Amplitude kth-power squeezing of k-photon coherent states

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We have defined a new nonclassical effect: amplitude kth-power squeezing. We have analyzed the squeezing properties of the k-photon coherent states as defined by D'Ariano, Rasetti, and Vadacchio [Phys. Rev. D **32**, 1034 (1985)]; Katriel *et al.* [Phys. Rev. D **34**, 332 (1986)]; and D'Ariano and Sterpi [Phys. Rev. A **39**, 1810 (1989)]. It has been shown that, due to the interaction with the nonlinear nonabsorbing medium modeled as an anharmonic oscillator, these states become amplitude kth power squeezed.

I. INTRODUCTION

Investigations of the nonclassical features[1–5] of the electromagnetic field have become one of the central topics in quantum optics. In particular, attention is paid to the effect of the sub-Poissonian statistics[1,2] in which the photon-number distribution is narrower than the Poissonian distribution. Another widely discussed effect is light squeezing[3–5] when the fluctuations in one of the canonically conjugated quadratures are reduced under the value of the vacuum fluctuations.

Since the pioneer works of Stoler[6] and Yuen[7] on squeezed states, several generalizations of the notion of squeezing have been studied. Particularly, the multiphoton squeezed states have been proposed. As shown by Fischer *et al.*,[8] the straightforward generalization of the two-photon unitary (squeeze) operators which generate the usual squeezed states to higher order is questionable due to the fact that the vacuum-to-vacuum matrix element of the generalized squeeze operator $\hat{U}_{(k)}$ defined as

$$\hat{U}_{(k)}=\exp[z_k(\hat{a}^\dagger)^k+R_e(\hat{a}^\dagger,\hat{a})-z_k^*(\hat{a})^k] \; ,$$

(where $R_e(\hat{a}^\dagger,\hat{a})$ is a polynomial in $\hat{a}$, and $\hat{a}^\dagger$ of order less than k) diverges for $k>2$. This difficulty can be partly overcome from the computational point of view using Padé approximants.[9] An alternative way to define k-photon states has been proposed by D'Ariano and coworkers.[10–12] They used the Brandt-Greenberg[13] multiphoton operators (see below) and the group-theoretical approach for constructing generalized coherent states.

Besides generalizing the ordinary coherent and squeezed states of the radiation field, it is possible to extend the notation of squeezing to higher-order moments of the field quadratures. In particular, Hong and Mandel[4] considered the Nth order moments $\langle(\Delta X_1)^N\rangle$ of the real part of the field amplitude X_1. $\langle(\Delta X_1)^N\rangle$ is said to be squeezed if it takes values less than its coherent state value.

Subsequently Hillery[14] has shown that the squeezing of the square of the field amplitude is also a nonclassical effect. Squeezing proposed by Hillery is not equivalent to that of Hong and Mandel. It is natural that under different definitions of squeezing the very same state of light can exhibit different properties.

In this paper we study the interaction of k-photon coherent states as proposed by D'Ariano *et al.*[10] with a nonlinear, nonabsorbing medium modeled as an anharmonic oscillator. We will study the time behavior of the variances of the quadrature operators corresponding to the Brandt-Greenberg k-photon operators. Using these multiboson operators the amplitude kth-power squeezing is defined.

II. k-PHOTON COHERENT STATES

We begin with some brief remarks on k-photon coherent states as discussed by D'Ariano and coworkers.[10–12] They have used Brandt-Greenberg operators[13] $\hat{A}_{(k)}$ and $\hat{A}_{(k)}^\dagger$:

$$\hat{A}_{(k)}^\dagger=\left[\left[\frac{\hat{n}}{k}\right]\frac{(\hat{n}-k)!}{\hat{n}!}\right]^{1/2}(\hat{a}^\dagger)^k \; , \qquad \hat{A}_{(k)}=(\hat{A}_{(k)}^\dagger)^\dagger \; , \tag{2.1}$$

satisfying the commutation relation

$$[\hat{A}_{(k)},\hat{A}_{(k)}^\dagger]=1 \; . \tag{2.2}$$

In (2.1) the function $[x]$ is defined as the greatest integer less or equal to x; $\hat{a}^\dagger$ and $\hat{a}$ are the usual bosonic creation and annihilation operators and $\hat{n}$ is the photon-number operator $\hat{n}=\hat{a}^\dagger\hat{a}$. For $k=1$ we obtain the usual creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.

