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Fundamental limit on energy transfer in k-photon down-conversion

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We show that in the k-photon down-conversion a fundamental limit on the energy transfer does exist providing the signal mode is initially in the vacuum state and the pump mode is excited. In particular, in the two-photon down-conversion less than $\frac{2}{3}$ of the pump energy can be transferred from the pump to the signal mode. With the increase of the order of the nonlinear process under consideration the efficiency is even smaller. On the contrary, we show that no restriction on the efficiency of the energy transfer in the kth harmonic generation does exist, i.e., in this process the total energy from the initially excited mode can be transferred into the mode which initially was in the vacuum state. We study restrictions implied by the fundamental limit on the energy transfer in the k-photon down-conversion on nonclassical effects which can be observed in the signal mode in this process.

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I. INTRODUCTION

In quantum optics due attention is paid to quantum nonlinear processes [1] in which nonclassical states of light are generated [2]. A parametric amplifier [3] represents a nonlinear optical system in which states of light exhibiting nonclassical effects such as squeezing of quadrature fluctuations [4,5] or sub-Poissonian photon statistics [5,6] can be produced. Moreover, the time evolution of mean photon numbers of the light modes in the parametric amplifier is characterized by a typical quantum collapse-revival pattern [7].

Field modes in the parametric amplifier (i.e., the pump and the signal modes) do not interact directly. This interaction is mediated via atoms of a nonlinear medium (crystal). Nevertheless, using the standard procedure one can derive an *effective* Hamiltonian describing a nonlinear coupling between the two modes of the parametric amplifier [see Eq. (1)]. This effective coupling between the modes is very small compared to the optical pump frequency, therefore the modes do not interact intensively enough. To enhance observable effects, one can place the nonlinear crystal into a perfect resonator (cavity) which supports the frequency of the signal and the pump modes. The cavity serves to enhance the interaction of the light with the nonlinear crystal. In this arrangement the quantum dynamics leads to a strong entanglement between the pump and the signal modes [8]. Moreover, a back action of the signal mode on the pump causes the appearance of nonclassical properties also in the pump mode. These effects were studied in detail recently in Ref. [8].

Recently Hillery and co-workers [9] have studied the degenerate parametric amplifier described in the interaction picture and in the rotating-wave approximation by the Hamiltonian (we consider a two-photon resonance between the pump mode with a frequency ω_b and the signal mode with a frequency ω_a, i.e., $\omega_b = 2\omega_a$)

$$\hat{H}_2 = \lambda_2[(\hat{a}^\dagger)^2\hat{b} + (\hat{a})^2\hat{b}^\dagger], \tag{1}$$

where $\hat{a}$ ($\hat{b}$) is the annihilation operator of the signal (pump) mode and λ_2 is a coupling constant proportional to the second order polarizability of the nonlinear medium. The dynamics governed by the Hamiltonian (1) is characterized by the integral of motion $\hat{L} = \hat{n}_a + 2\hat{n}_b$, i.e., the total number of excitations in the pump mode and the signal mode is conserved. In other words, the existence of the integral of motion $\hat{L}$ reflects a conservation of the flow of power between monochromatic fields which are coupled through a nonlinear medium (the so-called Manley-Rowe relations [10]). In particular, Hillery *et al.* in Ref. [9] have shown that if an initial state of the pump-signal modes is characterized by small fluctuations of the conserved quantity $\hat{L} = \hat{n}_a + 2\hat{n}_b$ then the number fluctuations in the signal (a) mode are approximately twice those in the pump (b) mode. In Ref. [9] the process governed by the Hamiltonian (1) with the pump (b) mode initially prepared in a number (Fock) state and the signal (a) mode in the vacuum state has been considered. It has been shown that in this nonlinear process (the so-called degenerate down-conversion [1]) the signal mode can exhibit a higher degree of sub-Poissonian photon statistics than the pump mode only in the case when the pump is significantly depleted, namely, only in the case when the mean photon number $\langle\hat{n}_b\rangle$ in the pump mode at time t is smaller than one third of the initial photon number (n_{b0}) in the pump, i.e., $\langle\hat{n}_b\rangle \le n_{b0}/3$. Here a natural

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question arises whether such significant depletion of the pump mode in the down-conversion process can be observed. In other words, is it possible that the pump mode loses 2/3 of its initial intensity? One of the main purposes of the present paper is to give an answer to this question.

To generalize the problem we consider a k-photon nonlinear process which is governed by the interaction Hamiltonian which in the rotating-wave approximation reads (we assume a k-photon resonance, i.e., $\omega_b = k\omega_a$)

$$\hat{H}_k = \lambda_k[(\hat{a}^\dagger)^k\hat{b} + (\hat{a})^k\hat{b}^\dagger]. \tag{2}$$

In this case the operator $\hat{L} = \hat{n}_a + k\hat{n}_b$ is an integral of motion and represents a quantum-mechanical interpretation of the Manley-Rowe relations in terms of photon numbers [10]. If the modes are initially prepared in number states then at any stage of the time evolution the number fluctuations of the a mode are k times those of the b mode, i.e.,

$$\begin{aligned} &\langle(\Delta\hat{n}_a)^2\rangle = k\langle(\Delta\hat{n}_b)^2\rangle, \\ &\langle(\Delta\hat{n}_x)^2\rangle = \langle\hat{n}_x^2(t)\rangle - \langle\hat{n}_x(t)\rangle^2, \quad x = a, b. \end{aligned} \tag{3}$$

