Quantum phase distributions of amplified Schrödinger-cat states of light

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We study the phase properties of quantum superpositions of two coherent states (Schrödinger-cat states) amplified by phase-sensitive (squeezed) amplifiers. We show that a phase-sensitive amplifier with a properly chosen phase can preserve the phase distribution of the Schrödinger-cat-state input.

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Quantum interference in the phase space between component states of a superposition state leads to appearance of various nonclassical effects such as quadrature squeezing or sub-Poissonian photon statistics [1]. The character of the quantum interference (and consequently the character of nonclassical behavior of the superposition state under consideration) depends on a relative phase between component states [2]. This phase dependence of nonclassical effects has stimulated a considerable effort directed towards better understanding of a concept of the phase operator and the phase probability distribution in quantum mechanics and quantum optics [3].

Recently, Garraway and Knight [4] (also see papers by Tanas and co-workers [5] and by Agarwal and co-workers [6]) have investigated the relation between two conceptually different approaches to phase distributions. One approach is based upon a phase dependence of quasidistributions (such as the Wigner function or the Husimi Q function [7]) that describe states of quantum-mechanical systems (such as light fields). The other approach is based on a definition of phase states and the Hermitian phase operator introduced by Pegg and Barnett [3]. Garraway and Knight [4] have shown that the two approaches give almost the same phase probability distributions for field states dominated by Fock states within a narrow distribution, but the phase distributions can differ for cases involving widely separated photon-number contributions. In particular, dominance by even Fock states can result in the Wigner phase quasiprobability distribution (PQD) with negative values (actually, this is a reason why we call the phase distribution emerging from the Wigner function the phase quasiprobability distribution). The fact that the Wigner PQD takes negative values can serve as a signature of a nonclassical behavior of the state under consideration [4]. In this Brief Report we concentrate our attention on nonclassical properties of quantum-mechanical superposition of coherent states, and therefore we will use the Wigner PDW for description of phase properties of these states.

A pure state of the quantum-mechanical harmonic oscillator corresponding to a single mode of a quantized electromagnetic field $|\Psi\rangle$, which in the Fock basis has a form

$$|\Psi\rangle=\sum_{n=0}^{\infty} Q_n|n\rangle, \quad \hat{\rho}=|\Psi\rangle\langle\Psi|=\sum_{n,m=0}^{\infty} Q_n Q_m^*|n\rangle\langle m| , \qquad (1)$$

can be uniquely described in terms of quasiprobability distribution functions such as the Wigner function or the Husimi function. The Wigner function is defined as [7]

$$W(\alpha)=\frac{1}{\pi^2}\int d^2\beta \exp(\alpha\beta^*-\alpha^*\beta)\mathrm{Tr}[\hat{\rho}\exp(\beta\hat{a}^\dagger-\beta^*\hat{a})] , \qquad (2)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic annihilation and creation operators ($[\hat{a},\hat{a}^\dagger]=1$; in what follows we use units such that $\hbar=1$). Using the phase dependence of the Wigner function (α is a complex parameter), we can define the Wigner PQD as [4–6]

$$P^{(W)}(\theta)\equiv\int_0^\infty rW(r,\theta)dr , \qquad (3)$$

where $W(r,\theta)$ is the Wigner function expressed in polar coordinates (i.e., $\alpha=re^{i\theta}$).

Recently, Garraway and Knight [4] have proved that the Wigner PQD corresponding to a superposition of even Fock states can take negative values. One particular example of a state for which the Wigner PQD takes negative values is the even coherent state (CS). This state can be represented as a superposition of two coherent states, $|\zeta\rangle$ and $|-\zeta\rangle$, which are 180° out of phase with respect to each other:

$$|\zeta\rangle_{\text{even}}=A^{1/2}(|\zeta\rangle+|-\zeta\rangle), \quad A^{-1}=2[1+\exp(-2|\zeta|^2)] , \qquad (4)$$

where $|\zeta\rangle$ is the coherent state defined in the usual way [i.e., $|\zeta\rangle=\exp(\zeta\hat{a}^\dagger-\zeta^*\hat{a})|0\rangle$]. In the Fock basis the even CS is represented as a superposition of *only* even-number states, which means that the photon-number distribution of the even CS exhibits significant oscillations [2]. Besides this nonclassical effect, the even CS also exhibits quadrature squeezing as well as higher-order squeezing [2].