In the Fock space $\hat{A}_{(k)}$ and $\hat{A}_{(k)}^\dagger$ act as k-photon annihilation and creation operators

$$\hat{A}_{(k)}|nk+m\rangle=\sqrt{n}\,|(n-1)k+m\rangle \; , \qquad \hat{A}_{(k)}^\dagger|nk+m\rangle=\sqrt{(n+1)}|(n+1)k+m\rangle \; , \tag{2.3}$$

where $0\le m\le k-1$. Due to the fact that there are k vacuum states $|n\rangle$ $(0\le n\le k-1)$ for the generalized annihilation operator $\hat{A}_{(k)}$, the Fock space splits into k orthogonal subspaces which are invariant under the action of the k-photon operators; the generic Fock space $|nk+m\rangle$ is thus labeled by the quantum number n and the degeneracy parameter m, which are the eigenvalues of the commuting operators $\hat{A}_{(k)}^\dagger\hat{A}_{(k)}$ and $\hat{M}_{(k)}=\hat{a}^\dagger\hat{a}-k\hat{A}_{(k)}^\dagger\hat{A}_{(k)}$:

$$\hat{A}_{(k)}^\dagger\hat{A}_{(k)}|nk+m\rangle=n|nk+m\rangle \; , \qquad \hat{M}_{(k)}|nk+m\rangle=m|nk+m\rangle \; . \tag{2.4}$$

To overcome the degeneracy problem it is convenient to suppose $m=0$, i.e., the vacuum state of the k-photon operator $\hat{A}_{(k)}$ is also the vacuum state of the photon operator $\hat{a}$. This assumption is quite natural if the k-photon state is supposed to be generated from the vacuum state $|0\rangle$.

Glauber's coherent state $|\alpha\rangle$ can be generated by shifting the vacuum with the unitary displacement operator $\hat{D}(\hat{a},\alpha)=\exp(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})$:

$$|\alpha\rangle=\hat{D}(\hat{a},\alpha)|\alpha\rangle=\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \ . \quad (2.5)$$

The k-photon coherent state can be obtained by shifting the vacuum with the unitary k-photon displacement operator $\hat{D}(\hat{A}_{(k)},\alpha)$:

$$|\alpha\rangle_k=\hat{D}(\hat{A}_{(k)},\alpha)|0\rangle \ ,$$

where

$$\hat{D}=\exp(\alpha\hat{A}^{\dagger}_{(k)}-\alpha^*\hat{A}_{(k)}) \ . \quad (2.6)$$

Since the algebraic properties of the operators $\hat{A}_{(k)}$, $\hat{A}^{\dagger}_{(k)}$ are the same as in the case of the operators $\hat{a},\hat{a}^{\dagger}$, for $|\alpha\rangle_k$ we obtain the following expression:

$$|\alpha\rangle_k=\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{(\alpha\hat{A}^{\dagger}_{k})^n}{n!}|0\rangle$$

$$=\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|nk\rangle \ . \quad (2.7)$$

From (2.7) it follows that for $k=1$ the generalized coherent state is equal to Glauber's coherent state (2.5).

III. AMPLITUDE kTH-POWER SQUEEZING

We introduce two canonically conjugated variables $\hat{X}^1_{(k)}$ and $\hat{X}^2_{(k)}$

$$\hat{X}^1_{(k)}=\frac{\hat{A}_{(k)}+\hat{A}^{\dagger}_{(k)}}{2}, \quad \hat{X}^2_{(k)}=\frac{\hat{A}_{(k)}-\hat{A}^{\dagger}_{(k)}}{2i} \quad (3.1)$$