In the process of the k-photon down-conversion with the b mode initially in a number state $|n_{b0}\rangle$ and the a mode in the vacuum state, we find the following relation between photon-number fluctuations in the pump mode $[\langle(\Delta\hat{n}_b)^2\rangle]$ and the signal mode $[\langle(\Delta\hat{n}_a)^2\rangle]$:

$$\begin{aligned} \frac{\langle(\Delta\hat{n}_a)^2\rangle}{\langle\hat{n}_a\rangle} &= \frac{\langle(\Delta\hat{n}_b)^2\rangle}{\langle\hat{n}_b\rangle}\frac{k\langle\hat{n}_b\rangle}{n_{b0} - \langle\hat{n}_b\rangle} \\ &\le \frac{\langle(\Delta\hat{n}_b)^2\rangle}{\langle\hat{n}_b\rangle} \Leftrightarrow \langle\hat{n}_b\rangle \le \frac{n_{b0}}{k+1}. \end{aligned} \tag{4}$$

From the above it directly follows that if we want to observe in the k-photon process the signal mode with the degree of sub-Poissonian photon statistics higher than in the pump mode then we have to transfer the fraction $\frac{k}{k+1}$ of the initial energy of the pump mode into the signal mode, i.e., the pump mode has to be significantly depleted. Moreover, with the increase of the order of the nonlinear process this fraction of the transferred energy has to be larger.

In this paper we will show that in the k-photon down-conversion there exists a fundamental limit on the efficiency of the energy transfer from the pump mode into the signal mode which is initially prepared in the vacuum state. Moreover, we will show that the higher the order k of the nonlinear process the smaller is the amount of energy (for a given initial photon number in the pump mode) which can be transferred from the pump to the signal mode. From here then it follows that the degree of sub-Poissonian photon statistics in the signal mode will always be smaller than in the pump mode.

Here we should note that in addition to the effect of the fundamental limit on the efficiency of the energy transfer the quantum interaction between the pump and the signal modes leads to a strong entanglement between the two modes under consideration [8]. Obviously if the pump mode is described as a classical field, then there is no entanglement between the modes and no pump depletion can be observed. Within this parametric approximation the interaction Hamiltonian (1) has the form (see [2,11])

$$\hat{H}_2^{par} = \lambda_2\beta[(\hat{a}^\dagger)^2 + (\hat{a})^2]. \tag{5}$$

The Hamiltonian (5) describes the production of highly correlated photon pairs with reduced quadrature fluctuations. The evolution operator $\hat{U}(t) = \exp(-i\hat{H}_2^{par}t)$ (in what follows we use units such that $\hbar = 1$) corresponding to the Hamiltonian (5) is equivalent to the squeeze operator $\hat{S} = \exp[\xi(\hat{a}^\dagger)^2 - \xi^*\hat{a}^2]$ (where $\xi = -i\lambda_2 t$). This squeeze operator describes the Bogoliubov transformation by means of which the initial vacuum state of the signal mode is transformed into the squeezed vacuum state [2]. The degree of squeezing depends on the parameter ξ, i.e., the longer the interaction time the more energy is transferred from the classical pump to the signal mode and the higher is the degree of squeezing of the signal mode (ideally, for times long enough an arbitrary amount of the energy can be transferred to the signal mode which is coupled to a classical current). Moreover, the classical pump mode is not affected by this nonlinear process. On the other hand, if the quantum nature of the pump mode is taken into account then a completely different picture is obtained. First, the pump and the signal modes become entangled due to the quantum nature of their interaction. One of the consequences of this entanglement is that the signal mode via back action on the pump mode can change the statistical properties of the pump as well as the signal [8]. In addition, the energy is transferred not only from the pump to the signal mode but also in the reverse way. This happens when the signal mode which has initially been in the vacuum state becomes significantly excited. Therefore we can consider the back action of the signal mode on the pump mode as a physical reason why just a fraction of the pump energy in the down-conversion process can be transferred from the pump to the signal.

In the present paper we will study the k-photon down-conversion as well as the process of the kth harmonic generation which is governed by the same interaction Hamiltonian (2), but the initial state of the modes a and b is opposite than in the case of down-conversion, i.e., the mode a is supposed to be in a highly excited state and the mode b is supposed to be initially in a vacuum state. We will show that, in spite of the fact that both the k-photon down-conversion and the kth harmonic generation process are governed by the same Hamiltonian, in the case of the down-conversion there exists a fundamental limit on the energy transfer, while in the case of the harmonic generation the *total* energy from the a mode can be transferred to the b mode.

II. THE k-PHOTON DOWN-CONVERSION

In this section we will study how the energy (i.e., photons) is transferred between the modes in the k-photon down conversion process (2) in which the annihilation of one photon of a quantized pump mode gives rise to k

photons of signal mode. The regime of k-photon down-conversion can be associated with an initial state of the form

$$|\Phi_0^b\rangle = |0\rangle_a|\Phi_0\rangle_b = \sum_N b_N|0\rangle_a|N\rangle_b \equiv \sum_N b_N|0,N\rangle,$$