In the last few years several schemes have been proposed for production of the even CS in micromasers [8,9]. In the micromaser experiments one can expect the amplitude $|\zeta|$ of component states to be rather small. Therefore, natural questions arise as to whether it is possible to amplify the superposition state (4) without destroying its phase properties.

It is well known that amplification degrades an optical signal and rapidly destroys quantum features that may have been associated with the input. In particular, for an arbitrarily quadrature-squeezed input the phase-insensitive amplifier provides a squeezed output only for a gain factor G smaller than 2 [10]. To overcome this cloning limit phase-sensitive amplifiers have been proposed for which squeezed output for a gain larger than 2 can be obtained [10]. It is well known that squeezing is a phase-dependent nonclassical effect [11]; therefore one can expect that phase-sensitive amplifiers will preserve phase properties of input fields. In this paper we assume the even CS [Eq. (4)] to be the input state to the amplifier. As we said, the noise added by the amplifier inevitably destroys nonclassical features of the input state [10]. Simultaneously, we know that the amount of noise transferred from the amplifier into the particular quadrature of the field mode depends on the nature of the amplifier. In what follows, we consider phase-sensitive amplifiers [12]. The dynamics of the field mode coupled to the phase-sensitive amplifier is, in the Born-Markov approximation, governed by the Fokker-Planck equation of the Q function $[Q(\alpha)\equiv\langle\alpha|\hat{\rho}|\alpha\rangle/\pi]$, which, in the interaction picture, can be written as [13]

$$\frac{\partial Q(\alpha,t)}{\partial t}=\gamma\left[N\frac{\partial^2}{\partial\alpha^*\partial\alpha}-\frac{1}{2}\left[\frac{\partial}{\partial\alpha^*}\alpha^*+\frac{\partial}{\partial\alpha}\alpha\right]+\frac{M^*}{2}\frac{\partial^2}{\partial\alpha^2}+\frac{M}{2}\frac{\partial^2}{\partial(\alpha^*)^2}\right]Q(\alpha,t)\,, \tag{5}$$

where γ is proportional to the coupling constant between the field mode and the amplifier. The number of mode excitations of the amplifier is denoted N, and M measures the strength of correlations between the amplifier modes. If this phase-sensitive parameter M is set equal to zero, the Fokker-Planck equation (5) then reduces to an equation describing the phase-insensitive amplification of the single-mode field [7]. The squeezing parameter M has the limit determined by the value of N [14]:

$$|M|^2\le N(N+1)\,. \tag{6}$$

The gain G of the amplifier is defined as [15]

$$G=\exp(\gamma t)\,. \tag{7}$$

In this paper we study in detail the evolution of the single-mode field that is initially prepared in the even CS [Eq. (4)] with the real amplitude ζ of the component states $|\pm\zeta\rangle$. The squeezing parameter M is, in general, complex, but to make our analytical results more transparent we will consider only the case of M real. The Q function for the even CS can be written as

$$Q(\alpha,0)=\frac{A}{\pi}[Q_{\rm mix}(\alpha,0)+Q_{\rm int}(\alpha,0)]\,, \tag{8}$$

with

$$Q_{\rm mix}(\alpha,0)=\exp(-\alpha_i^2)\{\exp[-(\alpha_r-\zeta)^2]+\exp[-(\alpha_r+\zeta)^2]\}\,, \tag{9}$$

$$Q_{\rm int}(\alpha,0)=2e^{-\zeta^2}\exp(-\alpha_r^2-\alpha_i^2)\cos(2\zeta\alpha_i)\,, \tag{10}$$

where α_r and α_i are the real and imaginary parts of α. The mixture part, $Q_{\rm mix}$, of the Q function of the even CS consists of two Gaussian peaks localized around $\alpha_r=\pm\zeta$. The interference part $Q_{\rm int}$ has oscillatory behavior and has its maximum at the origin of phase space, $\alpha=\{0,0\}$. This term arises as a direct consequence of the quantum interference between coherent states $|\zeta\rangle$ and $|-\zeta\rangle$ and is responsible for nonclassical behavior of the even CS.