with the commutation relation $[\hat{X}^1_{(k)},\hat{X}^2_{(k)}]=i/2$. The canonical variables $\hat{X}^1_{(k)}$ and $\hat{X}^2_{(k)}$ can be understood as a generalization of the canonical operators $Y^1=(\hat{a}^2+\hat{a}^{\dagger 2})/2$ and $Y^2=(\hat{a}^2-\hat{a}^{\dagger 2})/2i$ which represent the real and the imaginary parts of the square amplitude introduced by Hillery. Besides this, it should be underlined here that the operators $\hat{a}^2,\hat{a}^{\dagger 2}$ and $\hat{a}^{\dagger}\hat{a}$ used by Hillery are related to the particular bosonic realization of SU(1,1) Lie algebra, while operators $\hat{A}_{(k)}$ and $\hat{A}^{\dagger}_{(k)}$ are related to the Weyl-Heisenberg algebra. The uncertainty relations are, in these two cases, also different. For the amplitude squared operators we obtain

$$\langle(\Delta\hat{Y}_1)^2\rangle\langle(\Delta\hat{Y}_2)^2\rangle\geq\langle\hat{n}+\tfrac{1}{2}\rangle^2 \quad (3.2)$$

with the squeezing condition $\langle(\Delta\hat{Y}_i)^2\rangle<\langle\hat{n}+\frac{1}{2}\rangle$, while for amplitude k-power squeezing (A^kS) in terms of the operators $\hat{A}_{(k)}$ the uncertainty relation is

$$\langle(\Delta\hat{X}^1_{(k)})^2\rangle\langle(\Delta\hat{X}^2_{(k)})^2\rangle\geq\tfrac{1}{16} \quad (3.3)$$

and the squeezing condition is $\langle(\Delta\hat{X}^i_{(k)})^2\rangle<\frac{1}{4}$. Following the idea of Hong and Mandel[4] we can introduce the parameters $Q^i_{(k)}$ which measure the degree of squeezing in both quadratures ($i=1$ or 2)

$$Q^i_{(k)}=\frac{\langle(\Delta\hat{X}^i_{(k)})^2\rangle-\frac{1}{2}|\langle[\hat{X}^1_{(k)},\hat{X}^2_{(k)}]\rangle|}{\frac{1}{2}|\langle[\hat{X}^1,\hat{X}^2]|}$$

$$=4\langle:(\Delta\hat{X}^i_{(k)})^2:\rangle \ , \quad (3.4)$$

where $\langle:(\Delta\hat{X}^i_{(k)})^2:\rangle$ is the normally ordered variance.

The squeezing condition is, in this case, very simple

$$Q^i_{(k)}<0 \quad (3.5)$$

and maximum (100%) A^kS is obtained in the case $Q^i_k=-1$. One can easily check that for the states $|\alpha\rangle_k$ the parameters $Q^i_{(k)}$ are equal to zero which means that these states are not amplitude kth power squeezed. Nevertheless, as it has been shown by D'Ariano and co-workers,[10–12] these states exhibit ordinary squeezing (in our notation $Q^i_{(1)}$ is less than zero).

IV. AMPLITUDE kTH-POWER SQUEEZING BY AN ANHARMONIC OSCILLATOR

The schemes proposed for the generation of squeezed states[15–18] are essentially based on nonlinear-optical processes when the field interacts with matter characterized by kth-order susceptibility. In particular, Tanas[19] has shown that a high degree of ordinary squeezing may be obtained from the interaction of coherent light with a nonlinear, nonabsorbing medium modeled as an anharmonic oscillator with the Hamiltonian (we adopt $\hbar=1$)

$$\hat{H}=\omega\hat{a}^{\dagger}\hat{a}+\frac{\lambda}{2}(\hat{a}^{\dagger})^2\hat{a}^2 \ , \quad (4.1)$$

where λ is related to the third-order susceptibility of the medium. Subsequently, the model has been generalized by Gerry[20] to the case of the s-photon anharmonic oscillator

$$\hat{H}=\omega\hat{a}^{\dagger}\hat{a}+\frac{\lambda}{s}(\hat{a}^{\dagger})^s\hat{a}^s \ . \quad (4.2)$$