$$\sum_N |b_N|^2 = 1, \quad (6)$$

where the signal (a) mode is initially in the vacuum state $|0\rangle_a$ and the initial state of the pump (b) mode is described by the state vector $|\Phi_0\rangle_b$. In (6) we use notation such that $|m,n\rangle$ denotes the signal-pump state with m (n) photons in the signal (pump) mode. The dynamics governed by the Hamiltonian (2) cannot be described in a closed analytical form. Nevertheless, it can be solved numerically due to the existence of the integral of motion $\hat{L} = \hat{n}_a + k\hat{n}_b$ (see, for example, [12], and references therein). Due to this conservation law the whole Hilbert space is split into a direct sum of dynamically independent and finite-dimensional subspaces which are labeled by eigenvalues of the operator $\hat{L}$. Therefore we will first study the dynamics of the down-conversion on a chosen particular subspace corresponding to the initial state

$$|\Phi_0^b\rangle = |0,N\rangle. \quad (7)$$

Our results can then be straightforwardly generalized to the case of an arbitrary superposition state (6) because the mean values of observables (such as the photon number) over the state (6) are equal to the sum of their mean values over independent subspaces weighted with the initial probability to find a state within a given subspace.

Let us assume a subspace characterized by $L = kN$ and formed out of the $(N+1)$-dimensional basis $\{|0,N\rangle, |k,N-1\rangle, \ldots, |k(N-1),1\rangle, |kN,0\rangle\}$. The $N+1$ eigenvectors satisfy the stationary Schrödinger equation

$$\hat{H}_k|E_j\rangle = E_j|E_j\rangle \quad (8)$$

and can be expressed in the photon-number basis as

$$|E_j\rangle = \sum_{n=0}^{N} c_{jn}|kn, N-n\rangle. \quad (9)$$

We notice that if E_j is an eigenvalue in Eq. (8) then also $-E_j$ is an eigenvalue of the Hamiltonian $\hat{H}_k$. Moreover, the corresponding eigenstate has the form

$$|-E_j\rangle = \sum_{n=0}^{N} (-1)^n c_{jn}|kn, N-n\rangle. \quad (10)$$

Now the state vector $|\Phi^b(t)\rangle$ describing the signal-pump system at time t which initially has been prepared in the state (7) can be expressed through the eigenstates $|E_j\rangle$ in a standard way:

$$|\Phi^b(t)\rangle = \sum_j e^{-iE_jt}|E_j\rangle\langle E_j|\Phi_0^b\rangle, \quad (11)$$

and therefore the mean photon number in the pump mode at time t can be expressed as

$$n_b(t) = \langle\Phi^b(t)|\hat{n}_b|\Phi^b(t)\rangle = \sum_i w_{i,i}^b + 2\sum_{i<j} w_{i,j}^b \cos[(E_i - E_j)t], \quad (12)$$

where

$$w_{i,j}^b = \langle E_j|\Phi_0^b\rangle\langle\Phi_0^b|E_i\rangle\langle E_i|\hat{n}_b|E_j\rangle. \quad (13)$$

The projection $\langle E_j|\Phi_0^b\rangle$ of the initial state (7) on the eigenstate $|E_j\rangle$ equals c_{j0} [see Eq. (9)]. From Eq. (12) we can obtain an estimation of the minimum number of photons in the mode b. Because of the fact that the time variable t enters the expression for the mean photon number only via cosine terms, we can find that for any time t the relation

$$\sum_i w_{i,i}^b - 2\sum_{i<j} |w_{i,j}^b| \le n_b(t) \le \sum_i w_{i,i}^b + 2\sum_{i<j} |w_{i,j}^b| \quad (14)$$

is valid. As a consequence of the relation (14) it follows that in the process of k-photon down-conversion with the initial state (7) the b mode will always contain a number of photons which is not smaller than

$$w_b = \sum_i w_{i,i}^b - 2\sum_{i<j} |w_{i,j}^b|. \quad (15)$$

Therefore, if the signal-pump initial state is described by the vector (7) then it is in principle *impossible* to transfer from the pump to the signal more than $(n_{b0} - w_b)$ photons (here we remind the reader that n_{b0} denotes the initial photon number in the pump mode). In other words, the maximum number of photons which can be observed in the signal mode during the time evolution is equal to $k(n_{b0} - w_b)$.

From the fact that the parameter w_b given by Eq. (15) is larger than zero (see later) it follows that there exists a *fundamental* limit on the number of photons (i.e., energy) which can be transferred from the pump to the signal mode in the k-photon down-conversion process with the initial state given by Eq. (7). There is one exception to this general behavior; namely, if the pump mode is initially prepared in the Fock state with one photon, i.e., $|\Phi_0^b\rangle = |0,1\rangle$, then this photon can be completely transferred into the signal mode, which means that the pump can be completely depleted. Otherwise, as soon as the initial number of photons in the pump is larger than 1 a portion of the initial pump energy corresponding to the number of photons equal to w_b will be "trapped" in the pump.

To illustrate this "trapping" effect we have collected in Table I some particular values of the quantity $(n_{b0} - w_b)$, i.e., the maximum number of pump photons which can be transferred via k-photon down-conversion into the signal mode. To make this illustration even more transparent we plot in Fig. 1 the energy-transfer efficiency parameter ζ defined as

TABLE I. Maximum numbers of photons $n_{b0} - w_b$ [see Eq. (15)] which can be *in principle* transferred from the pump mode to the signal mode in the process of k-photon down-conversion. The pump mode is supposed to be prepared initially in the Fock state $|N = n_{b0}\rangle_b$ and the signal mode in the vacuum state.