One can find an explicit solution to the Fokker-Planck equation (5) for the Q function with the initial condition (8) from which the expression for the Wigner function of the output state at time t (i.e., for a given G) can be obtained [16]:

$$W(\alpha,t)=\frac{A}{\pi\sqrt{a_w b_w}}\exp\left[\frac{-\alpha_i^2}{a_w}\right]\left[\exp\left[\frac{-[\alpha_r-\zeta(t)]^2}{b_w}\right]+\exp\left[\frac{-[\alpha_r+\zeta(t)]^2}{b_w}\right]+2\exp\left[-2\zeta^2+\frac{\zeta^2(t)}{a_w}-\frac{\alpha_r^2}{b_w}\right]\cos\left[\frac{2\zeta(t)\alpha_i}{a_w}\right]\right]\,, \tag{11}$$

where the "noise" factors a_w and b_w are defined as

$$a_w=N(t)+\exp(\gamma t)-M(t)-\tfrac{1}{2}\,,$$
$$b_w=N(t)+\exp(\gamma t)+M(t)-\tfrac{1}{2}\,, \tag{12}$$

and

$$N(t)=N(G-1),\quad M(t)=M(G-1)\,. \tag{13}$$

The time-dependent amplitude of the component states at $t>0$ is

$$\zeta(t)=\zeta\sqrt{G}\,. \tag{14}$$

In Fig. 1(a) we plot the Wigner function of the initial even CS. This function describes two component states (two Gaussian peaks centered at $\alpha=\{\zeta,0\}$ and $\alpha=\{-\zeta,0\}$) and the interference term centered at the origin of phase space. The interference term arises as a direct consequence of quantum interference between component states [1,2] and it can take negative values. The fact that the interference term of the Wigner function takes negative values results in nonclassical behavior of the even CS. Moreover, the negativity of the interference part of the Wigner function represents a necessary condition for the Wigner PQD to take negative values [4] (see below). In Fig. 1(b) we plot the Wigner function of the even CS amplified by the phase-insensitive amplifier ($M=0$) for the gain factor $G=2.5$. From this figure we can learn two important results. First, the quantum-interference term in Wigner function is completely suppressed by the action of the amplifier, i.e., the noise transferred from the amplifier to the quantum system (field mode) completely destroys quantum coherences and the Wigner function becomes positive. Secondly, the width of the Gaussian peaks describing component states increases significantly for $G>1$, which reflects the increase of noise transferred from the amplifier. Nevertheless, in the case of the phase-sensitive amplifier ($M\neq0$) [see Figs. 1(c) and 1(d)] the amount of noise transferred from the amplifier into a given quadrature of the light field depends on the phase of the amplifier. In particular,