It has been shown that due to the nonlinear interaction, the ordinary squeezing can appear for the initial coherent state of the field. Besides, it has been shown that the systems under consideration can serve as a generator of amplitude-squared squeezed light.[21,22]

One of the possible generalizations of the nonlinear Hamiltonian (4.2) can be written in the form

$$\hat{H}=\omega\hat{a}^{\dagger}\hat{a}+\frac{\lambda}{s}(\hat{A}^{\dagger}_{(k)})^s\hat{A}^s_{(k)} \ , \quad (4.3)$$

where ω is the characteristic frequency of the field and λ is the coupling constant related to the nonlinear susceptibility and s is an integer. The Hamiltonian (4.3) given in terms of the multiboson operators $\hat{A}^{\dagger}_{(k)}$ and $\hat{A}_{(k)}$ preserves the structure of the Hamiltonian (4.2) and in the case $k=1$ reduces to (4.2). Another possible generalization of the nonlinear Hamiltonian (4.2) can be thought by considering the terms of the form $\hat{a}\hat{A}^{\dagger}_{(k)}$ or $\hat{a}\hat{A}_{(k)}$. For instance, one can study the dynamics of the nonlinear oscillator given by the Hamiltonian

$$\hat{H}=\omega\hat{a}^\dagger\hat{a}+\frac{\lambda}{s}(\hat{A}^\dagger_{(k)})^s(\hat{A}_{(k)})^s+\varepsilon(\hat{a}+\hat{a}^\dagger)(\hat{A}_{(k)}+\hat{A}^\dagger_{(k)}) .$$

In this case the generalized Hamiltonian does not preserve the subspace structure of the Hilbert space generated by the action of the operator (2.6) upon the vacuum state.

The time evolution of the operator $\hat{A}_{(k)}(t)$ is given by the Heisenberg equation with $\hat{H}$ given by (4.3)

$$i\frac{d}{dt}\hat{A}_{(k)}=[\hat{A}_{(k)},\hat{H}] , \tag{4.4}$$

and the solution of this equation can be written as

$$\hat{A}_{(k)}(t)=\exp(-it\{k\omega+\lambda[\hat{A}^\dagger_{(k)}(0)]^{s-1}\times[\hat{A}_{(k)}(0)]^{s-1}\})\hat{A}_{(k)}(0) . \tag{4.5}$$

Further, we will consider only the variance in the first quadrature for which we have

$$\langle(\Delta\hat{X}^1_{(k)})^2\rangle=\tfrac{1}{2}\langle\hat{A}^\dagger_{(k)}(t)\hat{A}_{(k)}(t)\rangle+\tfrac{1}{2}\mathrm{Re}\langle\hat{A}^2_{(k)}(t)\rangle-[\mathrm{Re}\langle\hat{A}_{(k)}(t)\rangle]^2+\tfrac{1}{4} . \tag{4.6}$$

Dropping the free-evolution term and assuming the light to be initially in the k-photon coherent state $|\alpha\rangle_k$, we obtain for the mean values of the relevant operators

$$\begin{aligned}
{}_k\langle\alpha|\hat{A}_{(k)}(t)|\alpha\rangle_k&=\alpha\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\exp\left[-i\tau\frac{n!}{(n-s+1)!}\right] ,\\
{}_k\langle\alpha|\hat{A}^2_{(k)}(t)|\alpha\rangle_k&=\alpha^2\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\exp\left[-i\tau\left[\frac{n!}{(n-s+1)!}+\frac{(n+1)!}{(n-s+2)!}\right]\right] ,\\
{}_k\langle\alpha|\hat{A}^\dagger_{(k)}(t)\hat{A}_{(k)}(t)|\alpha\rangle_k&=|\alpha|^2 ,
\end{aligned} \tag{4.7}$$

where we have set $\lambda t=\tau$. Using the expressions (4.6) and (4.7) we can evaluate the function

$$Q^1_{(k)}(t)=4\langle[\Delta\hat{X}^1_{(k)}(t)]^2\rangle-1 .$$

Further we will analyze the Hamiltonian (4.3) for $s=2$. In this case the function $Q^i_{(k)}$ can be obtained in a closed form. If α is chosen to be real, then we can write

$$Q^i_{(k)}(t)=2\alpha^2\{1+\exp[\alpha^2(\cos2\tau-1)]\cos(\tau+\alpha^2\sin2\tau)-2\exp[2\alpha^2(\cos\tau-1)]\cos^2(\alpha^2\sin\tau)\} . \tag{4.8}$$