$n_{b0} - w_b$	$N = 20$	$N = 21$	$N = 40$	$N = 41$	$N = 60$	$N = 61$	$N = 80$	$N = 81$
$k = 2$	14.356	15.372	28.629	29.701	42.908	44.021	57.190	58.337
$k = 3$	3.541	6.178	5.503	9.217	7.045	11.614	8.359	13.659
$k = 4$	0.543	2.770	0.645	3.222	0.704	3.486	0.745	3.673

$$\zeta = \frac{n_{b0} - w_b}{n_{b0}}, \tag{16}$$

as a function of the initial photon number n_{b0} in the pump mode. We analyze separately the case when the pump mode is initially prepared in the Fock state with an even number of photons [Fig. 1(a)] and the case when the pump mode is initially prepared in the odd Fock state [Fig. 1(b)]. In both cases we study the k-photon down conversion process with $k = 2, 3$, and 4. From Table I and Fig. 1 we see three important features of the process of k-photon down-conversion. *First*, the efficiency parameter ζ decreases with the increase of the initial photon number in the pump mode. Only the trivial case with $n_{b0} = N = 1$ is characterized by total energy transfer from the pump to the signal mode (i.e., only in this case does $\zeta = 1$). For any other initial Fock state $|n_{b0} = N\rangle_b$ with $N > 1$ some part of the initial energy is trapped in the pump for any $t > 0$, i.e., we can observe an effect of the inhibition of the pump depletion which is due to the effect of the back action of the signal on the pump mode. From Fig. 1 we clearly see that the efficiency parameter is a monotonically decreasing function of the initial number of photons in the pump mode. *Secondly*, we should stress that there exists a significant sensitivity with respect to the fact whether the initial number of photons in the pump is even or odd. It is important to note that this sensitivity of dynamics with respect to the "parity" of the initial Fock state is preserved also for very large numbers N. Therefore we present two figures [Fig. 1(a) and Fig. 1(b)] in which we consider either even or odd initial number of photons in the pump mode, respectively. From Table I and Fig. 1 we see that there is a significant difference between the maximum number of photons which can in principle be transferred from the pump to the signal mode which depends on the "parity" of the initial number of photons in the pump mode. In particular, in the case of the down-conversion process with $k > 2$ we find for two neighboring subspaces corresponding to $N = 2m$ and $N = 2m + 1$ that the efficiency parameter ζ is much smaller in the case when the pump contains initially an even number of photons. *Thirdly*, for a fixed number of photons in the pump mode at $t = 0$, the efficiency parameter ζ decreases with increase of the order of the nonlinear process under consideration. In other words, the larger k the smaller the fraction of the energy which is transferred from the pump to the signal. In particular, for $N = 2$ we can derive an explicit expression for the efficiency parameter ζ as a function of k:

$$4A_kB_k/(A_k + B_k)^2, \tag{17}$$

where $A_k = 2k!$ and $B_k = (2k)!/k!$ From this expression it is clearly seen that ζ monotonically decreases as a function of k.

To understand the phenomenon of energy trapping in k-photon down-conversion we will analyze the dynamics of the signal-pump system in the basis composed of the eigenvectors of the Hamiltonian (2). For illustration, we will consider only a particular subspace of the whole Hilbert space corresponding to the initial state vector $|0, N\rangle$. Let us assume the three-photon down-conversion process with the initial-state vector $|0, N = 41\rangle$. In Fig. 2 we plot absolute values of the matrix elements $w^b_{i,j}$ [see Eq. (13)] through which the number of pump photons w_b "trapped" in the pump mode can be expressed [see Eq. (15)]. Graphical representation of the number w_b simply corresponds to the sum of diagonal bars minus the sum of heights of the off-diagonal bars. The eigenstates $|E_j\rangle$ are labeled in order of increasing eigenvalues with $j = 0$

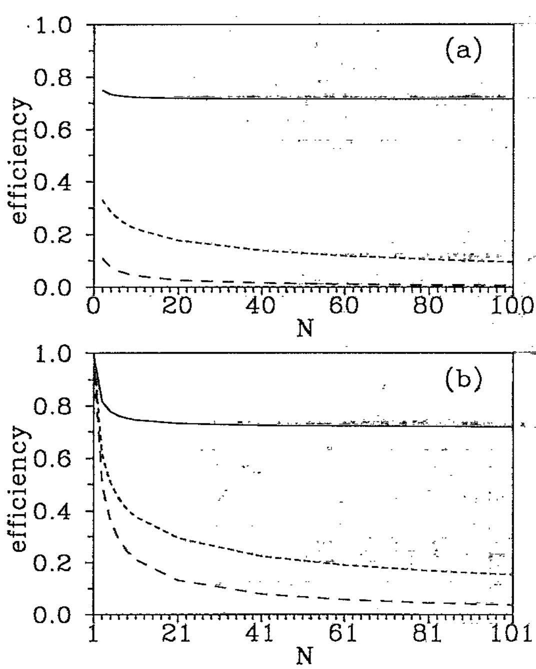

FIG. 1. The efficiency parameter ζ [see Eq. (16)] in k-photon down-conversion as a function of the initial number of photons n_{b0} in the pump mode which is initially prepared in the Fock state $|N = n_{b0}\rangle_b$. We plot separately two pictures for the case when initially the pump mode is prepared in the Fock state with an even number of photons (a) and an odd number of photons (b). The solid line corresponds to $k = 2$, the short-dashed line to $k = 3$, and the long-dashed line to $k = 4$. The signal mode is initially in the vacuum state.