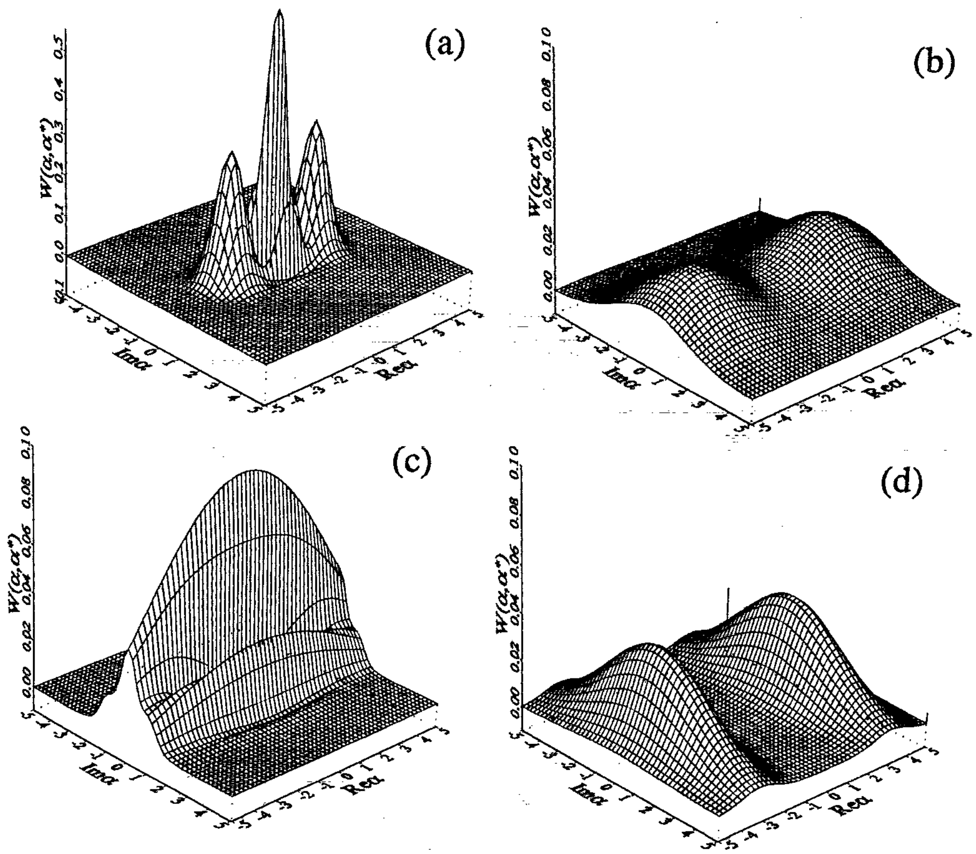

FIG. 1. (a) The Wigner function of the even CS with $\zeta=2$. The Wigner function of the even CS amplified by the phase-insensitive amplifier with (b) $N=3$ and $M=0$, by the phase-sensitive amplifier with (c) $N=3$ and $M=\sqrt{12}$, and (d) with $N=3$ and $M=-\sqrt{12}$. We assume the gain factor $G=2.5$.

for a proper choice of the relative phase between the phase-sensitive amplifier and the superposition state, one can preserve the quantum-interference term (i.e., quantum coherence and consequently nonclassical effects) for much larger gain factors than in the case of the phase-insensitive amplifier [compare Figs. 1(c) and 1(b), which are plotted for the same value of the gain, $G=2.5$]. In other words, we can conclude that the decay of quantum coherences can be significantly suppressed if the quantum-mechanical system (light field) is amplified by the phase-sensitive amplifier, which results in the fact that the Wigner function can be negative even for relatively large values of G. From Figs. 1(c) and 1(d) we also see that the noise transferred into the system increases quadrature fluctuations in one direction (quadrature) more rapidly than in the other. For a real ζ and $M>0$, quadrature fluctuations increase rapidly in the direction connecting two component states, while for $M<0$ the fluctuations increase in the direction orthogonal to the axis connecting two component states. The phase-sensitive transfer of the noise from the amplifier to the quantum system significantly affects phase properties of amplified superposition states.

At the initial moment ($G=1$) the Wigner PQD of the even CS has two maxima around $\theta=-\pi/2$ and $\theta=\pi/2$ that correspond to contributions from two component states, $|-\zeta\rangle$ and $|\zeta\rangle$ (here we have introduced a reference phase $\theta=-\pi/2$). Moreover, the function $P^{(W)}(\theta)$ has some negative values at the edges of two peaks, which is seen clearly in Fig. 2, where we have plotted the Wigner PQD of the even CS with $\zeta=2$. As pointed out by Garraway and Knight [4], the negativity of the Wigner PQD follows from the fact that the Wigner function of the even CS has an interference term that takes negative values [see Fig. 1(a)] and from the fact that the even CS can be represented as a superposition of even Fock states.

During the amplification process the noise added into the system destroys quantum coherence, and consequently the interference term in the Wigner function disappears, which means that the negative parts of the Wigner PQD disappear as well. The other, even more important feature that we can observe is that—under the influence of the phase-insensitive amplifier—the width of the two peaks of the function $P^{(W)}(\theta)$ becomes wider as the noise is transferred from the amplifier to the system during the amplification process. In other words, for sufficiently

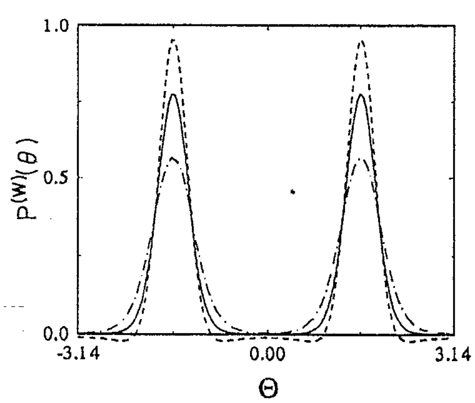

FIG. 2. The Wigner phase quasiprobability distribution $P^{(W)}(\theta)$ corresponding to the even CS. We assume ζ to be real and equal to 2.