This function describes the time evolution of amplitude kth-power squeezing govered by (4.3) of an initially k-photon coherent state (2.7). Now the question arises of whether the nonlinear interaction governed by (4.3) leads to the amplitude kth-power squeezing of the k-photon coherent state (2.7). To answer this question we evaluate the first and the second derivatives of the function $Q^i_{(k)}$ at the initial moment of the evolution. It can be easily found that $(\partial/\partial\tau)Q^1_{(k)}(\tau)|_{\tau=0}=0$ and

$$\frac{\partial^2}{\partial\tau^2}Q^1_{(k)}(\tau)|_{\tau=0}=-2\alpha^2(1+4\alpha^2) , \tag{4.9}$$

from which it follows that, due to the interaction with the material medium, the k-photon coherent state starts to be amplitude k power squeezed. Moreover, the higher the initial photon number ($\langle\hat{n}\rangle\simeq\alpha^2$) is, the more rapidly squeezing appears. The first minimum of the function $Q^1_{(k)}(t)$ for $\alpha^2\to\infty$ is equal to 68%. Then squeezing is revoked, but after some time it appears again and its maximum value is almost 100%. To analyze the high-intensity limit ($\alpha^2\gg1$) it is convenient to introduce new scaled time $\sigma=\alpha^2\tau$. Then the function $Q^1_{(k)}(\sigma)$ up to terms $0(1/\alpha^2)$ can be written in the following form:

$$Q^1_{(k)}(\sigma)\cong[2\sigma\sin(\sigma)-\cos(\sigma)]^2-\cos^2(\sigma) . \tag{4.10}$$

The time behavior of the function $Q^1_{(k)}(\sigma)$ is given in Fig. 1. From this picture it is seen that A^kS appears regularly with the period of π in the scaled time σ.

Here we can mention that the explicit expression for the functions $Q^i_{(k)}$ does not depend on k, from which it follows that the functions $Q^i_{(k)}$ describe the time evolution of the ordinary squeezing ($k=1$) of the field initially prepared in Glauber's coherent state[22] as well as A^kS of the k-photon coherent field for any k.

Finishing the paper, we can conclude the following.

(i) In the paper the definition of the amplitude kth-power squeezing has been given.

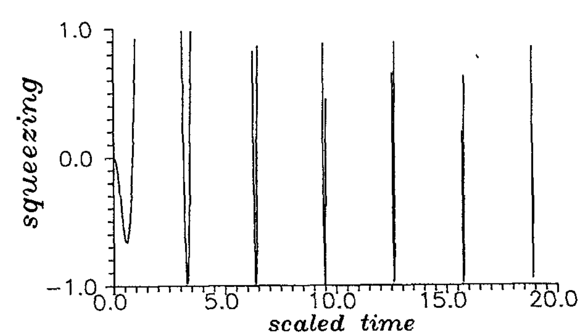

FIG. 1. The time dependence of squeezing given by the function $Q^1_{(k)}(\sigma)$. From the picture it is seen that only the first minimum differs considerably from 100% squeezing. Scaled time σ is equal to $\lambda\alpha^2t$.

(ii) The squeezing properties (particularly A^kS) of the k-photon coherent states (CS) have been analyzed. It has been shown that these states are not amplitude kth power squeezed.

(iii) The nonlinear interaction of the k-photon CS governed by the Hamiltonian (4.3) leads to A^kS of the k-photon CS. In this case, the time behavior of A^kS has been analyzed in detail. In particular, we have shown that the A^kS behaves periodically for high intensities of the initial field [see Eq. (4.10) and Fig. 1].

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