m}\bar{Q}(n,m) = 1. \qquad \text{(A6)}$$

In this case $\bar{Q}(n,m)$ can be interpreted as a proper probability distribution.

[1] E. P.Wigner, Phys. Rev. **40**, 749 (1932); see also in *Perspectives in Quantum Theory*, edited by W. Yourgrau and A. van der Merwe (Dover, New York, 1979), p. 25; H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950); V. I. Tatarskij, Usp. Fiz. Nauk **139**, 587 (1983) [Sov. Phys. Usp. **26**, 311 (1983)]; M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[2] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940); Y. Kano, J. Math. Phys. **6**, 1913 (1965); S. Stenholm, Ann. Phys. (N.Y.) **218**, 233 (1992).
[3] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963); E. C. G. Sudarshan, *ibid.* **10**, 277 (1963).
[4] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[5] A. K. Ekert and P. L. Knight, Phys. Rev. A **43**, 3934 (1991).
[6] A. E. Glassgold and D. Mallidey, Phys. Rev. **139**, 1717 (1965).
[7] K. Wódkiewicz, Phys. Rev. Lett. **52**, 1064 (1984); Phys. Lett. A **115**, 304 (1986); **129**, 1 (1988).
[8] E. Arthurs and J. L. Kelly, Jr., Bell Syst. Tech. J. **44**, 725 (1965); Y. Lai and H. A. Haus, Quantum Opt. **1**, 99 (1989); D. Lalović, D. M. Davidović, and N. Bijedić, Phys. Rev. A **46**, 1206 (1992); D. M. Davidović and D. Lalović, J. Phys. A **26**, 5099 (1993); S. Chaturverdi, G. S. Agarwal, and V. Srinivasan, *ibid.* **27**, L39 (1994).
[9] D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988); Phys. Rev. A **39**, 1665 (1989); D. T. Pegg, J. A. Vaccaro, and S. M. Barnett, J. Mod. Opt. **37**, 1703 (1990).
[10] W. K. Wootters, Ann. Phys. (N.Y.) **176**, 1 (1987).
[11] D. Galetti and A. F. R. de Toledo Piza, Physica A **149**, 267 (1988).
[12] J. A. Vaccaro and D. T. Pegg, Phys. Rev. A **41**, 5156 (1990).
[13] T. Opatrný, A. Miranowicz, and J. Bajer, J. Mod. Opt. (to be published).
[14] V. Bužek, C. H. Keitel, and P. L. Knight, Phys. Rev. A **51**, 2575 (1995).

[15] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Ref. [1]), pp. 272–280.
[16] T. S. Santhanam and A. R. Tekumalla, Found. Phys. **6**, 583 (1976); T. S. Santhanam, Phys. Lett. **56A**, 345 (1976); Found. Phys. **7**, 121 (1977).
[17] P. Šťovíček and J. Tolar, Rep. Math. Phys. **20**, 157 (1984).
[18] A. Bandilla and H. Paul, Ann. Phys. (Leipzig) **23**, 323 (1969).
[19] I. Białynicki-Birula, M. Freyberger, and W. Schleich, Phys. Scr. **T48**, 113 (1993).
[20] T. Opatrný, J. Phys. A **27**, 7201 (1994).
[21] V. Bužek, A. D. Wilson-Gordon, P. L. Knight, and W. K. Lai, Phys. Rev. A **45**, 8079 (1992).
[22] A. Miranowicz, K. Piątek, and R. Tanaś, Phys. Rev. A **50**, 3423 (1994).
[23] L. M. Kuang, F. B. Wang, and Y. G. Zhou, Phys. Lett. A **183**, 1, (1993); J. Mod. Opt. **41**, 1307 (1994).
[24] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
[25] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993); D. T. Smithey, M. Beck, J. Cooper, and M. G. Raymer, Phys. Scr. **T48**, 35 (1993).