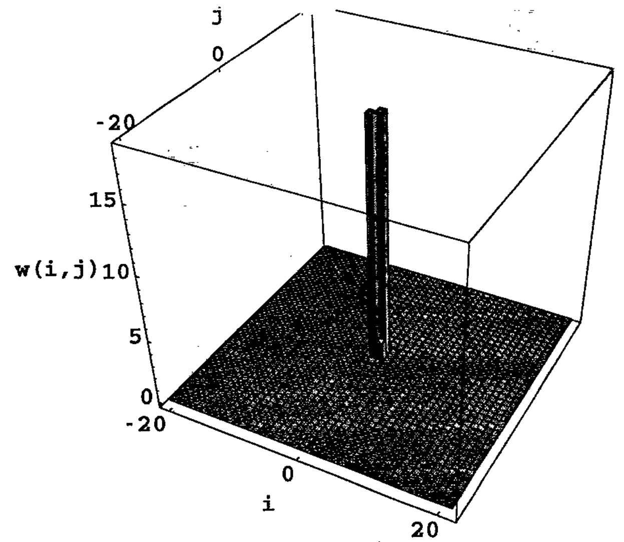

FIG. 2. Matrix elements $|w^b_{i,j}|$ [see Eq. (12)] in the three-photon down-conversion ($k = 3$). The initial state is taken to be $|0, N = 41\rangle$.

attached to the first state with a non-negative energy. It is seen from Fig. 2 that there are just two dominant contributions coming from two *diagonal* elements $w^b_{0,0}$ and $w^b_{-1,-1}$. The off-diagonal elements are strongly suppressed; therefore their contribution to the number w_b is negligible. This means that the dynamics of the down-conversion is determined just by *two* dominant eigenvectors $|E_0\rangle$ and $|E_{-1}\rangle$ for which we can find the relation $|E_{-1}\rangle \equiv |-E_0\rangle$. These two eigenstates have dominant overlap $\langle E_j|\Phi^b_0\rangle$ with the initial state $|0, N\rangle$ as can be seen in Fig. 3 where we plot $|\langle E_j|\Phi^b_0\rangle|^2$. The orthogonality of the eigenvectors implies the relation $\sum_n (-1)^n c^2_{0n} = 0$ [see Eqs. (9) and (10)] which suggests that $w^b_{-1,0} = w^b_{0,-1} \sim \langle E_{-1}|\hat{n}_b|E_0\rangle = \sum_n n(-1)^n c^2_{0n}$ is a small number (at least in comparison with $w^b_{-1,-1} = w^b_{0,0}$). The other initial-state weighted matrix elements $w^b_{i,j}$ are small owing to small overlaps of the remaining eigenvectors with the initial state. From the above we can conclude that the effect of the energy trapping in the pump mode is caused by the fact that the initial state $|0, N\rangle$ of the signal-pump system has a significant overlap with just two mutually orthogonal eigenvectors, so the dynamics can effectively be described on a two-dimensional subspace spanned by these two eigenvectors. Moreover, the off-diagonal terms w^b_{ij} [see Eq. (13) and Fig. 2] are negligible, so the parameter w_b is approximately equal to twice the value of w^b_{00}.

Earlier in our discussion we have noticed that the efficiency of the energy transfer from the pump to the signal mode strongly depends on the "parity" of the initial pump state. If initially the pump mode is in a Fock state with an even number of photons (let us say $N = 40$) then in three-photon down-conversion the initial-state vector of the signal-pump system, i.e., $|0, N = 40\rangle$, has a dominant overlap with just *one* eigenvector $|E_0\rangle$ (see Fig. 4) which has an eigenvalue equal to zero. Because of the fact that the initial state $|0, N = 40\rangle$ is almost an eigenstate of the Hamiltonian of the system under consideration we have to expect that much of the energy is trapped in the pump mode for any $t > 0$. For $k > 3$ this energy trapping is much more pronounced. On the other hand for two-photon down conversion ($k = 2$) the initial state with an even photon number (let us say $|0, N = 40\rangle$) has a comparable overlap with three eigenstates $|E_0\rangle$, $|E_1\rangle$, and $|E_{-1}\rangle \equiv |-E_1\rangle$. Because of this fact the dynamics of the two-photon down-conversion with the initial states $|0, N = 40\rangle$ and $|0, N = 41\rangle$ (or $|0, N = 39\rangle$) are rather similar. Nevertheless, if we compare two "neighboring" subspaces corresponding to initial pump photon numbers equal to N and $N + 1$ then we can conclude that in the case of even initial photon number more energy is confined in the pump mode, i.e., the efficiency of the energy transfer is smaller than in the case with odd number of pump photons.

We should note here that the parameter w_b represents a rather rough estimation of the number of photons trapped in the pump mode. Nevertheless, it clearly reveals the fact that in the case of k-photon down-conversion the efficiency ζ of the energy transfer decreases as the initial number of photons in the pump mode increases (we have to stress here that this does not

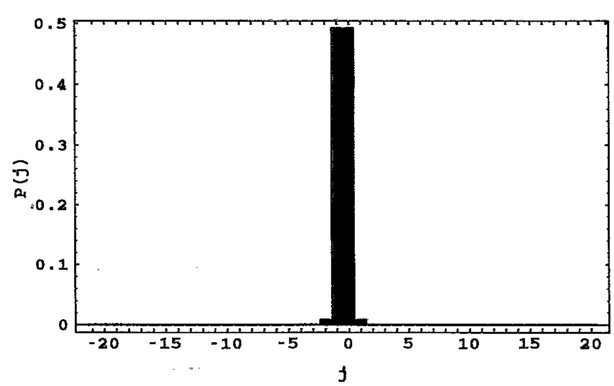

FIG. 3. Overlaps $P(j) = |\langle E_j|\Phi_0\rangle|^2$ of the initial-state vector $|\Phi^b_0\rangle = |0, N = 41\rangle$ with eigenvectors of the interaction Hamiltonian for three-photon down-conversion. Eigenvectors are indexed in the order of their increasing eigenenergies.