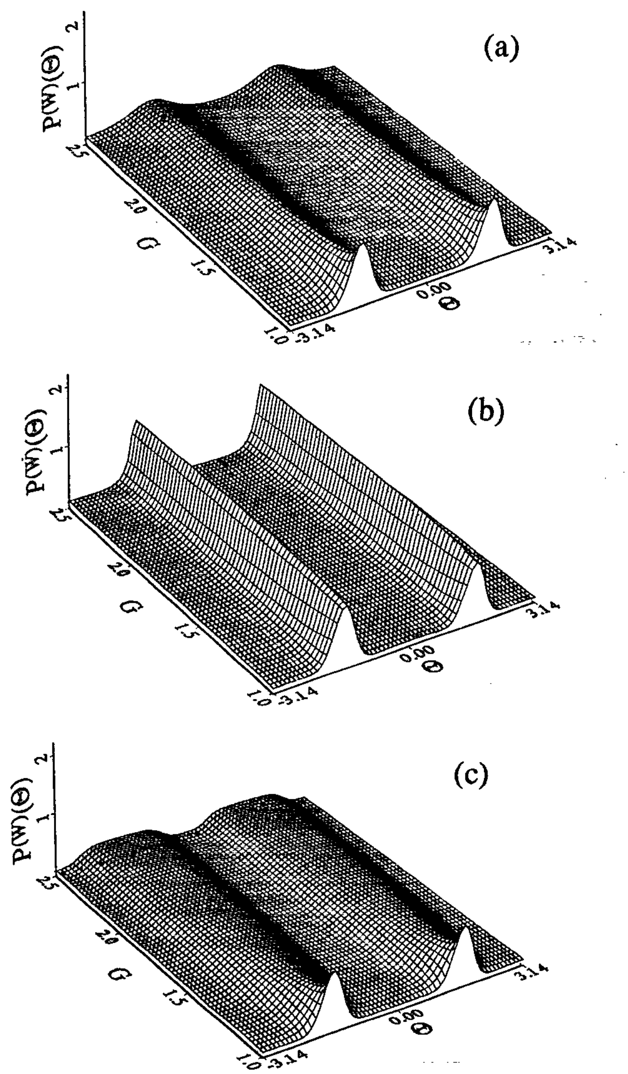

FIG. 3. (a) The Wigner PQD of the initial even CS ($\zeta=2$) as a function of G in the case of phase-insensitive amplifier with $N=3$ and $M=0$, and (b) in the case of the phase-sensitive amplifier with $N=3$ and $M=\sqrt{12}$ and (c) with $N=3$ and $M=-\sqrt{12}$.

large gain the phase distribution becomes almost completely flat (i.e., the phase is random) and we lose any information about phase properties of the initial state [see Fig. 3(a)]. On the other hand, using the phase-sensitive amplifier with a properly chosen phase, we can preserve the initial two-peak structure of the Wigner PQD for an arbitrarily large gain factor G [see Fig. 3(b)]. We should stress here that this preservation of the two-peak structure of the phase distribution does not mean that nonclassical effects associated with the even CS (such as squeezing) will be preserved for arbitrarily large gain. In particular, in the case of the even CS amplified by squeezed amplifier the quadrature squeezing can be observed for maximum gain factor G_{max} equal to 2.1 [16] (which slightly overcomes the cloning limit of $G=2$ for phase-insensitive amplifiers). We have just shown that a phase-sensitive amplifier with properly chosen phase can preserve the two-peak structure of the initial phase distribution of the even CS for any value of G. On the other hand, if the phase of the squeezed amplifier is such that $M<0$, then the two-peak structure of the Wigner PQD deteriorates much faster than in the case of the phase-insensitive amplifier [compare Figs. 3(a) and 3(c)].

In conclusion, we have shown that the two-peak structure of the phase distribution is preserved for any value of the gain factor, provided the initial even CS is amplified by the phase-sensitive amplifier with the properly chosen phase. It can be shown that the two-peak structure of the Wigner PQD is also preserved for any other superpositions of two-component states (not necessarily pure superpositions) that are out of phase by 180° and are amplified by phase-sensitive (squeezed) amplifiers with the properly chosen phase.

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