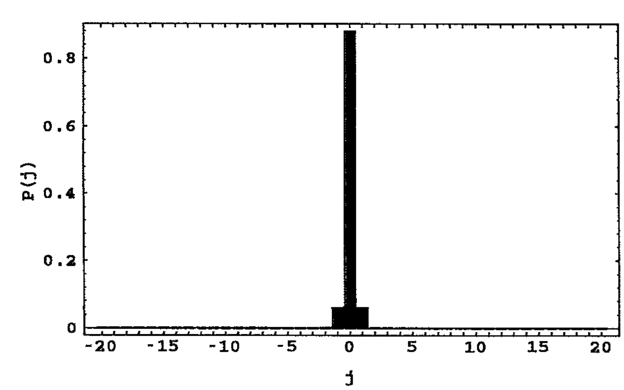

FIG. 4. Overlaps $P(j) = |\langle E_j|\Phi^b_0\rangle|^2$ of the initial-state vector $|\Phi_0\rangle = |0, N = 40\rangle$ with eigenvectors of the interaction Hamiltonian for $k = 3$.

mean that the maximum number of photons transferred from the pump to the signal decreases with the increase of N). In the case of the two-photon down-conversion ($k = 2$) the efficiency parameter ζ for large values of N does converge to a limiting value which is approximately equal to 0.7. Simultaneously we have to stress that w_b is just the lowest estimation of the number of photons confined in the pump mode. In fact, the number of "trapped" photons is much larger than w_b which is due to the fact that the eigenstates $|E_j\rangle$ are not equidistant, i.e., $E_{j+1} - E_j \neq$ const (see [3,7]). As a consequence, the off-diagonal terms which contribute to the mean photon number $n_b(t)$ given by Eq. (12) and which are "responsible" for the energy transfer to the signal mode oscillate on different frequencies. Because of this dephasing the maximum number of photons transferred into the signal mode is much smaller than w_b (here we assume that the initial number N of pump photons is larger than unity). Numerical calculations of the time evolution of the mean photon number presented in Fig. 5 show us that for the two-photon down-conversion there exists a fundamental limit on the efficiency of the energy transfer. For $N \gg 1$ this efficiency is less than 2/3.

Our results can be straightforwardly generalized to the case of the initial states (6) when the signal is still in the vacuum state but the pump mode is in an arbitrary superposition of Fock states. In k-photon down-conversion the resulting mean photon number n_b is given as the sum of mean photon numbers $n_b(N)$ from particular subspaces which are weighted with the initial probability to find a state within a given subspace:

$$n_b(t) = \sum_N |b_N|^2 n_b(N;t). \tag{18}$$

Usually the pump mode is assumed to be prepared in a highly excited coherent state with a Poissonian photon-number distribution (PND) with the mean photon number equal to n_{b0}. In this case the state $|N_0 \approx n_{b0}\rangle_b$ gives a dominant contribution to the initial superposition of Fock states of the pump mode. Due to the fact that the eigenvalues E_j corresponding to eigenstates $|E_j\rangle$ in different subspaces labeled by the parameter N are not equal, the mean photon numbers $n_b(N,t)$ which are given by Eq. (12) do not evolve "in phase." Therefore $n_b(N,t)$ corresponding to different values of N do not reach their minima simultaneously, from which it follows that the transfer of photons from the initial coherent pump with the initial mean photon number n_{b0} equal to N_0 to the signal mode is *less* efficient than the transfer of the energy when the pump is initially prepared in the Fock state $|N_0\rangle_b$. In addition, the transfer of the energy would be even weaker if the pump is initially prepared in the squeezed vacuum state with the mean photon number equal to N_0. This deterioration of the efficiency of the energy transfer is caused by two effects. First, the squeezed vacuum state has a super-Poissonian photon statistics, which means that the PND of the squeezed vacuum is much wider than the PND of the coherent state with the same mean photon number. Therefore the effect of the "dephasing" is more pronounced. Secondly, the squeezed vacuum state in the Fock basis is composed of only even Fock states. We have shown earlier that the energy transfer for initial even Fock states is much worse than for the odd Fock states; therefore we have to expect that the efficiency of the energy transfer with the initial superposition of even Fock states will be very small.

In this section we have shown that the efficiency of the energy transfer in the two-photon down-conversion is less than 2/3 and for higher k is even smaller. For example, for the three-photon process it is only 15% and for $k = 4$ less than 4% if the pump is initially in the number state with 101 photons. Because of this trapping of the pump energy the pump mode will be always more sub-Poissonian than the signal [see Eq. (4)]. In other words, the number fluctuations in the signal mode measured by the Mandel q parameter which is defined as $q_x = \langle(\Delta\hat{n}_x)^2\rangle/\langle\hat{n}_x\rangle - 1$ (see [13]) can never be smaller than the Mandel q parameter of the pump mode in the down-conversion process. Simultaneously, the maximum degree of quadrature squeezing one can observe in the signal mode is significantly restricted by the amount of energy which in principle can be transferred from the pump to the signal mode (see [7,14]). This illustrates a crucial consequence of the fundamental limitation of the efficiency of the energy transfer in the k-photon down-conversion on the statistical properties of the pump and the signal modes.

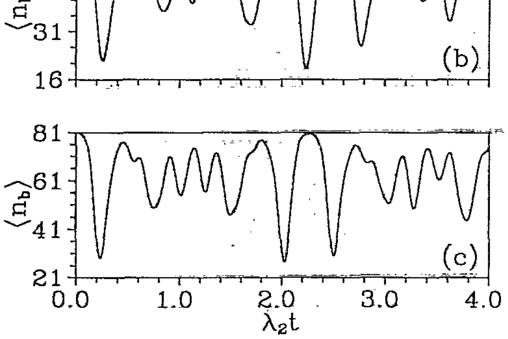

FIG. 5. The time evolution of the mean photon number of the pump b mode in the two-photon down-conversion. The initial state of the pump is equal to the Fock state $|N\rangle_b$ with (a) $N = 41$, (b) $N = 61$, and (c) $N = 81$.

III. THE kTH HARMONIC GENERATION

In the previous section we have analyzed dynamics governed by the Hamiltonian (2) when at $t = 0$ the mode b was in the excited state while the mode a was in the vacuum state. We have shown that in this process (the k-photon down conversion) a significant portion of the initial pump (mode b) energy is "trapped" in the pump and is not transferred into the signal mode. Now we turn our attention to the process which is governed by the same Hamiltonian but the initial state is such that the mode a is excited (for simplicity we will assume that the a mode is initially in the Fock state $|kN\rangle_a$) and the mode b is in the vacuum state $|0\rangle_b$. We will study whether in this "reverse" process (the so-called kth harmonic generation) "trapping" of the energy in the mode a takes place. One of the motivations for such an investigation lies in the fact that following the arguments presented by Hillery *et al.* [9] we can derive a relation

$$\frac{\langle(\Delta\hat{n}_b)^2\rangle}{\langle\hat{n}_b\rangle} = \frac{\langle(\Delta\hat{n}_a)^2\rangle}{\langle\hat{n}_a\rangle}\frac{\langle\hat{n}_a\rangle}{k(n_{a0}-\langle\hat{n}_a\rangle)} \leq \frac{\langle(\Delta\hat{n}_a)^2\rangle}{\langle\hat{n}_a\rangle} \Leftrightarrow \langle\hat{n}_a\rangle \leq \left(1-\frac{1}{k}\right)n_{a0}, \quad (19)$$

from which it follows that if the a mode loses $1/k$ of its initial intensity n_{a0} then the degree of number fluctuations in this mode will become larger than in the mode b.

To answer the question whether in the kth harmonic generation some limitation on the energy transfer from the mode a to the b mode does exist we study the mean photon number of the a mode for the initial state

$$|\Phi_0^a\rangle = |kN, 0\rangle. \quad (20)$$

This state is within the same subspace of the Hilbert space as the state $|0, N\rangle$ examined in the previous section. In analogy with Eq .(13) we can write the expression for the time evolution of the mean photon number in the mode a as

$$n_a(t) = \sum_i w_{i,i}^a + 2\sum_{i<j} w_{i,j}^a \cos[(E_i - E_j)t],$$
$$w_{i,j}^a = \langle E_j|\Phi_0^a\rangle\langle\Phi_0^a|E_i\rangle\langle E_i|\hat{n}_a|E_j\rangle. \quad (21)$$

According to Eq. (9), the overlap $\langle E_j|\Phi_0^a\rangle$ for the initial state (20) is equal to c_{jN}. Following the same arguments as in the case of the k-photon down-conversion we can introduce a quantity

$$w_a = \sum_i w_{i,i}^a - 2\sum_{i<j} |w_{i,j}^a| \quad (22)$$

which measures the number of photons which cannot *in principle* be transferred via the kth harmonic generation from the a mode to the b mode.

The magnitudes of the initial-state weighted matrix elements $w_{i,j}^a$ of the initial state given by Eq. (20) with $N = 41$ and $k = 3$ are presented in Fig. 6. The initial state $|kN = 123, 0\rangle$ belongs to the same dynamically independent subspace as the state $|0, N = 41\rangle$ described in Fig. 2 (the eigenstates are labeled in the same way). In contrast with the k-photon down-conversion (see Fig. 2), in the case of the kth harmonic generation (see Fig. 6) the sum of the off-diagonal elements $w_{i,j}^a$ compensates the sum of the diagonal elements $w_{i,i}^a$. We find that for any k and $N \gg 1$ in the kth harmonic generation $w_a \leq 0$ which means that *in principle all* photons initially stored in the a mode can be transferred to the mode b. The matrix elements $w_{i,j}^a$ reflect the overlaps $\langle E_i|\Phi_0^a\rangle$ of the initial state (20) with the eigenvectors (9). From Fig. 7 it is seen that the initial state given by Eq. (20) with $k = 3$ and $N = 41$ has significant overlaps with many eigenstates and therefore many eigenstates are involved in the dynamics, and consequently in principle all energy from the mode a can be transferred to the mode b (compare with Fig. 3).

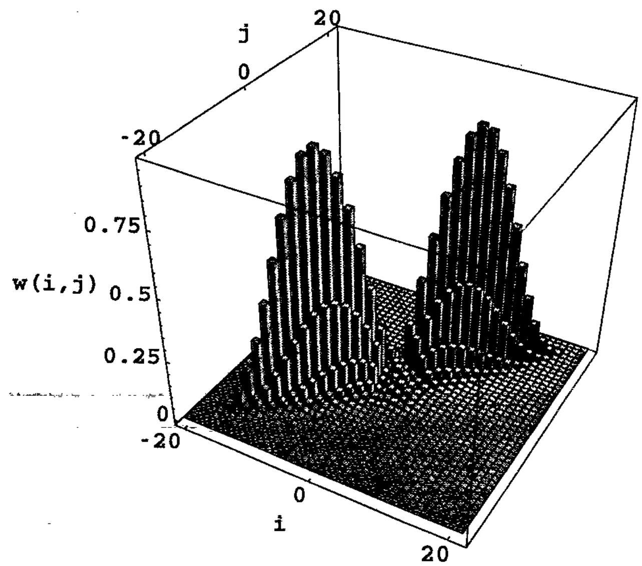

FIG. 6. Absolute values of the state-weighted matrix elements $w_{i,j}^a$ [see Eq. (21)] for the third harmonic generation ($k = 3$). The a mode is initially in the Fock state $|kN\rangle_a$ with $N = 41$. The dynamically independent Hilbert subspace is the same as in Fig. 2.

As there is *in principle* no restriction on the efficiency of the kth harmonic generation, one can expect that the situation described by Eq. (19) happens for higher k at very early stages of the time evolution, i.e., the a mode loses $1/k$ of its initial intensity and its Mandel q parameter exceeds the q parameter of the b mode (see [7]).

IV. CONCLUSIONS

We have analyzed an important question about the efficiency of the energy transfer from one mode to another

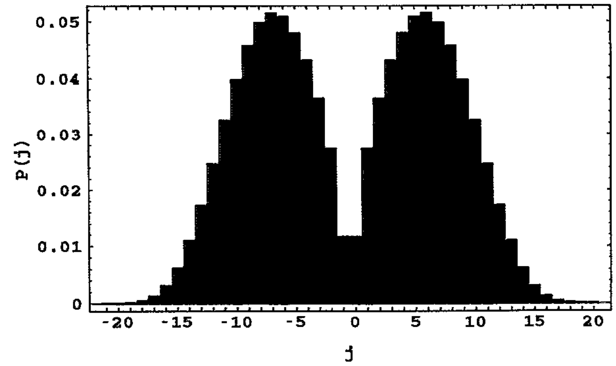

FIG. 7. Overlaps $P(j)$ of the initial-state vector $|\Phi_0^a\rangle = |kN, 0\rangle$ ($k = 3$, $N = 41$) with eigenvectors within the same subspace as in Fig. 3.

in degenerate k-photon processes. We have shown that in the k-photon down-conversion a fundamental limit on the energy transfer does exist, providing the signal mode is initially in the vacuum state and the pump mode is excited. In particular, in the two-photon down-conversion less than 2/3 of the pump energy can be transferred to the signal mode. With increase of the order k of the nonlinear process under consideration the efficiency is even smaller. On the contrary, we have found no restriction on the efficiency of the energy transfer in the kth harmonic generation, i.e., in this process the total energy of the mode a can be transferred to the mode b which initially was in the vacuum state.

The fundamental limitation on the energy transfer in the k-photon down-conversion implies very strong restrictions on nonclassical effects which can be observed in the signal mode. In particular, the degree of squeezing obtainable in the process under consideration is limited by the amount of the transferred energy. Analogously, the signal mode cannot exhibit a larger degree of sub-Poissonian photon statistics than the pump mode.

In this paper we have discussed the k-photon down-conversion process with the signal mode initially prepared in the vacuum state. We should note that as soon as the signal mode is initially excited (even though the mean photon number in the signal is much smaller than in the pump mode) the maximum number of photons which can be transferred from the pump to the signal increases significantly. This "stimulated" transfer of energy from the pump to the signal mode considerably affects the dynamics of the modes under consideration. Generally speaking, the efficiency of the transfer of the energy from one mode to the other mode depends on the overlap of the initial state of the signal-pump system with eigenstates of the nonlinear Hamiltonian under consideration. In particular, in the k-photon down-conversion (the signal mode is initially in a vacuum state) the initial signal-pump state has a significant overlap with only a few eigenstates and therefore the energy exchange between the pump and the signal mode is not efficient (obviously, if the initial state of the signal-pump system is an eigenstate of the Hamiltonian, no energy is transferred between the modes). On the contrary, the initial state of the signal-pump system corresponding to the kth harmonic generation has a considerable overlap with many eigenstates of the nonlinear Hamiltonian (2), which results in an intense interaction between the pump and the signal mode. As a consequence energy can be transferred between the modes with very high efficiency (in principle, close to 100%).

In the investigation of the degenerate k-photon processes we have neglected losses due to the dissipative coupling of the signal-pump system to an environment. In practice, environmental influence inevitably leads to deterioration of the efficiency of the energy transfer from one mode to the other because part of the energy is lost due to the dissipation process. Anyway, we can expect that for decay rates small enough and the pump mode initially highly excited our results remain valid.

ACKNOWLEDGMENTS

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